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LOWER SEMICONTINUOUS DIFFERENTIAL INCLUSIONS WITH STATE CONSTRAINS [1]

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Abstract. The main purpose of the present paper is to impose one-sided Lipschitz condition for differential inclusions on a closed and convex domain of a uniformly convex Banach space. Both differential inclusions with almost upper demicontinuous and almost lower semicontinuous right–hand sides are considered. The existence theorems are proved and it is shown that the set of solutions is connected.

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1. Introduction

In the paper we examine the solution set of the following Cauchy problem

$$(CP) \qquad \dot{x} \in F(t,x),\ x(0) = x_0,\ x(t) \in D.$$

We prove that it is nonempty and connected, when F is Almost Lower Semi-Continuous (ALSC), satisfying the one sided Lipschitz condition. The solutions of (CP) are AC (absolutely continuous) functions such that (CP) holds almost everywhere in $t \in I = [0,1]$. The multifunction (multimap) F is defined on $I \times D$ and has nonempty compact values. Here E is a uniformly convex Banach space with uniformly convex dual E^*, $D \subset E$ is locally closed convex. First we consider (CP) with almost Upper Demi–Continuous (UDC) right-hand side with convex compact values. In this case we show that the solution set of (CP) is R_δ set. Afterwards we consider the ALSC case and using theorem 2 of [2] we prove that the solution set of (CP) is nonempty and connected. We only note that the results for differential inclusions with state constrains known in the literature are obtained under additional compactness assumptions. For differential equations with state constrain, satisfying dissipative type assumptions we refer to [8].

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We avoid the problem of continuation of local solution by imposing the growth condition:

$$|F(t,x)| = \max\{|y| : \; y \in F(t,x) \leq \lambda(t)(1+|x|) \text{ on } I \times D, \tag{1}$$

where $\lambda(\cdot)$ is an L^1 function. In the paper we extend the techniques used in [5] and our main results (theorems 1 and 2) improve Theorems 1 and 2 of that article.

We will use the following notes and notations:
– 2^E is the set of all nonempty subsets of E,
– $B_r(x)$ is the open ball centered at x with radius r,
– $\mathrm{cl}A$, $\mathrm{co}A$ is the closed, respectively the convex hull of $A \in 2^E$,
– $\rho(x,A) = \inf\limits_{y\in A} |x-y|$ is the distance from x to the set A,
– $D_H(A,B) = \max\{\sup\limits_{a\in A} \rho(a,B), \sup\limits_{b\in B} \rho(b,A)\}$ is the Hausdorff distance between the bounded sets $A, B \in 2^E$,
– $D_H^+(A,B) = \sup\limits_{a\in A} \rho(a,B)$ is the one-sided Hausdorff distance between the bounded sets $A, B \in 2^E$,
– $Proj_D(a) = \{a \in D : |a-d| = \rho(a,D)\}$ is the metric projection of the point a to the set D,
– $\sigma(x,A) = \sup\limits_{a\in A} \langle x, a \rangle$ is the support function of $A \in 2^E$ for every $x \in E^*$.
– $J(x) = \{x* \in E^* : \langle x^*, x \rangle = |x|^2 = |x^*|^2\}$ is the duality map. When E^* is uniformly convex $J(\cdot)$ is single valued and uniformly continuous on the bounded sets (see [3] for details).
– $\chi_A(x)$ is the characteristic function, i.e. $\chi_A(x) = 1$ for $x \in A$ and 0 elsewhere,
– $\omega(\delta, x)$ is the modulus of continuity of the duality map $J(\cdot)$ at the point x, i.e.

$$\omega(\delta, x) = \sup_{|y|\leq\delta} |J(x) - J(x+y)|.$$

According to [3] (proposition 4.5) $\lim\limits_{\delta\to 0} \sup\limits_{x\in D} \omega(x,\delta) = 0$.

We call $F(\cdot,\cdot)$ one-sided Lipschitz continuous iff there exists L^1 function $k(\cdot)$ such that for every $x, y \in D$

$$\sigma(J(x-y), F(t,x)) - \sigma(J(x-y), F(t,y)) \leq k(t)|x-y|^2 \tag{2}$$

To avoid problems with measurability we suppose E is separable. The set $D \in 2^E$ is said to be locally closed if for every $x \in D$ there exists a neighborhood U_x such that $U_x \bigcap D$ is closed. In the sequel we suppose that D is locally closed.

