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QUASI–NORMED OPERATOR IDEALS ON BANACH SPACES AND INTERPOLATION

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Abstract. We prove that applying real methods of interpolation, more exactly the K-method, to the couples and triples of quasi–normed operator ideals on the Banach space, new operator ideals are obtained. Extending the results of C. Bennett and R. Sharpley (see [1]) from the function spaces to ideals, we present a variant of reiteration theorem for the couples of quasi–normed operator ideals.

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1. Introduction

We denote by $\mathcal{L}$ the class of all linear continuous operators acting between the Banach spaces and by $\mathcal{L}(E,F)$ those which act from Banach space E to F. It is known that $\mathcal{L}(E,F)$ is a Banach space with the usual operator norm.

Recall (after Pietsch [7]) that a subclass $\mathcal{A} \subset \mathcal{L}$ is an operator ideal on Banach spaces if its components $\mathcal{A}(E,F) := \mathcal{A} \cap \mathcal{L}(E,F)$ satisfy the following conditions:
(O.I.0) $I_{\mathbf{K}} \in \mathcal{A}(\mathbf{K},\mathbf{K})$, where $I_{\mathbf{K}}$ is the identity on the scalar field $\mathbf{K}$.
(O.I.1) It follows from $S_1, S_2 \in \mathcal{A}(E,F)$ that $S_1 + S_2 \in \mathcal{A}(E,F)$.
(O.I.2) $T \in \mathcal{L}(X,E)$, $S \in \mathcal{A}(E,F)$, $R \in \mathcal{L}(F,Y)$ then $RST \in \mathcal{A}(X,Y)$.

A positive function A defined on an operator ideal which satisfies the conditions:
(Q.O.I.0) $A(I_{\mathbf{K}}) = 1$.
(Q.O.I.1) There exists a constant $\lambda \geq 1$ such that

$$A(S_1 + S_2) \leq \lambda[A(S_1) + A(S_2)], \text{ for } S_1, S_2 \in \mathcal{A}(E,F).$$

(Q.O.I.2) If $T \in \mathcal{L}(X,E)$, $S \in \mathcal{A}(E,F)$ and $R \in \mathcal{L}(F,Y)$ then

$$A(RST) \leq \|R\| A(S) \|T\|$$

will be called a quasi-norm on $\mathcal{A}$. It is clear that $\mathcal{A}(E,F)$ endowed with the quasi–norm A is a linear topological Hausdorff space. The couple $(\mathcal{A}, A)$ will be

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called a quasi–normed operator ideal on Banach spaces if, for each pair (E,F), $\mathcal{A}(E,F)$ is complete.

Recall that a Banach couple $\overline{X}=(X_1,X_2)$ means two Banach spaces X_j $(j=1,2)$ continuously embedded in some linear topological Hausdorff space.

For a Banach couple $\overline{X}$ we define the spaces $X_\Delta = X_1\cap X_2$ and $X_\Sigma = = X_1+X_2$, which are Banach spaces with respect to the norms:

$$\|x\|_\Delta := \max\{\|x\|_{X_1}, \|x\|_{X_2}\}, \quad (x\in X_\Delta) \tag{1.1}$$

and

$$\|x\|_\Sigma := \inf\{\|x_1\|_{X_1}+\|x_2\|_{X_2} : x=x_1+x_2,\ x_i\in X_i\},\ (x\in E_\Sigma). \tag{1.2}$$

For a Banach couple $\overline{X}=(X_1,X_2)$ and $t>0$ we define the functional

$$K(t,a)=K(t,a;\overline{X})=\inf_{a=a_1+a_2}\{\|a_1\|_{X_1}+t\|a_2\|_{X_2}\}$$

which is an equivalent norm on X_Σ, for every $t>0$, fixed.

Let $\overline{X}=(X_1,X_2)$ be a given Banach couple. Then a Banach space X will be called an intermediate space between X_1 and X_2 (or with respect to $\overline{X}$) if $X_\Delta \hookrightarrow X \hookrightarrow X_\Sigma$.

Definition 1.1. *Let $\overline{X}=(X_1,X_2)$ be a Banach couple and $0<\theta<1$, $1\le q<\infty$ or $0\le\theta\le 1$, $q=\infty$. The space*

$$\overline{X}_{\theta,q}=(X_1,X_2)_{\theta,q}$$

consists of all elements $f\in X_1+X_2$ for which

$$\|f\|_{\theta,q} := \alpha \left(\begin{array}{ll} \left(\int\limits_0^\infty [t^{-\theta}K(t,f,\overline{X})]^q \dfrac{dt}{t}\right)^{1/q} & ,\ \text{if } 0<\theta<1,\ 1\le q<\infty \\ \sup\limits_{t>0} t^{-\theta}K(t,f,\overline{X}) & ,\ \text{if } 0\le\theta\le1,\ q=\infty \end{array}\right.$$

is finite.

