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LAPLACE TRANSFORM OF LAPLACE HYPERFUNCTIONS AND ITS APPLICATIONS

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Abstract. The aim of this paper is to touch a few aspects of the theory and applications of the Laplace transform of Laplace hyperfunctions. It is also shown how this theory can applied to the mathematical model of dynamics of a viscoelastic rod.

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1. Introduction

Laplace transform (in short LT) was originally employed to justify the Heaviside operational calculus [8]. Thereupon, many papers have been published in this sense. Let us mention only some of them [3], [4], [6],... Later on the Laplace transform has been elaborated as a powerful mathematical theory (see [6], [19]), very useful in practice and many time applied by engineers. And despite of the belief it has the following three theoretical shortcomings:

1. In order that the integral

$$\int_0^\infty e^{-st} f(t)dt \equiv \widehat{f}(s) \equiv \mathcal{L}f(s) \tag{1}$$

has a sense, the function f must satisfy

$$|f(x)| \leq ce^{Hx}, \quad x > 0, \tag{2}$$

with constants H and c (f being of exponential type). In short, applications of the LT call for some growth conditions of the originals.

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2. No simple characterisation of the Laplace image of functions satisfying (2) is know. So we do not know a priori whether or not a solution $\widehat{u}$ to an equation $P(\widehat{u}) = 0$, obtained using LT to a mathematical model $Q(u) = 0$, is the LT of a solution u to $Q(u)$.

3. The expression

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} \widehat{f}(s) ds, \quad c > H, \tag{3}$$

is the inversion formula for the LT only if we know that $\widehat{f}(s)$ is LT of f. Also, it converges in general case only as Cauchy's principal value. It does not converge absolutely. In short, we have no an easily applicable inversion formula.

To overcome these dificulties mathematicians invented different, foundations and theories of the Heaviside calculus. We can divide them in two groups by their aproach: analitic or algebraic.

To the first group belong those theories in which the LT is defined on a subspace of generalized functions, continuous functionals on appropriate test function space ([10], [17], [18], [20]) or other analytic approaches [5], [2].

The second group contains theories which use algebraic approaches [14], [15]. Of this group, the most popular in applications has been Mikusinski's operator calculus.

Mikusinski's fonudation of the Heaviside calculus is based on the fact that the convolution algebra $\mathcal{C}([0,\infty))$ of continuous functions on $[0,\infty)$ has no any divisor of zero. The quotient field $\mathcal{M}$ is defined to be the space of Mikusinski opetarors. $\mathcal{M}$ includes different spaces of distributions $\mathcal{D}'$ and ultradistributions $\mathcal{D}'^*_+$ ([12]). This follows because if $f \in \mathcal{D}'^*_{[a,\infty)}$ and $\varphi \in \mathcal{D}^*_{[-a,b]}$, then the regularization $f * \varphi \in \mathcal{C}_{[0,\infty)}$. In this way $\mathcal{D}'^*_{[0,\infty)}$ is imbedded in $\mathcal{M}$. $\mathcal{M}$ is large enough, perhaps too large. $\mathcal{M}$ overcomes the first of the shortcomings of the LT but the second and third are reinforced.

We will present the foundation of the theory of LT of Laplace hyperfunctions elaborated by H. Komatsu ([10], [11]). In our opinion his approach successfully overcomes all the three shortcomings of the classical LT and the well elaborated classical theory of LT can be used. However, this approach is not known sufficiently. This is the reason of my talk in this Conference.

2. Hyperfunctions and Laplace hyperfunctions

Let us denote by $\mathcal{O}(\Omega)$ the space of all holomorphic functions on an open set $\Omega \subset \mathbb{C}$.

Definition 1. *([16])*

$$\mathcal{B}_{[0,\infty)} = \mathcal{O}(\mathbb{C} \setminus [0,\infty))/\mathcal{O}(\mathbb{C})$$

is the space of hyperfunctions with support in $[0,\infty)$.

If $f \in \mathcal{B}_{[0,\infty)}$, then there exists an $F \in \mathcal{O}(\mathbb{C} \setminus [0,\infty))$ which defines a class $[F]$ such that $f = [F]$ and f is also written in the form

$$f = F_+(x + i0) - F_-(x - i0), \tag{4}$$

F is called a defining function of f. One can find in [9], Chapter 1, how to determine a defining function F for an f belonging to $\mathcal{A}(\mathbb{R})$, $C(\mathbb{R}), \mathcal{L}_{loc}(\mathbb{R})$ or $\mathcal{D}'(\mathbb{R})$.