Now we recall some definitions and notations and note that all the concepts not discussed in the sequel can be found in [3].

Definition 1. *The multimap F is called almost LSC iff there exists a sequence $\{J_m\}_{m=1}^\infty$ of mutually disjoint compact subsets of I such that*

$\text{meas}(I \setminus \bigcup_{i=1}^{\infty} J_i) = 0$ *and* F *is LSC on* $J_i \times Y$ *for every* i. F *is LSC at* (t,x) *when to* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $F(s,y) + \varepsilon U \supset F(t,x)$ *for all* (s,y) *with* $|t-s| + |x-y| < \delta$. *Here* $U = \{x \in E \mid |x| \le 1\}$.

Since F admits compact values this definition is equivalent to:

For every $u^0 \in F(t,x)$ *and every* $t^i \to t$, $x^i \to x$ *there exist* $u^i \in F(t^i, x^i)$ *with* $u^i \to u^0$.
The multimap $F(t,\cdot)$ *is called UDC when* $\sigma(l, F(t,\cdot))$ *is USC as a real valued function for all* $l \in E^*$. *Similarly,* F *is called almost UDC iff there exists a sequence* $\{J_m\}_{m=1}^{\infty}$ *of mutually disjoint compact subsets of* I *such that* $\text{meas}(I \setminus \bigcup_{i=1}^{\infty} J_i) = 0$ *and* F *is UDC on* $J_i \times Y$ *for all* i.

Given $M > 0$ we define the cone $\Gamma^M := \{(t,x) \in I \times E : t \ge 0; |x| \le Mt\}$.

Definition 2. *Let* $A \subset I \times E$. *The map* $f : A \to E$ *is said to be* Γ^M *continuous at* (t_0, x_0) *if for* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$|f(t,x) - f(t_0,x_0)| < \varepsilon, \text{ whenever } (t,x) \in B_\delta(t_0,x_0) \bigcap A \bigcap [t_0,x_0) + \Gamma^M].$$

The Bouligand contingent cone for $x \in D$ is

$$T_D(x) = \{y \in E \lim_{\lambda \to 0^+} \lambda^{-1} \rho(x + \lambda y, D) = 0\} = \text{cl}\{\lambda(y-x), \lambda \ge 0, y \in D\}$$

because D is locally closed and convex. To obtain the existence of the solutions we need of the subtangential condition

$$F(t,x) \subset T_D(x) \text{ for all } x \in D \tag{3}$$

The following lemma, proved in [2] will play a crucial role in the sequel.

Lemma 1. *For every* $M > 0$ *and* $\Omega \subset I \times E$ *the LSC multimap* $F : \Omega \to 2^E$ *with closed values admits* Γ^M *continuous selection.*

2. Main results

First we will prove the existence result for UDC multimaps. We use (with essential modifications) the main idea of [7]. We need two auxiliary lemmas.

Lemma 2. *Suppose* F *is almost UDC convex compact valued. If* F *satisfies (1) and if* $W(t,x) = F(t,x) \bigcap T_D(x) \ne \emptyset$ *satisfies (2) then one can reduce (1) and (2) to the case* $\lambda(t) \equiv k(t) \equiv 1$ *preserving the other hypotheses.*

Proof. Define $\varphi(t) = \max\{1, k(t), \lambda(t)\} > 0$. The map $t \to \int_0^t \varphi(s)\,ds$ is continuous and strongly monotonely increasing, i.e. invertible. Let $\Phi(\cdot)$ be its inverse, i.e. $\Phi\Big(\int_0^t \varphi(s)\,ds\Big) = t$. Define $\tilde{F}(t,x) = \frac{1}{\varphi(\Phi(t))} F(\Phi(t), x)$ for $(t,x) \in I \times D$. Evidently, $\tilde{F}$ satisfies all the conditions mentioned above with $k(t) \equiv \lambda(t) = 1$. Moreover the set of trajectories, as curves in the phase space, is preserved (see also [6]). $\square$