Theorem 1.1. *(T. Holmstedt's (see [1])). Let $\overline{X}=(X_1,X_2)$ be a Banach couple and the interpolation spaces $\overline{X}_{\theta_0}=(X_1,X_2)_{\theta_0,q_0}$, $\overline{X}_{\theta_1}=(X_1,X_2)_{\theta_1,q_1}$, where $0<\theta_0<\theta_1<1$ and $1\le q_0,\ q_1\le\infty$.*

Denoting by

$$K(t,f)=K(t,f,X_1,X_2), \quad \overline{K}(t,f)=K(t,f,\overline{X}_{\theta_0},\overline{X}_{\theta_1})$$

and $\delta=\theta_1-\theta_0$, we have

$$\overline{K}(t^\delta,f)\sim\left\{\int\limits_0^t [s^{-\theta_0}K(s,f)]^{q_0}\frac{ds}{s}\right\}^{1/q_0} + t^\delta\left\{\int\limits_t^\infty [s^{-\theta_1}K(s,f)]^{q_1}\frac{ds}{s}\right\}^{1/q_1} \tag{1.3}$$

for any $f \in \overline{X}_{\theta_0} + \overline{X}_{\theta_1}$ and any $t > 0$; if q_0 or q_1 are infinites the right-hand side of the relation (1.3) will be modified in a suitable way.

Definition 1.2. *Let $\overline{X}$ be a given Banach couple and X an intermediate space with respect to $\overline{X}$. Then we say that $X \in \mathcal{C}_K(\theta, \overline{X})$ if $K(t, a, \overline{X}) \leq$ $\leq c \cdot t^\theta \|a\|_X$, $a \in X$.*

Theorem 1.2 *Suppose that $0 < \theta < 1$. Then:*
(a) *$X \in \mathcal{C}_K(\theta, \overline{X})$ iff $X_\Delta \hookrightarrow X \hookrightarrow \overline{X}_{\theta,\infty}$.*
(b) *$X \in \mathcal{C}_K(\theta, \overline{X})$ if $(X_1, X_2)_{\theta,1} \hookrightarrow X \hookrightarrow (X_1, X_2)_{\theta,\infty}$.*
Obviously, $\overline{X}_{\theta,1} \hookrightarrow \overline{X}_{\theta,p} \hookrightarrow \overline{X}_{\theta,\infty}$.

Lemma 1.1. *(G. H. Hardy's). Let ψ be a measurable non-negative function on $(0, \infty)$, $-\infty < \lambda < 1$ and $1 \leq q < \infty$. Then:*

$$\left\{ \int_0^\infty \left(t^\lambda \cdot \frac{1}{t} \int_0^t \psi(s) ds \right)^q \frac{dt}{t} \right\}^{1/q} \leq \frac{1}{1-\lambda} \left\{ \int_0^\infty (t^\lambda \psi(t))^q \frac{dt}{t} \right\}^{1/q}$$

and

$$\left\{ \int_0^\infty \left(t^{1-\lambda} \int_t^\infty \psi(s) \frac{ds}{s} \right)^q \frac{dt}{t} \right\}^{1/q} \leq \frac{1}{1-\lambda} \left\{ \int_0^\infty (t^{1-\lambda} \psi(t))^q \frac{dt}{t} \right\}^{1/q}.$$

2. Interpolation of operator ideals

Considering two quasi-normed operator ideals on Banach spaces we define a new operator ideal in the following way:

Definition 2.1. *Let $(\mathcal{A}, a)$, $(\mathcal{B}, b)$ be two quasi-normed operator ideals on Banach spaces. For $1 \leq p < \infty$, $0 < \theta < 1$ we define:*

$$\mathcal{C}_{\theta,p} := (\mathcal{A}, \mathcal{B})_{\theta,p}$$

in the following way: for an arbitrary pair of Banach spaces (E, F)

$$\mathcal{C}_{\theta,p}(E, F) := (\mathcal{A}(E, F), \mathcal{B}(E, F))_{\theta,p} =$$

$$= \left\{ T \in \mathcal{A}(E, F) + \mathcal{B}(E, F) \mid \int_0^\infty \left(\frac{K(t, T, \mathcal{A}(E, F), \mathcal{B}(E, F))}{t^\theta} \right)^p \frac{dt}{t} < \infty \right\},$$

where $K(t, T, \mathcal{A}(E, F), \mathcal{B}(E, F)) = \inf_{T = T_1 + T_2} \{a(T_1) + t \cdot b(T_2)\}$, $t > 0$ (it will be denoted by $K(t, T)$).

Theorem 2.1. *$\mathcal{C}_{\theta,p}$ is an operator ideal on Banach spaces.*

Proof. We prove that the three conditions of the definition of ideals are satisfied.

(OI.0) $I_{\mathbf{K}} \in \mathcal{C}_{\theta,p}(\mathbf{K},\mathbf{K})$.

