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ON A SPECIAL ALMOST GEODESIC MAPPINGS OF THIRD TYPE OF AFFINE SPACES[1]

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Abstract. We investigate two kinds of the almost geodesic mappings of third type. Also we find some invariant geometric objects for special $\underset{1}{\widetilde{\pi}}_3$ and $\underset{2}{\widetilde{\pi}}_3$ mappings.

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0. Introduction

Let A_N and $\overline{A}_N$ be two differentiable manifolds endowed with symmetric affine connections L and $\overline{L}$ respectively. We suppose that there exists one-to-one correspondence between A_N and $\overline{A}_N$ such that if the point $M \in A_N$ has local coordinates (x^i) and the corresponding point $\overline{M} \in \overline{A}_N$ has local coordinates $(\overline{x}^i)$, then

$$\overline{x}^i = f^i(x^1, \cdots, x^N), \quad (i = 1, 2, \cdots, N)$$

such that the functions f^i are of the class $C^r \quad (r > 2)$ and

$$det\left(\frac{\partial f^i}{\partial x^j}\right) \neq 0.$$

Then we can choose the local coordinates such that the corresponding points have the same coordinates $(x^1, \cdots, x^N)$.

The curve $l : x^h = x^h(t)$ in $\overline{A}_N$ is said to be an *almost geodesic line* if its tangential vector $\lambda^h(t) = dx^h/dt \neq 0$ satisfies the equations

$$\overline{\lambda}^h_{(2)} = \overline{a}(t)\lambda^h + \overline{b}(t)\overline{\lambda}^h_{(1)}, \quad \overline{\lambda}^h_{(1)} = \lambda^h_{||\alpha}\lambda^\alpha, \quad \overline{\lambda}^h_{(2)} = \overline{\lambda}^h_{(1)||\alpha}\lambda^\alpha,$$

where $\overline{a}(t)$ and $\overline{b}(t)$ are functions of the parameter t, and $||$ denotes the covariant derivative with respect to the connection in $\overline{A}_N$.

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The mapping $f : A_N \to \overline{A}_N$ is called *almost geodesic mapping* if any *geodesic line* of the space A_N turns into an *almost geodesic line* of the space $\overline{A}_N$.

Investigating the almost geodesic mappings, Sinyukov [3] distinguished three types of them. Namely, if L^h_{ij} and $\overline{L}^h_{ij}$ are the components of the connections in A_N and $\overline{A}_N$ respectively, and

$$P^h_{ij}(x) = \overline{L}^h_{ij}(x) - L^h_{ij}(x),$$

then

$$(P^h_{\alpha\beta|\gamma} + P^h_{\delta\alpha}P^\delta_{\beta\gamma})\lambda^\alpha\lambda^\beta\lambda^\gamma = bP^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + a\lambda^h,$$

and the mapping is of type π_1, π_2 or π_3 according some conditions satisfied by the function b.

If a differentiable manifold is endowed with nonsymmetric affine connection L^h_{ij}, for a vector exist two kinds of covariant derivative [1], [2]

$$\lambda^h_{\underset{1}{|}m} = \lambda^h_{,m} + L^h_{pm}\lambda^p, \quad \lambda^h_{\underset{2}{|}m} = \lambda^h_{,m} + L^h_{mp}\lambda^p,$$

because of which we can define two kinds of almost geodesic lines and two kinds of almost geodesic mappings.

In the sequel, we shall denote by GA_N the N-dimensional differentiable manifold endowed with nonsymmetric affine connection.

1. Almost geodesic mappings of affine spaces

In an affine space GA_N (with nonsymmetric affine connection L^i_{jk}) for a vector one can define two kinds of a covariant derivative [1,2]. Let us denote by $\underset{\theta}{|}, \underset{\theta}{||}$ a covariant derivative of the kind θ, $(\theta = 1, 2)$ in GA_N and $G\overline{A}_N$ respectively. A curve is called *the first kind almost geodesic line* if its tangential vector $\lambda^h(t) = dx^h/dt \neq 0$ satisfies the equations

$$(1.1) \qquad \underset{1}{\overline{\lambda}}{}^h_{(2)} = \underset{1}{\overline{a}}(t)\lambda^h + \underset{1}{\overline{b}}(t)\underset{1}{\overline{\lambda}}{}^h_{(1)}, \quad \underset{1}{\overline{\lambda}}{}^h_{(1)} = \lambda^h_{\underset{1}{||}\alpha}\lambda^\alpha, \quad \underset{1}{\overline{\lambda}}{}^h_{(2)} = \underset{1}{\overline{\lambda}}{}^h_{(1)\underset{1}{||}\alpha}\lambda^\alpha,$$

where $\overline{a}_1(t)$ and $\overline{b}_1(t)$ are functions of a parameter t. A curve is called *the second kind almost geodesic line* if its tangential vector $\lambda^h(t) = dx^h/dt \neq 0$ satisfies the equations

$$(1.2) \qquad \underset{2}{\overline{\lambda}}{}^h_{(2)} = \underset{2}{\overline{a}}(t)\lambda^h + \underset{2}{\overline{b}}(t)\underset{2}{\overline{\lambda}}{}^h_{(1)}, \quad \underset{2}{\overline{\lambda}}{}^h_{(1)} = \lambda^h_{\underset{2}{||}\alpha}\lambda^\alpha, \quad \underset{2}{\overline{\lambda}}{}^h_{(2)} = \underset{2}{\overline{\lambda}}{}^h_{(1)\underset{2}{||}\alpha}\lambda^\alpha,$$

where $\overline{a}_2(t)$ and $\overline{b}_2(t)$ are functions of a parameter t.