Let $x(\cdot), x(0) = x_0$ be AC with $\rho(\dot{x}(t), F(t, x(t) + B_1(0)) \le 1$. Then the Gronwall inequality and (1) imply the existence of the constants M and N such that $|x(t)| \le N$, $|F(t, x(t))| \le M$. So we suppose $F(\cdot,\cdot)$ is bounded. Moreover the solution set of (CP) coincides with the solution set of

$$(SP) \qquad \dot{x}(t) \in W(t, x(t)) = F(t,x) \bigcap T_D(x), \quad x(0) = x_0, \quad x(t) \in D$$

Indeed, if $x(t) \in D$ then $\dot{x}(t) = \lim\limits_{h \to 0} \frac{x(t+h) - x(t)}{h} \in T_D(x)$.

Definition 3. *The function $y(\cdot)$ is called quasipolygonal μ–solution of (SP) with $y(0) = x_0$ iff:*

1) There exists a countable pairwise disjoint family $I_k = [\tau_k, \eta_k)$ of semiopen intervals for which $[0,1) = \bigcup\limits_{k=1}^{\infty} I_k$.

2) $y(t) = y(\tau_k) + \int_{\tau_k}^t f(\tau)\,d\tau$, where $f(t) \in W(t, y(\tau_k))$ on $[\tau_k, \eta_k)$.

3) $y(\tau_k) \in D$, $\rho(y(t), D) \le \mu/2$ and $y(\eta_k) = Proj_D\Big(y(\tau_k) + \int_{\tau_k}^{\eta_k} f(\tau)\,d\tau\Big)$.

. The quasipolygonal μ–solutions $y(\cdot)$ are right–hand side continuous, M–Lipschitz continuous on every interval I_k and possibly jump on the right ends of I_k.

Definition 4. *The M–Lipschitz continuous function $x(\cdot)$ is said to be (ε, μ)–solution iff:*

1) $x(t) = Proj_D(y(t))$ for some quasipolygonal solution μ–solution $y(\cdot)$.

2) $|J(x(\tau_i) - J(x(t))| \le \frac{\varepsilon}{3M}$ and $|x(t) - x(\tau_i)| \le \frac{\varepsilon}{2}$ for every $t \in I_i$.

Lemma 3. *Suppose F is almost UDC convex and compact valued. If $W(\cdot,\cdot)$ is nonempty valued satisfying (1), (2) and $W(I,\cdot)$ maps compact sets into relatively compact ones.*

Then for any fixed $\mu > 0, \varepsilon > 0$ there exist a subdivision P of I and its respective (ε, μ)–solution $x(\cdot)$. Furthermore, the set of all (ε, μ)–solutions is $C(I, E)$ compact.

Proof. Suppose the needed $x(\cdot)$ exists on $[0,T]$ (the interval is closed since $x(t)$ is M–Lipschitz continuous), $T<1$, T is an end point of the subdivision P. Let $x(T)\in A\subset D$, where A is a compact set. As long as $\mathrm{cl}(W(I,a))$ is compact for every $a\in A$ there exists $\lambda(\mu,a)>0$ such that $\lambda^{-1}D_H^+(a+\lambda\mathrm{cl}(W(I,a)),D)\le\dfrac{\mu}{2}$ for every $0<\lambda\le\lambda(\mu,a)$ and, obviously, for every integrable selection $f(t)\in W(t,a)$ one has $\lambda^{-1}\rho(a+\displaystyle\int_T^{T+\lambda}f(t)\,dt,D)\le\dfrac{\mu}{2}$ whenever $0<\lambda\le\lambda(\mu,a)$. Furthermore $\inf\limits_{a\in A}\lambda(\mu,a)=\lambda_\mu>0$ because A is a compact set and $\rho(b,D)$ is continuous in b. Hence for every $x\in A$ and every measurable selector $f(t)\in W(t,x)$ the following inequality holds $\lambda^{-1}\rho(x+\displaystyle\int_T^{T+\lambda}f(t)\,dt,D)\le\mu/2$ for every $0<\lambda<\lambda_\mu$. We set $y(t)=x(T)+\displaystyle\int_T^{T+t}f(\tau)\,d\tau$ and $x(t)=Proj_D(y(t))$.