This condition is satisfied because $I_{\mathbf{K}} \in \mathcal{A}(\mathbf{K},\mathbf{K})$ and $I_{\mathbf{K}} \in \mathcal{B}(\mathbf{K},\mathbf{K})$, $K(1, I_{\mathbf{K}}) \le \min(1,t)$ involves

$$\int_0^\infty \left(\frac{K(t,I_{\mathbf{K}})}{t^\theta}\right)^p \frac{dt}{t} \le \int_0^\infty \left(\frac{\min(1,t)}{t^\theta}\right)^p \frac{dt}{t} = \int_0^1 \left(\frac{t}{t^\theta}\right)^p \frac{dt}{t} + \int_1^\infty \left(\frac{1}{t^\theta}\right)^p \frac{dt}{t} =$$

$$= \frac{1}{p(1-\theta)} + \frac{1}{p\theta} < \infty.$$

(OI.1) It follows from $T_1, T_2 \in \mathcal{C}_{\theta,p}(E,F)$ that $T_1 + T_2 \in \mathcal{C}_{\theta,p}(E,F)$.

Obviously, we have $T_1 + T_2 \in \mathcal{A}(E,F) + \mathcal{B}(E,F)$ (being linear spaces) and $K(t, T_1 + T_2) \le \lambda_1 \max\left(1, \frac{\lambda_2}{\lambda_1}\right) [K(t,T_1) + K(t,T_2)]$ implies

$$\int_0^\infty \left(\frac{K(t,T_1+T_2)}{t^\theta}\right)^p \frac{dt}{t} \le \left[\lambda_1 \max\left(1, \frac{\lambda_2}{\lambda_1}\right)\right]^p \int_0^\infty \left(\frac{K(t,T_1)+K(t,T_2)}{t^\theta}\right)^p \frac{dt}{t} \le$$

$$\le c \int_0^\infty \left(\frac{\max(K(t,T_1),K(t,T_2))}{t^\theta}\right)^p \frac{dt}{t} < \infty$$

because $T_1, T_2 \in \mathcal{C}_{\theta,p}(E,F)$.

(OI.2) If $T \in \mathcal{L}(E_0,E)$, $S \in \mathcal{C}_{\theta,p}(E,F)$, $R \in \mathcal{L}(F,F_0)$, then $RST \in \in \mathcal{C}_{\theta,p}(E_0,F_0)$.

It follows from $RST \in \mathcal{A}(E_0,F_0) + \mathcal{B}(E_0,F_0)$, and $K(t,RST) \le \le \|R\| K(t,S) \|T\|$ that

$$\int_0^\infty \left(\frac{K(t,RST)}{t^\theta}\right)^p \frac{dt}{t} \le (\|R\| \cdot \|T\|)^p \int_0^\infty \left(\frac{K(t,S)}{t^\theta}\right)^p \frac{dt}{t} < \infty.$$

Theorem 2.2. *The couple $(\mathcal{C}_{\theta,p}, c_{\theta,p})$, where $c_{\theta,p}$ is defined by:*

$$c_{\theta,p}(T) := [p\theta(1-\theta)]^{1/p} \left(\int_0^\infty \left(\frac{K(t,T)}{t^\theta}\right)^p \frac{dt}{t}\right)^{1/p}, \ 1 \le p < \infty, \ 0 < \theta < 1,$$

is a quasi-normed operator ideal on Banach spaces.

Proof. (QOI.0) $c_{\theta,p}(I_{\mathbf{K}}) = 1$.

By definition we have

$$(c_{\theta,p}(I_{\mathbf{K}}))^p = p\theta(1-\theta)\int_0^\infty \left(\frac{K(t, I_{\mathbf{K}})}{t^\theta}\right)^p \frac{dt}{t} \leq \tag{2.1}$$

$$\leq p\theta(1-\theta)\int_0^\infty \left(\frac{\min(1,t)}{t^\theta}\right)^p \frac{dt}{t} = p\theta(1-\theta)\left[\int_0^1 \left(\frac{t}{t^\theta}\right)^p \frac{dt}{t} + \int_1^\infty \left(\frac{1}{t^\theta}\right)^p \frac{dt}{t}\right] =$$

$$= p\theta(1-\theta)\left[\frac{1}{p(1-\theta)} + \frac{1}{p\theta}\right] = 1, \text{ so } c_{\theta,p}(I_{\mathbf{K}}) \leq 1.$$

Let $I_{\mathbf{K}} = T_1 + T_2$, where $T_1 \in \mathcal{A}(\mathbf{K}, \mathbf{K})$ and $T_2 \in \mathcal{B}(\mathbf{K}, \mathbf{K})$. Then

$$1 = \|I_{\mathbf{K}}\| = \|T_1 + T_2\| \leq ||T_1\| + \|T_2\| \leq a(T_1) + b(T_2).$$