Let $\overline{G}A_N$ be a differentiable manifold endowed with nonsymmetric affine connection. A mapping $f : GA_N \to G\overline{A}_N$ is called *the first kind almost geodesic* *mapping* if any geodesic line of the space GA_N turns into the first kind almost geodesic line of the space $G\overline{A}_N$. A mapping f is called *the second kind almost geodesic mapping* if any geodesic line of the space GA_N turns into the second kind almost geodesic line of the space $G\overline{A}_N$.

We can put

$$\overline{L}^h_{ij}(x) = L^h_{ij}(x) + P^h_{ij}(x), \tag{1.3}$$

where $L^h_{ij}(x)$, $\overline{L}^h_{ij}(x)$ are connection coefficients of the space GA_N and $G\overline{A}_N$, $(N > 2)$, and $P^h_{ij}(x)$ is a deformation tensor. Then the following theorem holds ([4], Th. 1)

Theorem 1.1. *The mapping f of the space GA_N onto $G\overline{A}_N$ is almost geodesic mapping of the first kind if and only if the deformation tensor $P^h_{ij}(x)$ satisfies identically the conditions*

$$(P^h_{\alpha\beta\underset{1}{|}\gamma} + P^h_{\delta\alpha}P^\delta_{\beta\gamma})\lambda^\alpha\lambda^\beta\lambda^\gamma = \underset{1}{b}P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + \underset{1}{a}\lambda^h, \tag{1.4}$$

where $\underset{1}{a}$ and $\underset{1}{b}$ are invariants.

Also we have ([4], Th. 2)

Theorem 1.2. *The mapping f of the space GA_N onto $G\overline{A}_N$ is almost geodesic mapping of the second kind if and only if the deformation tensor $P^h_{ij}(x)$ satisfies identically the conditions*

$$(P^h_{\alpha\beta\underset{2}{|}\gamma} + P^h_{\alpha\delta}P^\delta_{\beta\gamma})\lambda^\alpha\lambda^\beta\lambda^\gamma = \underset{2}{b}P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + \underset{2}{a}\lambda^h, \tag{1.4'}$$

where $\underset{2}{a}$ and $\underset{2}{b}$ are invariant.

Analogously to the case of the spaces those connection are without torsion, we have three types of the first kind and three types of the second kind of almost geodesic mappings. In the previous papers [4], [5] we investigated almost geodesic mappings of the first and second type of almost geodesic mappings for the spaces with nonsymmetric affine connection. The objective of this paper was to investigate of the almost geodesic mappings of the third type.

2. The third type almost geodesic mappings

The third type almost geodesic mappings of the first kind is determined by condition for the function $\underset{1}{b}(x;\lambda)$ in (1.4):

$$\underset{1}{b} = \frac{\underset{1}{b}_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\lambda^\gamma}{\sigma_{\varepsilon\delta}\lambda^\varepsilon\lambda^\delta}, \tag{2.1}$$

where $\sigma_{\varepsilon\delta}\lambda^{\varepsilon}\lambda^{\delta} \neq 0$. Suppose that

$$P^h_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} = 2\psi_{\alpha}\lambda^{\alpha}\lambda^{h} + \varphi^{h}\sigma_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}. \tag{2.2}$$

Here $1 \to b_{ijk}$ and σ_{ij} are symmetric tensors of the type $\binom{0}{3}$ and $\binom{0}{2}$ respectively. Equation (2.2) can be written in the equivalent form

$$(P^h_{\alpha\beta} - 2\psi_{\alpha}\delta^h_{\beta} - \varphi^h\sigma_{\alpha\beta})\lambda^{\alpha}\lambda^{\beta} \equiv 0, \tag{2.3}$$

that is

$$P^h_{\underline{ij}} = \psi_i\delta^h_j + \psi_j\delta^h_i + \sigma_{ij}\varphi^h. \tag{2.4}$$

Denote by $\xi^h_{ij} = P^h_{\underset{\vee}{ij}}$. Then for the deformation tensor we get

$$P^h_{ij}(x) = \psi_i(x)\,\delta^h_j + \psi_j(x)\,\delta^h_i + \sigma_{ij}(x)\varphi^h(x) + \xi^h_{ij}(x). \tag{2.5}$$