The latter is well defined and single valued since D is locally closed and convex in a uniformly convex Banach space. Obviously, there exists $\lambda>0$ such that so defined $x(\cdot)$ is (ε,μ)–solution on $[0,T+\lambda]$.

We add the subinterval $[T,T+\lambda)$ in P.

Therefore, by simple application of the Zorn lemma $x(\cdot)$ exists on the whole I. It remains to show that the set of all such $x(\cdot)$ is $C(I,E)$ compact.

The set of (ε,μ)–solutions is bounded ($|x(\cdot)|\le M$) and every solution is M–Lipschitz continuous. By virtue of Arzela theorem we have to show that the attainable set $X(t)$ of all such $x(\cdot)$ is a precompact set for every $t\in I$. $X(0)\equiv x_0$ is obviously compact. On the first interval I_1 of P, $Y(t)=x_0+\displaystyle\int_0^t W(\tau,y_0)\,d\tau$ is compact for every $t\in I_1$. Thus, $X(t)=Proj_D(Y(t))$ is also compact.

By induction one can choose the intervals I_k so small that $|x(t)-x(\tau_k)|\le\dfrac{\varepsilon}{2}$ for every $t\in I_k$. The duality map $J(\cdot)$ is uniformly continuous on the bounded sets and E is uniformly convex Banach space. Therefore for an appropriate choice of the length of I_k we can write $|J(x(\tau_i)-J(x(t))|\le\dfrac{\varepsilon}{3M}$, $t\in I_k$.

One has only to apply the Zorn lemma to obtain the result. □

Recall ([3] p. 83) that A is said to be a (metric) absolute retract if, given any metric space Ω and closed $B\subset\Omega$ and a continuous $f:B\to A$, there exists a continuous extension $\tilde f:\Omega\to A$ of f. A is said to be a (metric) R_δ if A is compact and $A=\bigcap\limits_{k\ge1}A_k$ for decreasing sequence of compact (metric) absolute retracts A_k. However, the following conception is more convenient.

The set B is said to be contractible iff there is $x_0\in B$ and continuous $h:[0,1]\times B\to B$ such that $h(0,x)=x$ and $h(1,x)=x_0$ on B. From Proposition 5.1 of [1] we know that A is R_δ iff $A=\bigcap\limits_{n\ge1}B_n$ with decreasing sequence of closed contractible with $\alpha(B_n)\to0$, where α is the Kuratovski measure of noncompactness.

Theorem 1. *Suppose $F(\cdot,\cdot)$ is almost UDC convex compact valued. Let for every $\varepsilon > 0$ there exist $I_\varepsilon \subset I$ with $\mathrm{meas} I_\varepsilon > 1-\varepsilon$ such that $F(I_\varepsilon, X)$ is precompact for every compact $X \subset D$. If F satisfies (1) and $W(t,x) = F(t,x) \bigcap T_D(x) \neq \emptyset$ satisfies (2) then the solution set of (CP) is nonempty R_δ set.*

Proof. First we prove the existence of solutions. Let $\{\varepsilon_i\}_{i=1}^\infty$ and $\{\mu_i\}_{i=1}^\infty$ be two sequences of positive numbers such that the series $\sum_{i=1}^{\infty} \sqrt{\varepsilon_i + \mu_i}$ converges. We claim there exist sequences $\{x_i\}_{i=1}^\infty$ of M–Lipschitz continuous functions and $\{P_j\}_{j=1}^\infty$ of partitions of I such that

a) There exists integrable $\alpha_i(t)$ such that $\rho(\alpha_i(t), W(t, x_i(t) + B_{\varepsilon_i}(0)) \le \varepsilon_i$ and $|x_i(t) - \int_\tau^t \alpha_i(s)\, ds - x_i(\tau)| \le \mu_i$ for a.a. $t \in I$.

b) $|x_i(t) - x_{i+1}(t)|^2 \le r_i(t)$, where $\dot r_i(t) \le r_i(t) + 2(\varepsilon_i + \varepsilon_{i+1} + \mu_i + \mu_{i+1})$, $r(0) = 0$.

c) $|J(x) - J(x+y)| \le \varepsilon_i/(3M)$ for $|x| \le N$ $|y| \le (\tau_j^i - k_j^i)$, where τ_j^i, k_j^i are two successive points of P_j.

d) $|x_i(\tau_j^i) - x_i(t)| \le \varepsilon_i$, for every $t \in [\tau_j^i, k_j^i]$.