Taking the infimum after all decompositions of $I_{\mathbf{K}}$, we obtain:

$$1 \leq K(1, I_{\mathbf{K}}).$$

But $K(t, I_{\mathbf{K}}) \geq \min(1,t)K(1, I_{\mathbf{K}}) \geq \min(1,t)$; we conclude that

$$c_{\theta,p}(I_{\mathbf{K}}) = [p\theta(1-\theta)]^{\frac{1}{p}} \left(\int_0^\infty \left(\frac{K(t, I_{\mathbf{K}})}{t^\theta}\right)^p \frac{dt}{t}\right)^{\frac{1}{p}} \geq \tag{2.2}$$

$$\geq [p\theta(1-\theta)]^{\frac{1}{p}} \left(\int_0^\infty \left(\frac{\min(1,t)}{t^\theta}\right)^p \frac{dt}{t}\right)^{\frac{1}{p}} = 1.$$

Using (2.1) and (2.2) we obtain $c_{\theta,p}(I_{\mathbf{K}}) = 1$.

(QOI.1) There exists a constant $\lambda \geq 1$ such that

$$c_{\theta,p}(T_1 + T_2) \leq \lambda[c_{\theta,p}(T_1) + c_{\theta,p}(T_2)]$$

for every $T_1, T_2 \in C_{\theta,p}(E, F)$.

Because $(\mathcal{A}, a)$, $(\mathcal{B}, b)$ are two quasi-normed operator ideals, there are $\lambda_1, \lambda_2 \geq 1$ so that

$$a(T_1 + T_2) \leq \lambda_1[a(T_1) + a(T_2)]$$

and

$$b(T_1 + T_2) \leq \lambda_2[b(T_1) + b(T_2)].$$

But

$$c_{\theta,p}(T_1 + T_2) = [p\theta(1-\theta)]^{1/p} \left(\int_0^\infty \left(\frac{K(t, T_1 + T_2)}{t^\theta}\right)^p \frac{dt}{t}\right)^{1/p}.$$

Let $T_1 = S_1 + R_1$, $T_2 = S_2 + R_2$, where $S_i \in \mathcal{A}(E,F)$, $R_i \in \mathcal{B}(E,F)$, $i = 1,2$. Then:

$$K(t, T_1+T_2) \le a(S_1+S_2)+tb(R_1+R_2) \le \lambda_1[a(S_1)+a(S_2)]+t\lambda_2[b(R_1)+b(R_2)] =$$
$$= \lambda_1 \left\{ \left[a(S_1) + t\frac{\lambda_2}{\lambda_1}b(R_1)\right] + \left[a(S_2) + t\frac{\lambda_2}{\lambda_1}b(R_2)\right] \right\}$$

and passing to infimum for all decompositions of T_1, T_2, we obtain:

$$K(t, T_1 + T_2) \le \lambda_1 \left[K\left(\frac{\lambda_2}{\lambda_1}t, T_1\right) + K\left(\frac{\lambda_2}{\lambda_1}t, T_2\right) \right] \le$$
$$\le \lambda_1 \max\left(1, \frac{\lambda_2}{\lambda_1}\right) [K(t, T_1) + K(t, T_2)]\,.$$

Then

$$c_{\theta,p}(T_1 + T_2) \le$$
$$\le [p\theta(1-\theta)]^{1/p} \left\{ \int_0^\infty \left(\lambda_1 \max\left(1, \frac{\lambda_2}{\lambda_1}\right)\right)^p \left(\frac{K(t,T_1) + K(t,T_2)}{t^\theta}\right)^p \frac{dt}{t} \right\}^{1/p}$$

and applying Minkowski's inequality, we have

$$c_{\theta,p}(T_1 + T_2) \le \lambda_1 \max\left(1, \frac{\lambda_2}{\lambda_1}\right) [c_{\theta,p}(T_1) + c_{\theta,p}(T_2)],$$

where $\lambda = \lambda_1 \max\left(1, \frac{\lambda_2}{\lambda_1}\right) \ge 1$.

(QOI.2) Let $T \in \mathcal{L}(E_0, E)$, $S \in \mathcal{C}_{\theta,p}(E,F)$, $R \in \mathcal{L}(F, F_0)$.

Then $c_{\theta,p}(RST) = [p\theta(1-\theta)]^{1/p} \left(\int_0^\infty \left(\frac{K(t,RST)}{t^\theta}\right)^p \frac{dt}{t} \right)^{1/p}$.

But $K(t, RST) \le a(RS_1T) + tb(RS_2T) \le \|R\|a(S_1)\|T\| + t\|R\|b(S_2)\|T\|$ for $S = S_1 + S_2$, $S_1 \in \mathcal{A}(E,F)$, $S_2 \in \mathcal{B}(E,F)$.