Conversely, if we have a defining function of a hyperfunction f we can characterize the subspace of hyperfunctions to which it belongs (see Theorem 2.4. in [13]). Specially, the following theorem is often used

Theorem 1. *(Theorem 1.3.2 in [9]). Let f be a continuous function. Let F denote a defining function of the hyperfunction lf defined by f. Then $F_+(x + i\varepsilon) - F_-(x - i\varepsilon)$ converges locally uniformly to f as $\varepsilon \to 0$.*

If $f \in \mathcal{L}_{loc}(0,\infty)$, then $F_+(x + i\varepsilon) - F_-(x - i\varepsilon)$ converges for almost all x to f, when $\varepsilon \to 0$.

We will define a sheaf $\mathcal{O}^{\exp}$ on the radial compactification $\mathbf{O} = \mathbb{C} \cup S^1_\infty$ of the complex plane as follows. For each open set V in $\mathbf{O}$ the section space $\mathcal{O}^{\exp}(V)$ is defined to be the space of all the holomorphic functions $F(x)$ on $V \cap \mathbb{C}$ such that on each closed sector

$$\Sigma = \{z \in \mathbb{C} : \alpha \leq arg(z - b) \leq \beta\}$$

whose closure $\overline{\Sigma}_0$ in $\mathbf{O}$ is included in V, we have the estimates

$$|F(z)| \leq ce^{H|z|}, \qquad z \in \Sigma,$$

with constants H and c.

Definition 2. *([11]).*

$$\mathcal{B}^{\exp}_{[0,\infty]} = \mathcal{O}^{\exp}(\mathbf{O} \setminus [0,\infty])/\mathcal{O}^{\exp}(\mathbf{O})$$

is the space of Laplace hyperfunctions with support in $[0,\infty]$.

A Laplace hyperfunction f with support in $[0,\infty]$, $f \in \mathcal{B}^{\exp}_{[0,\infty]}$, is represented by the class $[F]$, where F is a holomorphic function, $F \in \mathcal{O}^{\exp}(\mathbf{O} \setminus [0,\infty])$, or by

$$f(x) = F_+(x+i0) - F_-(x-i0).$$

The next theorem gives a interesting relation between $\mathcal{B}^{\exp}_{[0,\infty]}$ and $\mathcal{B}_{[0,\infty)}$.

Theorem 2. *(Theorem 2 in [11]). The restriction mapping $\mathcal{O}^{\exp}(\mathbf{O}\setminus[0,\infty]) \to \mathcal{O}(\mathbb{C}\setminus[0,\infty))$ induces a natural mapping ρ :*

$$\rho : \mathcal{B}^{\exp}_{[0,\infty]} \to \mathcal{B}_{[0,\infty)}. \tag{5}$$

The mapping ρ is surjective but not injective. Its kernel equals $\mathcal{B}^{\exp}_{[\infty]}$. Hence we have the natural isomorphism:

$$\mathcal{B}_{[0,\infty)} = \mathcal{B}^{\exp}_{[0,\infty]}/\mathcal{B}^{\exp}_{[\infty]}. \tag{6}$$

We will use a subspace of $\mathcal{B}^{\exp}_{[0,\infty]}$ denoted by $\tilde{\mathcal{L}}_{[0,\infty]}$: $\tilde{g} \in \mathcal{B}^{\exp}_{[0,\infty]}$ belongs to $\tilde{\mathcal{L}}_{[0,\infty]}$ if and only if there exists $g \in \mathcal{L}_{loc}([0,\infty))$ such that $\tilde{g}$ extends lg on $[0,\infty]$ we have

$$\mathcal{L}_{loc}([0,\infty)) \cong \tilde{\mathcal{L}}_{[0,\infty]}/\mathcal{B}^{\exp}_{[\infty]}.$$

3. Laplace transform of Laplace hyperfunctions

Definition 3. *([11]). LT $\hat{f} \equiv \mathcal{L}f$ of an $f = [F] \in \mathcal{B}^{exp}_{[0,\infty]}$ is defined by*

$$\hat{f}(s) = \int_C e^{-sz} F(z)dz, \quad s \in \Omega,$$

where C is a path composed of a ray from $e^{i\alpha}_{\infty}$ to a point $c<0$ and a ray from c to $e^{i\beta}_{\infty}$ with $-\pi/2 < \alpha < 0 < \beta < \pi/2$.