Due to Lemma 3 for (ε_1, μ_1) there exists a (ε_1, μ_1) solution $x_1(t)$ of (SP) and a respective partition P_1. Of course, c) and d) are fulfilled. As is done in the proof of Lemma 3 we can choose an integrable selection $\alpha_1(t)$ for which $\alpha_1(t) \in W(t, x_1(\tau))$ and

$$|x_1(t) - x_1(\tau) - \int_\tau^t \alpha_1(s)\, ds| \le |x_1(t) - x_1(\tau)| + |\int_\tau^t \alpha_1(s)\, ds| \le \varepsilon_1(M+1) < \mu_1,$$

where τ is any end point of the intervals of P_1. Thus, $x_1(t)$ and P_1 satisfy a), c) and d). We set $y_1(t) = x_1(\tau_k) + \int_{\tau_k}^t \alpha_1(s)\, ds$ for every $t \in [0,1]$. Now we are going to construct x_{i+1} if x_i is known.

First we consider the compact sets I_{δ_i} such that $\mathrm{meas}(I \backslash I_{\delta_i}) \le \delta_i$, $F(I_{\delta_i}, x)$ is compact for every $x \in D$ and $I_{\delta_i} \supset I_{\delta_{i+1}}$. Moreover, we define δ_i such that $M\delta_i \le \mu_i/4$. We show the existence of $x_{i+1}(\cdot)$ satisfying a) - d) if $x_i(\cdot)$ already exists. Extend $F(\cdot,\cdot)$ on $I_{\delta_{i+1}} \times D$ so that the extension has compact range on I for every fixed $x \in D$ and satisfies (1). Suppose as in the previous step that the needed x_{i+1} exists on $[0,T]$. If $T < 1$ then it belongs to some interval $[t_j^i, t_{j+1}^i)$, where the end points are two successive ones for which $y_i(\tau) = x_i(\tau)$. Here τ is an arbitrary end point of the above interval. If $T \in I_{\delta_i}$ we get a measurable selector $f(t) \in W(t, x_{i+1}(T))$ such that

$$\left\langle J(y_i(t) - y_{i+1}(T)), \alpha_i(t) - f(t) \right\rangle \le \{|y_i(t) - y_{i+1}(T)| + \varepsilon_i\}^2.$$

If $T \notin I_{\delta_i}$ we choose $f(t) \in W(t, x_{i+1}(T))$ arbitrary. Let $y_{i+1}(t) = x_{i+1}(T) + \int_T^t f(s)\, ds$ and let $x_{i+1}(t) = Proj_D(y_{i+1}(t))$. Therefore there exists $\tau > T$ such that $|x_{i+1}(t) - y_{i+1}(t)| \leq \mu_{i+1}$ for all $t \in [T, \tau]$ and

$$\left\langle J(y_i(t) - y_{i+1}(t)), \alpha_i(t) - f(t) \right\rangle \leq \left\{ |y_i(t) - y_{i+1}(T)| + \varepsilon_i + \varepsilon_{i+1} \right\}^2.$$

When $T \notin I_{\delta_i}$ we choose $\tau = \min\{t \in I_{\delta_i} : t > T\}$. Denote $|y_i(t) - y_{i+1}(t)|^2 = r_i(t)$. One can show that $\dot{r}_i(t) \leq r_i + 2(\varepsilon_i + \varepsilon_{i+1}) + m_i(t)$, for a.a. $t \in [T, \tau]$, where $m_i(t) = 0$ for $t \in I_{\delta_i}$ and $m_i(t) = M$ elsewhere. Therefore $|x_i(t) - x_{i+1}|^2 \leq r(t) + \mu_i + \mu_{i+1}$. Thus $x_{i+1}(\cdot)$ can be defined on the all I. The claim is proved.

We finish the proof using the sequence $\{x_i(\cdot)\}_{i=1}^{\infty}$. By the Gronwall inequality there exists a constant C, with $|x_i(t) - x_{i+1}(t)| \leq C\sqrt{\varepsilon_i + \varepsilon_{i+1} + \mu_i + \mu_{i+1}}$.