Here ξ^h_{ij} is an antisymmetric tensor of the type $\binom{1}{2}$. Substituting (2.5) in (1.4) we have

$$\begin{aligned}&\big[\psi_{\underset{1}{\alpha|\gamma}}\delta^h_{\beta} + \phi_{\underset{1}{\beta|\gamma}}\delta^h_{\alpha} + \sigma_{\underset{1}{\alpha\beta|\gamma}}\varphi^h + \sigma_{\alpha\beta}\varphi^h_{\underset{1}{|\gamma}} + \xi_{\underset{1}{\alpha\beta|\gamma}}\\ &+(\psi_{\varepsilon}\delta^h_{\alpha} + \psi_{\alpha}\delta^h_{\varepsilon} + \sigma_{\varepsilon\alpha}\varphi^h + \xi^h_{\varepsilon\alpha})(\psi_{\beta}\delta^{\varepsilon}_{\gamma} + \psi_{\gamma}\delta^{\varepsilon}_{\beta} + \sigma_{\beta\gamma}\varphi^{\varepsilon} + \xi^{\varepsilon}_{\beta\gamma})\big]\lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma}\\ &= \underset{1}{b}(2\psi_{\alpha}\lambda^{\alpha}\lambda^{h} + \varphi^h\sigma_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}) + \underset{1}{a}\lambda^h,\end{aligned}$$

that is

$$\begin{aligned}&\big[2\,\phi_{\underset{1}{\alpha|\beta}}\delta^h_{\gamma} + \sigma_{\underset{1}{\alpha\beta|\gamma}}\varphi^h + \sigma_{\alpha\beta}\varphi^h_{\underset{1}{|\gamma}} + \xi^h_{\underset{1}{\alpha\beta|\gamma}}\\ &+\psi_{\varepsilon}\delta^h_{\alpha}\,\psi_{\beta}\delta^{\varepsilon}_{\gamma} + \psi_{\varepsilon}\delta^h_{\alpha}\,\psi_{\gamma}\delta^{\varepsilon}_{\beta} + \psi_{\varepsilon}\delta^h_{\alpha}\,\sigma_{\beta\gamma}\varphi^{\varepsilon} + \psi_{\varepsilon}\delta^h_{\alpha}\,\xi^{\varepsilon}_{\beta\gamma}\\ &+\psi_{\alpha}\delta^h_{\varepsilon}\,\psi_{\beta}\delta^{\varepsilon}_{\gamma} + \psi_{\alpha}\delta^h_{\varepsilon}\,\psi_{\gamma}\delta^{\varepsilon}_{\beta} + \psi_{\alpha}\delta^h_{\varepsilon}\,\sigma_{\beta}\gamma\varphi^{\varepsilon} + \psi_{\alpha}\delta^h_{\varepsilon}\,\xi^{\varepsilon}_{\beta\gamma}\\ &+\sigma_{\varepsilon\alpha}\varphi^h\psi_{\beta}\delta^{\varepsilon}_{\gamma} + \sigma_{\varepsilon\alpha}\varphi^h\psi_{\gamma}\delta^{\varepsilon}_{\gamma} + \sigma_{\varepsilon\alpha}\varphi^h\sigma_{\beta\gamma}\varphi^{\varepsilon} + \sigma_{\varepsilon\alpha}\varphi^h\xi^{\varepsilon}_{\beta\gamma}\\ &+\xi^h_{\varepsilon\alpha}\psi_{\beta}\delta^{\varepsilon}_{\gamma} + \xi^h_{\varepsilon\alpha}\psi_{\gamma}\delta^{\varepsilon}_{\beta} + \xi^h_{\varepsilon\alpha}\sigma_{\beta\gamma}\varphi^{\varepsilon} + \xi^h_{\varepsilon\alpha}\xi^{\varepsilon}_{\beta\gamma}\big]\lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma}\\ &= 2\underset{1}{b}\psi_{\alpha}\lambda^{\alpha}\lambda^h + \underset{1}{b}\varphi^h\sigma_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} + \underset{1}{a}\lambda^h.\end{aligned}$$

From $\xi^h_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} = 0$ we have

(2.6)

$$\begin{aligned}\sigma_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}\varphi_{\underset{1}{|\gamma}}\lambda^{\gamma} + \xi^h_{\varepsilon\alpha}\sigma_{\beta\gamma}\varphi^{\varepsilon}\lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma} &= \big[\underset{1}{b}_{\alpha\beta\gamma}\lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma} - \sigma_{\underset{1}{\alpha\beta|\gamma}}\lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma}\\ &-\sigma_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}(3\psi_{\gamma}\lambda^{\gamma} + \sigma_{\varepsilon\gamma}\varphi^{\varepsilon}\lambda^{\gamma})\big]\varphi^h + (\underset{1}{a} + 2\psi_{\alpha}\lambda^{\alpha}\underset{1}{b} - 2\phi_{\underset{1}{\alpha|\beta}}\lambda^{\alpha}\lambda^{\beta}\\ &-4\psi_{\alpha}\psi_{\beta}\lambda^{\alpha}\lambda^{\beta} - \psi_{\varepsilon}\varphi^{\varepsilon}\sigma_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta})\lambda^h.\end{aligned}$$