So

$$K(t, RST) \le \|R\|(a(S_1) + tb(S_2))\|T\|$$

and by passing to infimum for all decompositions of S, it follows

$$K(t, RST) \le \|R\| \cdot K(t, S)\|T\|$$

and

$$\begin{aligned} c_{\theta,p}(RST) &\le [p\theta(1-\theta)]^{1/p}\|R\| \left(\int_0^\infty \left(\frac{K(t,S)}{t^\theta}\right)^p \frac{dt}{t} \right)^{1/p} \|T\| = \\ &= \|R\|c_{\theta,p}(S) \cdot \|T\| \end{aligned}$$

and the proof is complete.

Definition 2.2. *Let $(\mathcal{A}, a)$, $(\mathcal{B}, b)$, $(\mathcal{C}, c)$ be three quasi-normed operator ideals on Banach spaces. For $1 \le p < \infty$, $0 < \theta_1, \theta_2$; $\theta_1 + \theta_2 < 1$, we define*

$$\mathcal{D}_{\theta_1,\theta_2,p} := (\mathcal{A}, \mathcal{B}, \mathcal{C})_{\theta_1,\theta_2,p}$$

as follows: for an arbitrary pair of Banach spaces (E, F), the component

$$\mathcal{D}_{\theta_1,\theta_2,p}(E,F) := (\mathcal{A}(E,F), \mathcal{B}(E,F), \mathcal{C}(E,F))_{\theta_1,\theta_2,p} =$$

$$= \left\{ T \in \mathcal{A}(E,F) + \mathcal{B}(E,F) + \mathcal{C}(E,F) \mid \int_0^\infty \int_0^\infty \left(\frac{K(t_1,t_2,T)}{t_1^{\theta_1} t_2^{\theta_2}} \right)^p \frac{dt_1}{t_1} \frac{dt_2}{t_2} < \infty \right\},$$

where

$$K(t_1,t_2,T) = \inf_{T=T_1+T_2+T_3} (a(T_1) + t_1 b(T_2) + t_2 c(T_3)), \quad (t_1,t_2) \in \mathbf{R}_+^2.$$

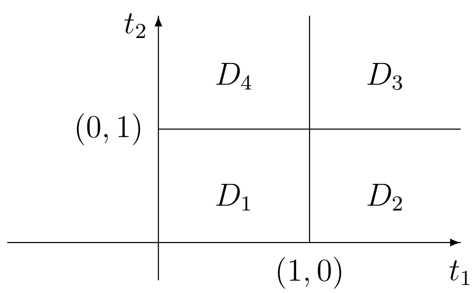
Theorem 2.3. *$\mathcal{D}_{\theta_1,\theta_2,p}$ is an operator ideal on Banach spaces.*

Proof. (OI.0) $I_{\mathbf{K}} \in \mathcal{D}_{\theta_1,\theta_2,p}(\mathbf{K},\mathbf{K})$.

Obviously, $I_{\mathbf{K}} \in \mathcal{A}(\mathbf{K},\mathbf{K}) + \mathcal{B}(\mathbf{K},\mathbf{K}) + \mathcal{C}(\mathbf{K},\mathbf{K})$

$$\int_0^\infty \int_0^\infty \left(\frac{K(t_1,t_2,I_{\mathbf{K}})}{t_1^{\theta_1} t_2^{\theta_2}} \right)^p \frac{dt_1}{t_1} \frac{dt_2}{t_2} \le \int_0^\infty \int_0^\infty \left(\frac{\min(1,t_1,t_2)}{t_1^{\theta_1} t_2^{\theta_2}} \right)^p \frac{dt_1}{t_1} \frac{dt_2}{t_2} = I.$$

Decomposing $\mathbf{R}_+^2$ in the following way:



we have:

$$I = \iint_{D_1} \left(\frac{\min(t_1,t_2)}{t_1^{\theta_1} t_2^{\theta_2}} \right)^p \frac{dt_1}{t_1} \frac{dt_2}{t_2} + \iint_{D_2} \left(\frac{t_2}{t_1^{\theta_1} t_2^{\theta_2}} \right)^p \frac{dt_1}{t_1} \frac{dt_2}{t_2} +$$

$$+ \iint_{D_3} \left(\frac{1}{t_1^{\theta_1} t_2^{\theta_2}} \right)^p \frac{dt_1}{t_1} \frac{dt_2}{t_2} + \iint_{D_4} \left(\frac{t_1}{t_1^{\theta_1} \cdot t_2^{\theta_2}} \right)^p \frac{dt_1}{t_1} \frac{dt_2}{t_2} = I_1 + I_2 + I_3 + I_4.$$

I_1 is convergent, because it is a Riemann integral

$$I_2 = \int_0^1 t_2^{p-\theta_2 p-1} \left(\int_1^\infty t_1^{-\theta_1 p-1} dt_1 \right) dt_2 = \frac{1}{p^2\theta_1(1-\theta_2)}.$$

Analogously,

$$I_3 = \frac{1}{p^2\theta_1\theta_2}; \quad I_4 = \frac{1}{p^2\theta_2(1-\theta_1)}.$$

Decomposing D_1 and computing the integral, we obtain:

$$I_1 = \frac{2-\theta_1-\theta_2}{p^2(1-\theta_1)(1-\theta_2)(1-\theta_1-\theta_2)}$$

therefore I is convergent, whence it results that $I_{\mathbf{K}} \in \mathcal{D}_{\theta_1,\theta_2,p}$.