Therefore $\sum_{i=1}^{\infty} \sqrt{\varepsilon_i + \mu_i + \varepsilon_{i+1} + \mu_{i+1}}$ converges and hence $\{x_i(\cdot)\}_{i=1}^{\infty}$ is a Cauchy sequence in $C(I, E)$. If $x(t)$ is its limit, then it is routine to prove that $x(\cdot)$ is in fact a solution of (CP).

Let $R_i(t)$ be the reachable set of all ε_i- solutions, satisfying a), c), d). The latter is compact for every $t \in I$. Furthermore, taking the sequence $\{x_i(t)\}_{i=1}^{\infty}$ of arbitrary ε_i-solutions (satisfying a), b), c), d)) one has that it is $C(I, E)$ precompact. Thus passing to subsequences if necessary $x_i(t) \to x(t)$ as $i \to \infty$, where $x(\cdot)$ is a solution of (CP).

Let $x(\cdot)$ be a solution of (CP). Consider the corresponding to ε_i subdivision $\{\tau_j^i\}_{i=1}^{\infty}$ of the interval $I = [0, 1]$ such that $x_i(\cdot)$ satisfies a), c), d). We get $f_i(t) \in W(t, x(\tau_j^i))$ such that

$$\left\langle J(x(t) - x_i(\tau_j^i)), \dot{x}(t) - f_i(t) \right\rangle \leq (M + N)|x(t) - x_i(\tau_j^i)|^2$$

consequently,

$$\left\langle J(x(t) - x_i(t)), \dot{x}(t) - \alpha_i(t) \right\rangle \leq C\{|x(t) - x_i(t)|^2 + \varepsilon_i + (\tau_{j+1}^i - \tau_j^i)\},$$

where C is a constant, dependent on M, N, but not on ε. Thus $\lim_{i \to \infty} D_H(R_{CP}, R_i) = 0$. Here X_1 is as in step 1. As in the proof of Theorem 5.2 of [1] we consider the sequence of the locally Lipschitz $F_n(t, x) \supset F(t, x)$ on $I \times D$. Denote $\tilde{F}_n(t, x) = \overline{F_n(t, x) + B_{c_n}(0)}$, where $c_n = 2^{-n}$. The solution set S_n of (CP) with $\tilde{F}_n$ instead of F is closed contractible and $\bigcap_{n \geq 1} S_n = S$ the solution set of (CP) is compact and $\lim_{n \to \infty} \alpha(S_n) = 0$, where α is the Kuratowski measure for noncompactness. Hence S is nonempty R_δ (see [1] for details). $\square$

Now we are ready to prove the main result of the paper.

Theorem 2. *Let $F(\cdot, \cdot)$ be an ALSC compact valued multimap satisfying (1), (2) and (3). Suppose moreover that for every $\delta > 0$ there exists $I_\delta \subset I$ such that*

$H(I_\delta, X)$ is precompact for every compact $X \subset D$, where the map $H_i(t,x) :=$ $\operatorname{clco}\left(\bigcap_{\varepsilon>0} F(t, B_\varepsilon(x) \bigcap D)\right)$. Then the solution set of (CP) is nonempty and connected.

Proof. Consider the differential inclusion

$$\dot{x}(t) \in H(t,x), \quad x(t) = x_0; x(t) \in D \tag{4}$$

Where $H(t,x) = H_i(t,x)$ for $t \in A_i$. Obviously $H(\cdot,\cdot)$ satisfies all the conditions of Theorem 1. Let now $f_i(t,x)$ be Γ^{M+1} continuous selection of $F(t,x)$ on $A_i \times D$. Denote $h_i(t,x) := \operatorname{cl}\operatorname{co} \bigcap_{\varepsilon>0} f(A_i \bigcap [t-\varepsilon, t+\varepsilon], B_\varepsilon(x) \bigcap D)$. We set $h(t,x) = h_i(t,x)$ for $t \in A_I$. Thus $h(t,x) \subset H(t,x)$. Obviously, $h(t,\cdot)$ is UDC, $h(\cdot,x)$ is measurable (see the proof of Theorem 6.2 of [3]) and $h(t,x) \bigcap T_D(x) \neq \emptyset$. One can consider the sequences $\{\varepsilon_i\}_{i=i}^\infty$ and $\{x_i(\cdot)\}_{i=i}^\infty$ (as in the proof of Theorem 1) such that $x_i(\cdot)$ satisfies a), c), d) in the proof of the claim but $F(t,x)$ is replaced by $h(t,x)$. As shown $\{x_i(\cdot)\}_{i=i}^\infty$ is $C(I,E)$ precompact. Therefore passing to subsequences if needed one obtains the existence of $\lim_{i\to\infty} x_i(t) = x^0(t)$ which is a solution of (CP) with $F(t,x)$ replaced by $h(t,x)$. As in the proof of Theorem 6.1 of [3] $\dot{x}^0(t) \in f(t, x^0(t))$. Thus (CP) admits a solution.