We can put

$$\begin{aligned}\underset{1}{\widetilde{b}} &= \underset{1}{b}_{\alpha\beta\gamma}\lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma} - \sigma_{\underset{1}{\alpha\beta|\gamma}}\lambda^{\alpha}\lambda^{\beta}\lambda^{\gamma} - \sigma_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}(3\psi_{\gamma}\lambda^{\gamma} + \sigma_{\varepsilon\gamma}\varphi^{\varepsilon}\lambda^{\gamma}),\\ \underset{1}{\widetilde{b}} &= \underset{1}{a} + 2\psi_{\alpha}\lambda^{\alpha}\underset{1}{b} - 2\phi_{\underset{1}{\alpha|\beta}}\lambda^{\alpha}\lambda^{\beta} - 4\psi_{\alpha}\psi_{\beta}\lambda^{\alpha}\lambda^{\beta} - \psi_{\varepsilon}\varphi^{\varepsilon}\sigma_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}.\end{aligned} \tag{2.7}$$

Then

$$\sigma_{\alpha\beta}\lambda^{\alpha}\,\lambda^{\beta}\,\varphi^{h}_{\underset{1}{|}\gamma}\,\lambda^{\gamma}\;+\xi^{h}_{\varepsilon\alpha}\sigma_{\beta\gamma}\varphi^{\varepsilon}\lambda^{\alpha}\,\lambda^{\beta}\,\lambda^{\gamma}\;=\underset{1}{\widetilde{b}}\varphi^{h}+\underset{1}{\widetilde{a}}\lambda^{h}\,. \tag{2.8}$$

The equations (2.8) can be written in the form

$$\varphi_{\underset{1}{|}\gamma}\,\lambda^{\gamma}\;+\xi^{h}_{\varepsilon\alpha}\varphi^{\varepsilon}\lambda^{\alpha}\;=\underset{1}{\nu}\varphi^{h}+\underset{1}{\mu}\lambda^{h}\,. \tag{2.9}$$

Let be $\underset{1}{\nu}=\nu_{\gamma}\lambda^{\gamma}$. Then (2.9) we can present in the form

$$(\varphi_{\underset{1}{|}\gamma}\;+\xi^{h}_{\varepsilon\gamma}\varphi^{\varepsilon})\lambda^{\gamma}\;=(\underset{1}{\nu}_{\gamma}\varphi^{h}+\underset{1}{\mu}\delta^{h}_{\gamma}\,)\lambda^{\gamma}\,, \tag{2.10}$$

wherefrom

$$\varphi^{h}_{\underset{1}{|}m}\;+\xi^{h}_{\varepsilon m}\varphi^{\varepsilon}=\underset{1}{\nu}_{m}\varphi^{h}+\underset{1}{\mu}\delta^{h}_{m}\,. \tag{2.11}$$

Here $\underset{1}{\nu}_{m}$ is a covariant vector, $\underset{1}{\mu}$ is an invariant and ξ^{h}_{ij} is an antisymmetric tensor.

From (2.1) we can see that

$$\underset{1}{\widetilde{a}}=\underset{1}{\mu}\sigma_{\alpha\beta}\lambda^{\alpha}\,\lambda^{\beta}\,,\qquad \underset{1}{\widetilde{b}}=\underset{1}{\nu}_{\alpha}\lambda^{\alpha}\,\sigma_{\beta\gamma}\lambda^{\beta}\,\lambda^{\gamma}\,.$$

The equations (2.7) and (2.11) are characterized by the third type almost geodesic mapping of the first kind $\underset{1}{\pi}_{3}$.

Almost geodesic mapping of the second kind of the space GA_N into a space $G\overline{A}_N$ is *the third kind* $\underset{2}{\pi}_{3}$ if the function $\underset{2}{b}(x;\,\lambda)$ in (1.4') has the form

$$\underset{2}{b}=\frac{\underset{2}{b}_{\alpha\beta\gamma}\lambda^{\alpha}\,\lambda^{\beta}\,\lambda^{\gamma}}{\sigma_{\varepsilon\delta}\lambda^{\varepsilon}\,\lambda^{\delta}}, \tag{2.1$'$}$$

where is $\sigma_{\varepsilon\delta}\lambda^{\varepsilon}\,\lambda^{\delta}\neq 0$. The deformation tensor in this case has the form (2.5). From (1.4) and (2.5) we have

$$\sigma_{\alpha\beta}\lambda^{\alpha}\,\lambda^{\beta}\,\varphi^{h}_{\underset{2}{|}\gamma}\,\lambda^{\gamma}\;+\xi^{h}_{\alpha\varepsilon}\sigma_{\beta\gamma}\varphi^{\varepsilon}\lambda^{\alpha}\,\lambda^{\beta}\,\lambda^{\gamma}\;=\underset{2}{\widetilde{b}}\varphi^{h}+\underset{2}{\widetilde{a}}\lambda^{h}\,, \tag{2.8$'$}$$

i.e.

$$\varphi_{\underset{2}{|}m}\;+\xi^{h}_{m\varepsilon}\varphi^{\varepsilon}=\underset{2}{\nu}_{m}\varphi^{h}+\underset{2}{\mu}\delta^{h}_{m}\,. \tag{2.11$'$}$$

Here $\underset{2}{\nu}_{m}$ is a covariant vector, $\underset{2}{\mu}$ is an invariant and ξ^{h}_{ij} is an antisymmetric tensor.