(OI.1) Let $S, T \in \mathcal{D}_{\theta_1,\theta_2,p}(E,F)$. We prove that $S+T \in \mathcal{D}_{\theta_1,\theta_2,p}(E,F)$. Obviously, $S+T \in \mathcal{A}(E,F)+\mathcal{B}(E,F)+\mathcal{C}(E,F)$. Let $S = S_1+S_2+S_3$, $T = T_1+T_2+T_3$

$$K(t_1,t_2,S+T) \le a(S_1+T_1)+t_1 b(S_2+T_2)+t_2 c(S_3+T_3) \le$$

$$\le \lambda_1[a(S_1)+a(T_1)]+t_1\lambda_2[b(S_2)+b(T_2)]+t_3\lambda_3[c(S_3)+c(T_3)] =$$

$$= [\lambda_1 a(S_1)+t_1\lambda_2 b(S_2)+t_3\lambda_3 c(S_3)]+[\lambda_1 a(T_1)+t_1\lambda_2 b(T_2)+t_2\lambda_3 c(T_3)] =$$

$$= \lambda_1 \left\{ \left[a(S_1)+t_1\frac{\lambda_2}{\lambda_1}b(S_2)+t_2\frac{\lambda_3}{\lambda_1}c(S_3) \right] + \left[a(T_1)+t_1\frac{\lambda_2}{\lambda_1}b(T_2)+t_2\frac{\lambda_3}{\lambda_1}c(T_3) \right] \right\}$$

whence it results:

$$K(t_1,t_2,S+T) \le \lambda_1 \left[K\left(\frac{\lambda_2}{\lambda_1}t_1, \frac{\lambda_3}{\lambda_1}t_2, S\right) + K\left(\frac{\lambda_2}{\lambda_1}t_1, \frac{\lambda_3}{\lambda_1}t_2, T\right) \right] \le$$

$$\le \lambda_1 \max\left(1, \frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1}\right) [K(t_1,t_2,S)+K(t_1,t_2,T)]$$

and

$$\int_0^\infty \int_0^\infty \left(\frac{K(t_1,t_2,S+T)}{t_1^{\theta_1} t_2^{\theta_2}} \right)^p \frac{dt_1}{t_1}\frac{dt_2}{t_2} \le$$

$$\le c \cdot \int_0^\infty \int_0^\infty \left(\frac{\max((K(t_1,t_2,S), K(t_1,t_2,T))}{t_1^{\theta_1} t_2^{\theta_2}} \right)^p \frac{dt_1}{t_1}\frac{dt_2}{t_2}$$

which is finite. Therefore $S+T \in \mathcal{D}_{\theta_1,\theta_2,p}(E,F)$.

(OI.2) If $T \in \mathcal{L}(E_0,F)$, $S \in \mathcal{D}_{\theta_1,\theta_2,p}(E,F)$, $R \in \mathcal{L}(F,F_0)$, then

$$RST \in \mathcal{A}(E_0,F_0)+B(E_0,F_0)+\mathcal{C}(E_0,F_0),$$

and

$$K(t_1, t_2, RST) \le \|R\| K(t_1, t_2, S) \|T\|$$

involves

$$\int_0^\infty \int_0^\infty \left(\frac{K(t_1, t_2, RST)}{t_1^{\theta_1} t_2^{\theta_2}} \right)^p \frac{dt_1}{t_1} \frac{dt_2}{t_2} \le \|R\|^p \cdot \|T\|^p \int_0^\infty \int_0^\infty \left(\frac{K(t_1, t_2, S)}{t_1^{\theta_1} t_2^{\theta_2}} \right)^p \frac{dt_1}{t_1} \frac{dt_2}{t_2} < \infty.$$

Whence it results $RST \in \mathcal{D}_{\theta_1,\theta_2,p}(E_0, F_0)$.