Let u_2, u_2 be two solutions of (CP). Let $f_i(\cdot)$ be a measurable selection of $F(\cdot, u_i(\cdot))$ $i = 1,2$. For $i = 1,2$ consider the map

$$F^i(t,x) := \begin{cases} f_i(t) & \text{for } x = u_i(t) \\ F(t,x) & \text{otherwise.} \end{cases}$$

Since F is ALSC by the Lusin theorem there exists a sequence of mutually disjoint compacts $J_n \subset I$ with $\operatorname{meas}(I \setminus \bigcup J_n) = 0$ such that $\dot{u}_i(\cdot)$ are continuous on J_n and $F^i(\cdot,\cdot)$ is LSC on $J_n \times E$. Thus there exists Γ^{M+1} continuous selection $f_n^i(t,x) \in F^i(t,x)$, $t \in J_n, n = 1,2,...,\infty$. Define $h^i(t,x) = f_n^i(t,x) \quad t \in J_n$. Set $F_n^i(t,x) = \bigcap_{\varepsilon>0} \operatorname{cl}\operatorname{co}\{f_n^i(s,y) \textit{ for } |x-y| < \varepsilon;\ s \in [t, t+\varepsilon) \bigcap J_n\}$. Also define $H^i(t,x) = F_n^i(t,x)$, $t \in J_n$. For $\lambda \in [0,1]$ consider:

$$r_\lambda(t,x) = \chi_{[0,\lambda)}(t) h^1(t,x) + \chi_{[\lambda,1]}(t) h^2(t,x) \tag{5}$$

$$R_\lambda(t,x) = \chi_{[0,\lambda)}(t) H^1(t,x) + \chi_{[\lambda,1]}(t) H^2(t,x).$$

First note that $R_\lambda(t,x) \subset H(t,x)$. Let S_λ be the solution set of (CP) with $R_\lambda(\cdot,\cdot)$ instead of $F(\cdot,\cdot)$. From Theorem 1 S_λ is compact and connected. Obviously, S_λ is also the solution set of (5). Moreover $\lambda \to S_\lambda$ is USC. Thus $\bigcup_{\lambda\in[0,1]} S_\lambda \subset S(x)$ is compact and connected containing u_1 and u_2. Therefore R_{CP} is connected itself. □

Remark 1. The conditions of Theorem 2 are natural except for the requirement of the precompactness of $H(I_\delta, X)$ for every compact $X \subset D$. The latter obviously holds when $H(\cdot,\cdot)$ is almost USC or clco $F(\cdot,\cdot)$ is almost continuous and hence $H(t,x) \equiv \text{clco } F(t,x)$.

Remark 2. The main difficulties here come from the fact that the right-hand side F is defined only on D. If F is defined on the whole space E the convexity of D can be dispensed with. Furthermore, one can relax the assumptions to

A1. $F(\cdot,\cdot)$ is almost LSC and H maps compact sets into relatively compact ones.

A2. $F(t,x) \le \lambda(t)\{1+|x|\}$ and F satisfies (2).

A3. $F(t,x) \subset T_D(x)$ for every $x \in D$ (D is locally closed not necessarily convex).

Theorem 3. *If A1-A3 hold, then the differential inclusion (CP) admits nonempty solution set. (When D is convex the solution set is also connected)*

Proof. We claim that under A1-A3 the Cauchy problem

$$\dot{x}(t) \in H(t,x), \quad x(0) = x_0 \tag{6}$$

admits a compact solution set and any sequence of approximated solutions admits an accumulation point.