We note that the domain Ω of $\hat{f}$ depends on the choice of the defining function F. Therefore the LT $\hat{f}$ should be regarded as a germ of functions as the domain Ω of opening $(-\pi/2, \pi/2)$ shrinks.

Next theorem characterizes the LT and the space $\mathcal{L}\mathcal{B}^{\exp}_{[0,\infty]}$.

Theorem 3. *(Theorem 1 in [11]). $\mathcal{L}$ is an isomorphism*

$$\mathcal{L} : \mathcal{B}^{\exp}_{[0,\infty]} \to \mathcal{L}\mathcal{B}^{\exp}_{[0,\infty]},$$

where $\mathcal{L}\mathbf{B}^{\exp}_{[0,\infty]}$ *is the space of holomorphic functions* $\hat{f}$ *of exponential type defined on a neighbourhood* Ω *of the semi-circle* $S = \{e^{i\theta}_{\infty}; |\theta| < \pi/2\}$ *in* $\mathbf{O}$ *such that*

$$\overline{\lim}_{\rho\to\infty} \frac{\log|\widehat{f}(\rho e^{i\theta})|}{\rho} \leq -a\cos\theta, |\theta| < \pi/2. \tag{7}$$

If $\widehat{f} \in \mathcal{L}\mathcal{B}^{\exp}_{[0,\infty]}$, then a defining function F of its inverse image is given by

$$F(z) = \frac{1}{2\pi i}\int_{u}^{\infty} e^{zx}\widehat{f}(s)ds, \quad z \in \mathbb{C}\setminus[0,\infty), \tag{8}$$

where u is a fixed point in Ω and the path of integration is a convex curve in Ω.

F belongs to $\mathcal{O}^{\exp}(\mathbb{C}\setminus[0,\infty))$ and

$$f(x) = F_+(x+i0) - F_-(x-i0).$$

In connection with (6) we have that $\mathcal{L}$ induces the isomorphism

$$\mathcal{L} : \mathcal{B}_{[0,\infty)} \cong \mathcal{L}\mathcal{B}^{\exp}_{[0,\infty]}/\mathcal{L}\mathcal{B}^{\exp}_{[\infty]}.$$

Since every continuous function or locally integrable function on $[0,\infty)$ is indentified with a byperfunction in $\mathcal{B}_{[0,\infty)}$, its LT makes sense as a class of holomorphic functions. But some classes of functions can be directly imbedded into $\mathcal{B}^{\exp}_{[0,\infty]}$ using LT Such a class is $\mathcal{C}^{\exp}([0,\infty))$, the space of all continuous functions f on $[0,\infty)$ satisfying

$$|f(x)| \leq Ce^{Hx}, \quad x \geq 0. \tag{9}$$

If $G \in \mathcal{C}^{\exp}([0,\infty))$, then its Laplace transform

$$\widehat{g}(s) = \int_0^\infty e^{-sx}G(x)dx, \quad Res > H,$$

represents a holomorphic function which satisfies the estimate

$$|\widehat{g}(s)| \leq \frac{C}{Res - H}, \quad Res > H.$$

Because of (7) $\widehat{g}$ belongs to $\mathcal{L}\mathcal{B}^{\exp}_{[0,\infty]}$. Hence, the inverse image of $\widehat{g}$ gives by (8) the defining function F of $f \in \mathcal{B}^{\exp}_{[0,\infty]}$ and by Theorem 1 we obtain that f extends G on $[0,\infty]$. In this way the space $\mathcal{C}^{\exp}([0,\infty))$ is naturally imbedded in $\mathcal{B}^{\exp}_{[0,\infty]}$. If $G \in \mathcal{C}^{\exp}([0,\infty))$, then we denote by θG the corresponding Laplace hyperfunction. (θ stands for the Heaviside function). Similarly, we can imbed measurable functions satisfying exponential type condition (9). A direct consequence is that the classical LT of an $G \in \mathcal{C}^{\exp}_{([0,\infty))}$ and the LT of $\theta G \in \mathcal{B}^{\exp}_{[0,\infty]}$ coincide. This makes possible the use of well elaborated classical theory of LT ([6], [19]).