3. The property of reciprocity of the third type almost geodesic mappings

Let the mapping $\underset{1}{\pi}_3 : GA_N \to G\overline{A}_N$ has *the property of reciprocity*, i.e. let its inverse mapping be of type $\underset{1}{\pi}_3$, too. Then we have

$$\varphi^h_{\underset{1}{||}m} - \xi^h_{\alpha m}\varphi^\alpha = \underset{1}{\widetilde{\nu}}_m\varphi^h + \underset{1}{\widetilde{\mu}}\delta^h_m\,, \tag{3.1}$$

where $\tilde{\nu}_m$ is a vector and $\tilde{\mu}$ is an invariant.

From here we have

$$\xi^h_{\alpha m}\varphi^\alpha = \underset{1}{\theta}_m\varphi^h + \overline{\rho}\delta^h_m\,, \tag{3.2}$$

where

$$\underset{1}{\theta}_m = \nu_m - \widetilde{\nu}_m + \sigma_{\alpha m}\varphi^\alpha + \psi_m, \qquad \overline{\rho} = \mu - \widetilde{\mu} + \psi_\alpha\varphi^\alpha.$$

Here $\overline{\rho}$ is an invariant, and $\underset{1}{\theta}_m$ is a vector.

Suppose that the conditions (3.2) are satisfied identically with respect to φ^h. Then we have a special class of almost geodesic mappings $\underset{1}{\widetilde{\pi}}_3$. The basic equations of this mapping has the form

$$\begin{aligned}\overline{L}^h_{ij}(x) = L^h_{ij}(x) + \psi_i(x)\,\delta^h_j + \psi_j(x)\,\delta^h_i + \sigma_{ij}(x)\varphi^h(x)\\ +\underset{1}{\theta}_j(x)\delta^h_i - \underset{1}{\theta}_i(x)\delta^h_j\,,\end{aligned} \tag{3.3}$$

$$\varphi^h_{\underset{1}{|}m} = \underset{1}{\eta}_m\varphi^h + \underset{1}{\rho}\delta^i_m\,, \tag{3.4}$$

where $\underset{1}{\theta}_i$, $\underset{1}{\eta}_i$ are vectors and $\underset{1}{\rho}$ is an invariant.

Analogously to previous case, if the mapping $\underset{2}{\pi}_3$ has the property of reciprocity, then for the tensor ξ^h_{ij} we get

$$\varphi^h_{\underset{2}{||}m} - \xi^h_{m\alpha}\varphi^\alpha = \underset{2}{\widetilde{\nu}}_m\varphi^h + \underset{2}{\widetilde{\mu}}\delta^h_m\,, \tag{3.1$'$}$$

where $\underset{2}{\widetilde{\nu}}_m$ is a covariant vector and $\underset{2}{\widetilde{\mu}}$ is an invariant.

In this case, for the mapping $\underset{2}{\pi}_3$ we get the basic equations

$$\begin{aligned}\overline{L}^h_{ij}(x) = L^h_{ij}(x) + \psi_i(x)\,\delta^h_j + \psi_j(x)\,\delta^h_i + \sigma_{ij}(x)\varphi^h(x)\\ +\underset{2}{\theta}_i(x)\delta^h_j - \underset{2}{\theta}_j(x)\delta^h_i\,,\end{aligned} \tag{3.3$'$}$$

$$\varphi^h_{\underset{2}{|}m} = \underset{2}{\eta}_m\varphi^h + \underset{2}{\rho}\delta^i_m\,. \tag{3.4$'$}$$

Here θ_i, $\underset{2}{\eta}_m$ are vectors and $\underset{2}{\rho}$ is an invariant.

4. Mapping $\underset{1}{\widetilde{\pi}}_3(e,\theta)$ and $\underset{2}{\widetilde{\pi}}_3(e,\theta)$

Let $\underset{1}{\widetilde{\pi}}_3 : GA_N \to G\overline{A}_N$ be an almost geodesic mapping and q_i a vector such that

$$q_\alpha \varphi^\alpha = e, \qquad (e = \pm 1, 0). \tag{4.1}$$

From (3.3) i.e. from

$$\overline{L}^h_{ij} = L^h_{ij} + \psi_i \delta^h_j + \psi_j \delta^h_i + \sigma_{ij}\varphi^h + \underset{1}{\theta}_j \delta^h_i - \underset{1}{\theta}_i \delta^h_j$$

the composition with q_i gives

$$\overline{L}^\alpha_{ij} q_\alpha = L^\alpha_{ij} q_\alpha + \psi_i \delta^\alpha_j q_\alpha + \psi_j \delta^\alpha_i q_\alpha + \sigma_{ij}\varphi^\alpha q_\alpha + \underset{1}{\theta}_j \delta^\alpha_i q_\alpha - \underset{1}{\theta}_i \delta^\alpha_j q_\alpha,$$

wherefrom

$$q_{\underset{1}{i||j}} = q_{\underset{1}{i|j}} - \psi_i q_j - \psi_j q_i - e\sigma_{ij} - \underset{1}{\theta}_j q_i + \underset{1}{\theta}_i q_j. \tag{4.2}$$