We use with modifications the method of [4] to prove the claim. Indeed, consider the subdivision $\Delta = \{t_i\}_{i=0}^N; t_i - t_{i-1} = h_y = \dfrac{1}{N}$. Let

$$\dot{y}(t) \in H(t, y(t_i)),\ t \in [t_i, t_{i+1}),\ i = 0, \cdots, K-1 \tag{7}$$

For small h_z consider

$$\dot{z}(\tau) \in H(\tau, z(\tau_j)),\ \tau \in [\tau_j, \tau_{j+1}),\ j = 0, \cdots, K-1,\ h_z = \frac{1}{K} \tag{8}$$

such that $\big\langle J(z(t_j) - y(\tau_i)), \dot{z}(t) - \dot{y}(t)\big\rangle \le \omega(t, z(t_j) - y(\tau_i))|z(t_j) - y(\tau_i)|$ for $\tau_j \in [t_i, t_{i+1})$. In this case

$$\begin{aligned}
&\big\langle J(z(t) - y(t)), \dot{z}(t) - \dot{y}(t)\big\rangle \le \omega(t, z(t) - y(t))|z(t) - y(t)| \\
&+|\omega(t, z(t) - y(t))|z(t) - y(t)| - \omega(t, z(t_j) - y(\tau_i))|z(t_j) - y(\tau_i)| \\
&+|J(z(t_j) - y(\tau_i))||\dot{z}(t) - \dot{y}(t)|
\end{aligned}$$

Since H is bounded and since ω and J are continuous one has that for every $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exist h_z and h_y such that

$$\big\langle J(z(t) - y(t)), \dot{z}(t) - \dot{y}(t)\big\rangle \le u(t, |z(t) - y(t)|)|z(t) - y(t)| + \varepsilon_1 + \varepsilon_2.$$

Denote by $\{y_j(\cdot)\}_{j=1}^\infty$ the sequence of solutions of (7) with t_i replaced by t_i^j and K – by K^j. So it is not difficult to show using the same arguments as in the proof of Theorem 1 in [5] that there exist $\{\varepsilon_j\}_{j=1}^\infty$ and $\{y_j(\cdot)\}_{j=1}^\infty$ is a Cauchy sequence in $C(I,E)$. It is also routine to prove that if $y_j(t) \to y(t)$ then $y(\cdot)$ is a solution of (6). Let $x(\cdot)$ be a solution of (6). Consider the solution $z(\cdot)$ of (8) with $z(0) = x_0$ and $\langle J(x(t) - z(t_i)), \dot{x}(t) - f(t)\rangle \leq \omega(t, x(t) - z(t_i))|x(t) - z(t_i)|$ on $[t_i, t_{i+1}]$ and $z(t) = z(t_i) + \int_{t_i}^{t} f(s)\, ds$, for $i = 0, 1, \cdots, N-1$. Therefore

$$\begin{aligned}
&\langle J(z(t) - x(t)), f(t) - \dot{x}(t)\rangle \leq \omega(t, z(t) - x(t))|z(t) - x(t)| \\
&+|\omega(t, z(t) - x(t))|z(t) - x(t)| - \omega(t, z(t_i) - x(t))|z(t_i) - x(t)| + \\
&|J(z(t_i) - x(t))||f(t) - \dot{y}(t)|
\end{aligned}$$

Obviously, one has that $\lim\limits_{K\to\infty} D_H(R_K, R_{Ch}) = 0$ where we have denoted the solution set of (6) by R_{Ch} and the solution set of (8) by R_K. Therefore the claim is proved.

Let $f(t,x) \in F(t,x)$ be as in the proof of Theorem 2. Consider the differential inclusion

$$\dot{x}(t) \in h(t,x), \quad x(0) = x_0, \quad x(t) \in D$$

One can continue as in the proof of Theorem 4.1 of [1]. □

Remark 3. Suppose (2) holds with $k(t) \equiv k$ (constant). Then one can replace (1) by

$F(\cdot,\cdot)$ *is bounded on bounded sets.*

Indeed, in this case one can easily show that the map $x \to F(t, x + B_1(0))$ is also one-sided Lipschitz with a constant k.

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