We give some properties of the LT of Laplace hyperfunctions (see [11]). For $f \in \mathcal{B}^{\exp}_{[0,\infty]}$

$$\mathcal{L}(x^n e^{ax} f(x))(s) = \Big(-\frac{d}{ds}\Big)^n \mathcal{L}f(s-a),\ n \in \mathbb{N}\cup\{0\},\ a \in \mathbb{C}.$$

$$\mathcal{L}\Big(\frac{d^n}{dx^n} f(x+c)\Big)(s) = s^n e^{cs}\mathcal{L}f(s),\ u \in \mathbb{N}\cup\{0\},\ c \in \mathbb{R}.$$

$\mathcal{L}\delta^{(\alpha)}(x) = s^\alpha$ where $\delta^{(\alpha)}(x) = \delta^{(\alpha)}(x),\ \alpha = 0,1,2,\ldots$ and

$$\delta^{(\alpha)}(x) = x_+^{-\alpha-1}/\Gamma(-\alpha),\ \ \alpha \neq 0,1,,2,\ldots,\alpha \in \mathbb{C}.$$

We can define the convolution of Laplace hyperfunctions of two elements f, g belonging to $\mathcal{B}^{\exp}_{[0,\infty]}$ without any restriction. Since $\widehat{f}\cdot\widehat{g} \in \mathcal{L}\mathcal{B}^{\exp}_{[0,\infty]}$, the convolution

$$\mathcal{L}(f * g)(x) = \widehat{f}(s)\widehat{g}(s).$$

Hence, $f * g \in \mathcal{B}^{\exp}_{[0,\infty]}$. If $\theta f, \theta g \in \theta\mathcal{C}^{\exp}_{[0,\infty)}$, then $\theta f * \theta g = \theta \int_0^t f(t-\tau)g(\tau)d\tau$.

4. Dynamics of a fractional derivative type of a viscoelastic rod

We illustrate how the LT can be applied to a mathematical model of the dinamics of a fractional derivative type of a viscoelastic rod ([1]). This model is the initial value problem

$$(10)\qquad T^{(2)}(t) + \gamma T^{(\alpha)}(t) + g(t)T(t) = 0,\ t > 0,\ T(0) = 0,\ T'(0) = T_0' \neq 0$$

where $\gamma > 0,\ 0 < \alpha < 1,\ g \in \mathcal{O}^{\exp}(\mathbf{O}),\ \ g$ is bounded on $[0,\infty)$ and $T^{(\alpha)}(t)$ is the fractional derivative

$$T^{(\alpha)}(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{T(t-\tau)}{\tau^\alpha}d\tau,\ \ t \geq 0.$$

We have first to imbed equation (10) with initial data, in $\mathcal{B}^{\exp}_{[0,\infty]}$.

In view of Green's formula

$$\begin{aligned} D_x^2(\theta(x)T(x)) &= T^{(2)}(x)\theta(x) + T^{(1)}(0)\delta(x) + T(0)D_x\delta(x) \\ &= T^{(2)}(x)\theta(x) + T_0'\delta(x) \end{aligned}$$

and

$$\begin{aligned} D^\alpha(\theta(x)T(x)) &= \frac{d}{dx}\frac{1}{\Gamma(1-\alpha)}\int_0^x (x-\tau)^{-\alpha}T(\tau)\theta(\tau)d\tau \\ &= \frac{d}{dx}\frac{1}{\Gamma(1-\alpha)}\theta(x)\int_0^x (x-\tau)^{-\alpha}T(\tau)d\tau \\ &= \theta(x)T^{(\alpha)}(x). \end{aligned}$$

The initial value problem (10) is reduced to the equation in $\mathcal{B}^{\exp}_{[0,\infty]}$

(11) $$D^2(\theta(x)T(x)) + \gamma D^\alpha(\theta(x)T(x)) + g(x)(\theta(x)T(x)) = T_0'\delta(x).$$

Applying the LT we have

$$(s^2+\gamma s^\alpha) * \mathcal{L}T(s) = T_0' - \mathcal{L}gT(s)$$

or

(12) $$\mathcal{L}T(s) = \frac{T_0'}{s^2+\gamma s^\alpha} - \frac{1}{s^2+\gamma s^\alpha}\mathcal{L}gT(s).$$