Now, by composition with φ^i in (4.2) we get

$$\varphi^\alpha (q_{\underset{1}{\alpha||j}} - q_{\underset{1}{\alpha|j}}) = -\psi_\alpha \varphi^\alpha q_j - \psi_j q_\alpha \varphi^\alpha - e\sigma_{\alpha j}\varphi^\alpha - \underset{1}{\theta}_j q_\alpha \varphi^\alpha + \underset{1}{\theta}_\alpha \varphi^\alpha q_j,$$

wherefrom in view of (4.1) we have

$$\varphi^\alpha (q_{\underset{1}{\alpha||j}} - q_{\underset{1}{\alpha|j}}) = (-\psi_\alpha \varphi^\alpha + \underset{1}{\theta}_\alpha \varphi^\alpha) q_j - e(\psi_j + \underset{1}{\theta}_j) - e\sigma_{\alpha j}\varphi^\alpha. \tag{4.3}$$

Contracting with respect to h, i in (3.3) we get

$$\overline{L}^\alpha_{\alpha j} = L^\alpha_{\alpha j} + (N+1)\psi_j + \sigma_{\alpha j}\varphi^\alpha + (N-1)\underset{1}{\theta}_j. \tag{4.4}$$

From (4.3) and (4.4) we have

$$e\varphi^\alpha (q_{\underset{1}{\alpha||j}} - q_{\underset{1}{\alpha|j}}) = e(-\psi_\alpha \varphi^\alpha + \underset{1}{\theta} P \varphi^\alpha) q_j - \overline{L}^\alpha_{\alpha j} + L^\alpha_{\alpha j} + N\psi_j + (N-2)\underset{1}{\theta}_j,$$

that is

$$N\psi_j = \overline{L}^\alpha_{\alpha j} - L^\alpha_{\alpha j} + e(\psi_\alpha \varphi^\alpha - \underset{1}{\theta}_\alpha \varphi^\alpha) q_j - (N-2)\underset{1}{\theta}_j + e\varphi^\alpha (q_{\underset{1}{\alpha||j}} - q_{\underset{1}{\alpha|j}}), \tag{4.5}$$

wherefrom

$$(N-1)\psi_\alpha \varphi^\alpha = \varphi^\beta (\overline{L}^\alpha_{\alpha\beta} - L^\alpha_{\alpha\beta}) - (N-1)\underset{1}{\theta}_\alpha \varphi^\alpha + e\varphi^\alpha \varphi^\beta (q_{\underset{1}{\alpha||\beta}} - q_{\underset{1}{\alpha|\beta}}). \tag{4.6}$$

From (4.5) and (4.6) we get

$$N\psi_j = \overline{L}^\alpha_{\alpha j} - L^\alpha_{\alpha j} + \frac{e}{N-1} q_j \big[\varphi^\beta(\overline{L}^\alpha_{\alpha\beta} - L^\alpha_{\alpha\beta}) + e\varphi^\alpha\varphi^\beta(q_{\underset{1}{\alpha||\beta}} - q_{\underset{1}{\alpha|\beta}})\big] + e\varphi^\alpha(q_{\underset{1}{\alpha||j}} - q_{\underset{1}{\alpha|j}}) + N\underset{1}{\varepsilon}_j, \tag{4.7}$$

where

$$N\underset{1}{\varepsilon}_i = -2eq_i\underset{1}{\theta}_\alpha\varphi^\alpha - (N-2)\underset{1}{\theta}_i. \tag{4.8}$$

From (4.2) we have

$$\sigma_{ij} = e(q_{\underset{1}{i|j}} - q_{\underset{1}{i||j}}) - e\psi_i q_j - e\psi_j q_i - e\underset{1}{\theta}_j q_i + e\underset{1}{\theta}_i q_j,$$

wherefrom in view of (4.7) it follows

$$\begin{aligned} \sigma_{ij} &= e(q_{\underset{1}{i|j}} - q_{\underset{1}{i||j}}) - \frac{1}{N} q_j \big\{\overline{L}^\alpha_{\alpha i} - L^\alpha_{\alpha i} + \frac{e}{N-1} q_i \big[\varphi^\beta(\overline{L}^\alpha_{\alpha\beta} - L^\alpha_{\alpha\beta}) \\ &+ e\varphi^\alpha\varphi^\beta(q_{\underset{1}{\alpha||\beta}} - q_{\underset{1}{\alpha|\beta}}))\big] + e\varphi^\alpha(q_{\underset{1}{\alpha||i}} - q_{\underset{1}{\alpha|i}})\big\} - eq_j\underset{1}{\varepsilon}_i \\ &- \frac{1}{N} q_i \big\{\overline{L}^\alpha_{\alpha j} - L^\alpha_{\alpha j} + \frac{e}{N-1} q_j \big[\varphi^\beta(\overline{L}^\alpha_{\alpha\beta} - L^\alpha_{\alpha\beta}) + e\varphi^\alpha\varphi^\beta(q_{\underset{1}{\alpha||\beta}} - q_{\underset{1}{\alpha|\beta}})\big] \\ &+ e\varphi^\alpha(q_{\underset{1}{\alpha||j}} - q_{\underset{1}{\alpha|j}})\big\} - eq_i\underset{1}{\varepsilon}_j - e\underset{1}{\theta}_j q_i + e\underset{1}{\theta}_i q_j \end{aligned} \tag{4.9}$$