We define the function

$$d_{\theta_1,\theta_2,p} : \mathcal{D}_{\theta_1,\theta_2,p} \to \mathbf{R}_+$$

by

$$d_{\theta_1,\theta_2,p}(T) := \left(\lambda \int_0^\infty \int_0^\infty \left(\frac{K(t_1, t_2, T)}{t_1^{\theta_1} t_2^{\theta_2}} \right)^p \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{1/p}, \tag{2.3}$$

where $1 \le p < \infty$, $0 < \theta_1, \theta_2$; $\theta_1 + \theta_2 < 1$, and

$$\lambda = \left(\frac{1 - \theta_1 - \theta_2 + \theta_1\theta_2}{(p^2\theta_1\theta_2(1-\theta_1)(1-\theta_2)(1-\theta_1-\theta_2)} \right)^{-1}.$$

Theorem 2.4. *The couple* $(\mathcal{D}_{\theta_1,\theta_2,p}, d_{\theta_1,\theta_2,p})$*, where* $1 \le p < \infty$*,* $0 < \theta_1, \theta_2$*;* $\theta_1 + \theta_2 < 1$*, is a quasi-normed operator ideal on Banach spaces.*

Proof. It is shown that the function defined by (2.3) satisfies the three conditions of the definition of quasi-norm.

Remark. The results obtained in Theorems 2.2, 2.4 can be extended to the n-operator ideals on Banach spaces, with a suitable change of the constant that appears in the definition of quasinorm.

Theorem 2.5. *The reiteration theorem). Let* $(\mathcal{A}, a)$*,* $(\mathcal{B}, b)$ *be two quasi-normed operator ideals on Banach spaces, and* $\mathcal{C}_{\theta_0,p_0} = (\mathcal{A}, \mathcal{B})_{\theta_0,p_0}$*,* $\mathcal{C}_{\theta_1,p_1} = (\mathcal{A}, \mathcal{B})_{\theta_1,p_1}$*, where* $0 < \theta_i < 1$*,* $1 \le p_i < \infty$*,* $(i = 0, 1)$*. Then:*

$$(\mathcal{C}_{\theta_0,p_0}, \mathcal{C}_{\theta_1,p_1})_{\eta,p} = \mathcal{C}_{\theta,p},$$

with equivalent norms, where $\theta = (1-\eta)\theta_0 + \eta\theta_1$*,* $0 < \eta < 1$*,* $1 \le p < \infty$*.*

Proof. We remark that the ideal $\mathcal{C}_{\theta_0,p_0}$ is of class $\mathcal{C}(\theta_0, \mathcal{A}, \mathcal{B})$ (namely for any pair of Banach spaces (E, F), the component $\mathcal{C}_{\theta_0,p_0}(E, F) \in \mathcal{C}(\theta_0, \mathcal{A}(E, F), \mathcal{B}(E, F))$, and $\mathcal{C}_{\theta_1,p_1} \in \mathcal{C}(\theta_1, \mathcal{A}, \mathcal{B})$.

Let $T \in (\mathcal{C}_{\theta_0,p_0};\mathcal{C}_{\theta_1,p_1})_{\eta,p}(E,F)$. Then

$$\int_0^\infty \left(\frac{K(s,T,\mathcal{C}_{\theta_0,p_0},\mathcal{C}_{\theta_1,p_1}}{s^\eta}\right)^p \frac{ds}{s} < \infty.$$

If $T = T_0 + T_1$, $T_0 \in \mathcal{A}(E,F)$, $T_1 \in \mathcal{B}(E,F)$, then

$$K(t,T,\mathcal{A},\mathcal{B}) \leq K(t,T_0,\mathcal{A},\mathcal{B}) + K(t,T_1,\mathcal{A},\mathcal{B}) \leq$$
$$\leq c[t^{\theta_0}c_{\theta_0,p_0}(T_0) + t^{\theta_1}c_{\theta_1,p_1}(T_1)] = ct^{\theta_0}[c_{\theta_0,p_0}(T_0) + t^{\theta_1-\theta_0}c_{\theta_1,p_1}(T_1)] \leq$$
$$\leq c\cdot t^{\theta_0}K(t^{\theta_1-\theta_0},T,\mathcal{C}_{\theta_0,p_0},\mathcal{C}_{\theta_1,p_1}).$$

So,

$$c_{\theta,p}(T) = c\left(\int_0^\infty \left(\frac{K(t,T,\mathcal{A},\mathcal{B})}{t^\theta}\right)^p \frac{dt}{t}\right)^{1/p} \leq$$
$$\leq c'\left(\int_0^\infty (t^{-(\theta-\theta_0)}K(t^{\theta_1-\theta_0},T,\mathcal{C}_{\theta_0,p_0},\mathcal{C}_{\theta_1,p_1}))^p\frac{dt}{t}\right)^{1/p} =$$
$$= c''\left(\int_0^\infty [s^{-\eta}K(s,T,\mathcal{C}_{\theta_0,p_0},\mathcal{C}_{\theta_1,p_1})]^p\frac{ds}{s}\right)^{1/p} = c''\cdot c_{\eta,p}(T) < \infty$$

where $s = t^{\theta_1-\theta_0}$ and $\eta = \dfrac{\theta-\theta_0}{\theta_1-\theta_0}$.