The inverse Laplace transform of $\frac{1}{s^2+\gamma s^\alpha}$, $Re s > \gamma^{1/(2-\alpha)}$ is the function $xE_{2-\alpha,2}(-\gamma x^{2-\alpha})$, where $E_{\mu,\nu}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\mu+\nu)}$ is Mittag-Leffler's function of two parameters and $x^{2-\alpha}$, $x \geq 0$, is the principal branch. This follows from the properties of the function $E_{\mu,\nu}(x)$ (see [7], p. 210). Since

(13) $$E_{2-\alpha,2}(-\gamma x^{2-\alpha}) = \frac{1}{\gamma x^{2-\alpha}\Gamma(\alpha)} + \mathbf{O}((\gamma x^{2-\alpha})^{-2}), \quad x\to\infty,$$

where $0<\alpha<2$ (see [7], p. 210), $xE_{2-\alpha,2}(-\gamma x^{2-\alpha}) \in \mathcal{C}^{\exp}_{[0,\infty)}$ and to the function $xE_{2-\alpha,2}(-\gamma x^{2-\alpha})$ corresponds $\theta(x)E_{2-\alpha,2}(-\gamma x^{2-\alpha}) \in \theta\mathcal{C}^{\exp}_{[0,\infty)} \subset \mathcal{B}^{\exp}_{[0,\infty]}$.

The inverse Laplace transform of (12) gives

(14) $$\begin{aligned} \theta T(x) &= ((\theta(t)tE_{2-\alpha,2}(\gamma t^{2-\alpha}) * T_0')(x) \\ &\quad -((\theta(t)tE_{2-\alpha,2}(-\gamma t^{2-\alpha})) * (g\theta T(t))(x). \end{aligned}$$

Equation (14) is equivalent to (11) in $\mathcal{B}^{\exp}_{[0,\infty]}$.

If we consider equation (14) in $\theta\mathcal{C}^{\exp}_{[0,\infty)}$ it has the form of an integral equation of Volterra's type of the second kinde.

Theorem 4. *([1]). Denote by* $H(t) = tE_{2-\alpha}(-\gamma t^{2-\alpha})$, *where* $E_{\mu,\nu}(z)$ *is Mittag-Leffler's function (see [7]), and by*

$$K_1(t,\tau) = H(t-\tau)g(\tau),\ K_{n+1}(t,\tau) = \int_\tau^t K_n(t,\sigma),\ K_1(\sigma,\tau)d\sigma,\ n \in \mathbb{N}.$$

The initial value problem (10) has a solution

$$T(t) = T_0'H(t) + T_0'\sum_{n=0}^{\infty}(-1)^{n+1}\int_0^t K_{n+1}(t,\tau)H(\tau)d\tau \tag{15}$$

which belongs to $\mathcal{C}^2[0,\infty) \cap \mathcal{C}^{\exp}_{[0,\infty)}$.

If g *is a constant, then* $T \in \mathcal{C}^2_{[0,\infty)} \cap \mathcal{C}^{\exp}_{[0,\infty)} \cap \mathcal{C}^{\infty}_{(0,\infty)}$; *for* $g \geq 0$ *the solution* T *is asymptotic stable.*

We introduce a subspace $\tilde{\mathcal{L}}_{loc}([0,\infty])$ of $B^{\exp}_{[0,\infty]}$. A $\tilde{g} \in \mathcal{B}^{\exp}_{[0,\infty]}$ belongs to $\tilde{\mathcal{L}}_{loc}([0,\infty])$ if and only if $\tilde{g}$ is an extension to $[0,\infty]$ of a $g \in \mathcal{L}_{loc}([0,\infty))$ where g is bounded on $[0,r]$ for an $r > 0$.

Then, all the solutions to equation (11) in $\tilde{\mathcal{L}}_{loc}([0,\infty])$ have the form $\theta T + V$, where T is given by (15) and $V \in \mathcal{B}^{\exp}_{[\infty]}$.

If in (10) $t \geq 0$ and $T(0) \neq 0$, then T, given by (15), belongs to $C^1_{[0,\infty)} \cap C^{\exp}_{[0,\infty)}$ and there is no elassieal solutions to (10).

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