Let the vector $\underset{1}{\theta}_i$ in (3.3) satisfy the condition

$$\underset{1}{\varepsilon}_{(i}\delta^h_{j)} - e\underset{1}{\varepsilon}_{(i}q_{j)}\varphi^h + e\underset{1}{\theta}_{[i}q_{j]}\varphi^h - \underset{1}{\theta}_{[i}\delta^h_{j]} \equiv 0, \tag{4.10}$$

then we call it $\underset{1}{\widetilde{\pi}}_3(e,\theta)$ mapping. Here (i,j) and $[i,j]$ denote symmetrisation and antysymmetrisation without division, respectively. In view of (4.7)-(4.10) the relation (3.3) can be presented in the form

$$\underset{3}{\widetilde{\overline{T}}}{}^h_{ij}(x) = \underset{3}{\widetilde{T}}{}^h_{ij}(x), \tag{4.11}$$

where

$$\begin{aligned} \underset{3}{\widetilde{T}}{}^h_{ij} &= L^h_{ij} + eq_{\underset{1}{i|j}}\varphi^h \\ &- \frac{1}{N}(\delta^h_j - e\varphi^h q_j)\big[L^\alpha_{\alpha i} + eq_{\underset{1}{\alpha|i}}\varphi^\alpha + \frac{q_i}{N-1}(e\varphi^\beta L^\alpha_{\alpha\beta} + \varphi^\alpha\varphi^\beta q_{\underset{1}{\alpha|\beta}})\big] \\ &- \frac{1}{N}(\delta^h_i - e\varphi^h q_i)\big[L^\alpha_{\alpha j} + eq_{\underset{1}{\alpha|j}}\varphi^\alpha + \frac{q_j}{N-1}(e\varphi^\beta L^\alpha_{\alpha\beta} + \varphi^\alpha\varphi^\beta q_{\underset{1}{\alpha|\beta}})\big]. \end{aligned} \tag{4.12}$$

Analogously we define the object $\underset{3}{\widetilde{\overline{T}}}{}^h_{ij}$ of the basis $G\overline{A}_N$. On the basis of the facts given above, we get

Theorem 4.1. *Geometric objects* (4.12) *of the space* GA_N *are invariant of the mapping* $\underset{1}{\widetilde{\pi}}_3(e,\theta) : GA_N \to G\overline{A}_N$ *with respect to* φ^h *for any vector* q_i *satisfying the condition* (4.10).

Let $\underset{2}{\widetilde{\pi}}_3 : GA_N \to G\overline{A}_N$ be an almost geodesic mapping and q_i a vector such that

$$q_\alpha \varphi^\alpha = e, \qquad (e = \pm 1, 0). \tag{4.1'}$$

From (3.3)

$$q_{\underset{2}{j||i}} = q_{\underset{1}{j|i}} - \psi_i q_j - \psi_j q_i - e\sigma_{ij} - \underset{2}{\theta}_i q_j + \underset{2}{\theta}_j q_i. \tag{4.2'}$$

Now, by composition with φ^i in (4.2') we get

$$\varphi^\alpha(q_{\underset{2}{\alpha||i}} - q_{\underset{2}{\alpha|i}}) = (-\psi_\alpha \varphi^\alpha + \underset{2}{\theta}_\alpha \varphi^\alpha) q_i - e(\psi_i + \underset{2}{\theta} i) - e\sigma_{\alpha i}\varphi^\alpha. \tag{4.3'}$$

Contracting with respect to h, i in (3.3') we get

$$\overline{L}^\alpha_{i\alpha} = L^\alpha_{i\alpha} + (N+1)\psi_i + \sigma_{\alpha i}\varphi^\alpha + (N-1)\underset{2}{\theta}_i. \tag{4.4'}$$

From (4.3') and (4.4') we get

$$N\psi_i = \overline{L}^\alpha_{i\alpha} - L^\alpha_{i\alpha} - e(-\psi_\alpha\varphi^\alpha + \underset{2}{\theta}_\alpha \varphi^\alpha) q_i - (N-2)\underset{2}{\theta}_i + e\varphi^\alpha(q_{\underset{2}{\alpha||i}} - q_{\underset{2}{\alpha|i}}), \tag{4.5'}$$

wherefrom

$$(N-1)\psi_\alpha\varphi^\alpha = \varphi^\beta(\overline{L}^\alpha_{\beta\alpha} - L^\alpha_{\beta\alpha}) - (N-1)\underset{2}{\theta}_\alpha \varphi^\alpha + e\varphi^\alpha\varphi^\beta(q_{\underset{2}{\alpha||\beta}} - q_{\underset{2}{\alpha|\beta}}). \tag{4.6'}$$