It follows that $T \in (\mathcal{A},\mathcal{B})_{\theta,p}$ and so

$$(\mathcal{C}_{\theta_0,p_0},\mathcal{C}_{\theta_1,p_1})_{\eta,p} \hookrightarrow (\mathcal{A},\mathcal{B})_{\theta,p}.$$

In order to prove the converse inclusion, we remark that for an arbitrary Banach couple $\overline{X} = (X_0,X_1)$ we have

$$\overline{X}_{\theta_j,1} \hookrightarrow \overline{X}_{\theta_j,q_j} \hookrightarrow \overline{X}_{\theta_j,\infty}, \quad (j=\overline{0,1}) \tag{2.4}$$

and

$$\|x\|_{\theta_j,q_j} \leq c\cdot\|x\|_{\theta_j,1}.$$

We intend to show that

$$c_{\eta,p}(T) \leq c\cdot c_{\theta,p}(T).$$

Using the first inclusion of the relation (2.4) and Holmstedt's theorem for $q_0 = q_1 = 1$ and $\delta = \theta_1 - \theta_0$, we have:

$$\overline{K}(t^\delta,T) = K(t^\delta,T,(\mathcal{A},\mathcal{B})_{\theta_0,q_0},(\mathcal{A},\mathcal{B})_{\theta_1,q_1}) \leq c\cdot K(t^\delta,T,(\mathcal{A},\mathcal{B})_{\theta_0,1},(\mathcal{A},\mathcal{B})_{\theta_1,1}) \leq$$

$$\leq c\left\{\int_0^t [s^{-\theta_0}\cdot K(s,T,\mathcal{A},\mathcal{B})]\frac{ds}{s}+t^{\delta}\int_t^{\infty}[s^{-\theta_1}K(s,T,\mathcal{A},\mathcal{B})]\frac{ds}{s}\right\}$$

for any $T\in(\mathcal{A},\mathcal{B})_{\theta_0,q_0}+(\mathcal{A},\mathcal{B})_{\theta_1,q_1}$ and any $t>0$.

Using the above inequality, making the change of the variable $t=s^{\delta}$ and applying Minkowski's inequality, we obtain:

$$\left(\int_0^{\infty}\left[t^{-\eta}\overline{K}(t,T)\right]^p\frac{dt}{t}\right)^{1/p}=c'\left(\int_0^{\infty}\left[t^{-\eta\delta}\overline{K}(t^{\delta},T)\right]^p\frac{dt}{t}\right)^{1/p}\leq$$

$$\leq c''\left\{\left(\int_0^{\infty}\left[t^{-\eta\delta}\int_0^t s^{-\theta_0}K(s,T)\frac{ds}{s}\right]^p\frac{dt}{t}\right)^{1/p}+\right.$$

$$\left.+\left(\int_0^{\infty}\left[t^{\delta(1-\eta)}\int_t^{\infty}s^{-\theta_1}K(s,T)\frac{ds}{s}\right]^p\frac{dt}{t}\right)^{1/p}\right\}.$$

Applying Hardy's inequalities for the two integrales of the right side, we obtain:

$$I_1=\left(\int_0^{\infty}\left[t^{-\eta\delta}\int_0^t s^{-\theta_0}K(s,T)\frac{ds}{s}\right]^p\frac{dt}{t}\right)^{1/p}=$$

$$=\left(\int_0^{\infty}\left[t^{-\eta\delta+1}\cdot\frac{1}{t}\int_0^t s^{-\theta_0}K(s,T)\frac{ds}{s}\right]^p\frac{dt}{t}\right)^{1/p}\leq$$

$$\leq\frac{1}{\eta\delta}\left\{\int_0^{\infty}(t^{1-\eta(\theta_1-\theta_0)}\cdot t^{-\theta_0-1}\cdot K(t,T))^p\frac{dt}{t}\right\}^{1/p}=c'\left\{\int_0^{\infty}(t^{-\theta}K(t,T))^p\frac{dt}{t}\right\}^{1/p}$$

where $\theta=(1-\eta)\theta_0+\eta\theta_1$.

Analogously,

$$I_2=\left(\int_0^{\infty}\left[t^{\delta(1-\eta)}\int_t^{\infty}s^{-\theta_1}K(s,T)\frac{ds}{s}\right]^p\frac{dt}{t}\right)^{1/p}\leq c''\left\{\int_0^{\infty}(t^{-\theta}K(t,T))^p\frac{dt}{t}\right\}^{1/p}.$$

Therefore:

$$c_{\eta,p}(T)=\left(\int_0^{\infty}[t^{-\eta}\overline{K}(t,T)]^p\frac{dt}{t}\right)^{1/p}\leq c\left\{\int_0^{\infty}[t^{-\theta}K(t,T)]^p\frac{dt}{t}\right\}^{1/p}=c\cdot c_{\theta,p}(T).$$

It follows that

$$(\mathcal{A},\mathcal{B})_{\theta,p} \hookrightarrow (\mathcal{C}_{\theta_0,p_0},\mathcal{C}_{\theta_1,p_1})_{\eta,p}$$

and the theorem is proved.

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