From (4.5') and (4.6') we get

$$\begin{aligned} N\psi_i = \overline{L}^\alpha_{i\alpha} - L^\alpha_{i\alpha} + \frac{e}{N-1} q_i \big[\varphi^\beta(\overline{L}^\alpha_{\beta\alpha} - L^\alpha_{\beta\alpha}) \\ + e\varphi^\alpha\varphi^\beta(q_{\underset{2}{\alpha||\beta}} - q_{\underset{2}{\alpha|\beta}})\big] + e\varphi^\alpha(q_{\underset{2}{\alpha||i}} - q_{\underset{2}{\alpha|i}}) + N\underset{2}{\varepsilon}_i, \end{aligned} \tag{4.7'}$$

where

$$N\underset{2}{\varepsilon}_i = -2eq_i\underset{2}{\theta}_\alpha\varphi^\alpha - (N-2)\underset{2}{\theta}_i. \tag{4.8'}$$

From (4.2') we have

$$\sigma_{ij} = e(q_{\underset{2}{j|i}} - q_{\underset{2}{j||i}}) - e\psi_i q_j - e\psi_j q_i - e\underset{2}{\theta}_i q_j + e\underset{2}{\theta}_j q_i,$$

wherefrom in view of (4.7’) we have

(4.9′)

$$\begin{aligned}\sigma_{ij} &= e(q_{(}q_{\underset{2}{j|i}} - q_{\underset{2}{j||i}}) - -\frac{eq_j}{N}\{\overline{L}^{\alpha}_{i\alpha} - L^{\alpha}_{i\alpha} + \frac{e}{N-1}q_i\big[\varphi^{\beta}(\overline{L}^{\alpha}_{\beta\alpha} - L^{\alpha}_{\beta\alpha})\\ &+e\varphi^{\alpha}\varphi^{\beta}(q_{\underset{2}{\alpha||\beta}} - q_{\underset{2}{\alpha|\beta}})\big] + e\varphi^{\alpha}(q_{\underset{2}{\alpha||i}} - q_{\underset{2}{\alpha|i}})\} - eq_j\underset{2}{\varepsilon}\\ &-\frac{eq_i}{N}\{\overline{L}^{\alpha}_{j\alpha} - L^{\alpha}_{j\alpha} + \frac{e}{N-1}q_j\big[\varphi^{\beta}(\overline{L}^{\alpha}_{\beta\alpha} - L^{\alpha}_{\beta\alpha}) + e\varphi^{\alpha}\varphi^{\beta}(q_{\underset{2}{\alpha||\beta}} - q_{\underset{2}{\alpha|\beta}})\big]\\ &+e\varphi^{\alpha}(q_{\underset{2}{\alpha||j}} - q_{\underset{2}{\alpha|j}})\} - eq_i\underset{2}{\varepsilon}_j - e\underset{2}{\theta}_i q_j + e\underset{2}{\theta}_j q_i\end{aligned}$$

Let the vector θ_i in (3.3’) satisfy the condition

$$\underset{2}{\varepsilon}_{(i}\delta^h_{j)} - e\,\underset{2}{\varepsilon}_{(i}q_{j)}\varphi^h - e\,\underset{2}{\theta}_{[i}q_{j]}\varphi^h + \underset{2}{\theta}_{[i}\delta^h_{j]} \equiv 0, \tag{4.10′}$$

then we call it $\underset{2}{\pi}_3(e,\theta)$ mapping. Here (i,j) and $[i,j]$ denote symmetrisation and antysymmetrisation without division, respectively. In view of (4.7’)-(4.10’) the relation (3.3’) can be presented in the form

$$\overline{\underset{3}{\widetilde{\widetilde{T}}}}{}^h_{ij}(x) = \underset{3}{\widetilde{\widetilde{T}}}{}^h_{ij}(x)\ , \tag{4.11′}$$

where

(4.12′)

$$\begin{aligned}\underset{3}{\widetilde{\widetilde{T}}}{}^h_{ij}(x) &= L^h_{ij} + eq_{\underset{2}{j|i}}\varphi^h\\ &\quad -\frac{1}{N}(\delta^h_j - e\varphi^h q_j)\big[L^{\alpha}_{i\alpha} + eq_{\underset{2}{\alpha|i}}\varphi^{\alpha} + \frac{q_i}{N-1}(e\varphi^{\beta}L^{\alpha}_{\beta\alpha} + \varphi^{\alpha}\varphi^{\beta}q_{\underset{2}{\alpha|\beta}})\big]\\ &\quad -\frac{1}{N}(\delta^h_i - e\varphi^h q_i)\big[L^{\alpha}_{j\alpha} + eq_{\underset{2}{\alpha|j}}\varphi^{\alpha} + \frac{q_j}{N-1}(e\varphi^{\beta}L^{\alpha}_{\beta\alpha} + \varphi^{\alpha}\varphi^{\beta}q_{\underset{2}{\alpha|\beta}})\big].\end{aligned}$$

On the basis of the facts given above we get

Theorem 4.2. *Geometric objects* (4.12′) *of the space* GA_N *are invariant of the mapping* $\underset{2}{\pi}_3(e,\theta): GA_N \to G\overline{A}_N$ *with respect to* φ^h *for any vector* q_i *satisfying the condition* (4.10′).

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