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NONLINEAR SPP WITH NONLOCAL BOUNDARY CONDITIONS AND SPECTRAL APPROXIMATION

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Abstract. We consider boundary layer problem described by nonlinear second order differential equation with a small parameter multiplying the highest derivative and with nonlocal boundary conditions. The approximate solutions inside the layers are constructed iteratively, using monotone iterations in the form of truncated Chebyshev series. The layer subinterval is determined in terms of the degree of the spectral approximation and the perturbation parameter.

The solution outside the layers is approximated by the solution of the reduced problem.

Numerical example is included.

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1. Transformation of the problem

We shall consider the boundary layer problem described by the nonlinear differential equation

$$L_\varepsilon y \equiv -\varepsilon^2 y''(x) + b(x,y) = 0, \quad 0 \le x \le 1, \tag{1}$$

$$b(x,y) \in \mathbf{C}^2([0,1] \times \mathbf{R}), \quad b_y(x,y) \ge B^2 > 0, \quad B > 0, \tag{2}$$

and nonlocal boundary conditions

$$y(0) = 0, \; c_0 y(1) = \sum_{i=1}^{r} c_i y(s_i) + c_{r+1} \int_0^1 y(x)dx + d, \; s_i \in (0,1), \; c_0 \ge 0. \tag{3}$$

Condition (3) represents the generalization of the following special cases:

1. Samarski–Bicadze simple condition $y(1) = c_1 y(s) + d, \quad 0 < s < 1$,

2. Samarski–Bicadze general condition $y(1) = \sum_{i=1}^{r} c_i y(s_i) + d, \quad s_i \in (0,1)$,

3. Integral condition $\int_0^1 y(x)dx = d$.

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In Chegis, [4], for the linear problem

$$L_\varepsilon y \equiv -\varepsilon^2 y''(x) + g(x)y = f(x),\ f,g \in C^2([0,1]), g(x) \geq B^2 > 0,$$

we can find the sufficient condition for the existence of the unique solution:

$$\sum_{i=1}^{r} c_i u_0(s_i) + c_{r+1} \int_0^1 u_0(x)dx < c_0, \quad u_0(x) = \frac{\mathrm{sh}\frac{Bx}{\varepsilon}}{\mathrm{sh}\frac{B}{\varepsilon}}. \tag{4}$$

It is well known that if (2) holds the solution of the reduced problem $b(x,y) = 0$ has the unique solution $z(x) \in C^2[0,1]$ and in general, the exact solution displays two boundary layers of order $\mathcal{O}(\varepsilon)$. This brings us to the idea to approximate the exact solution by

$$y(x) \approx u(x) = \begin{cases} u_L(x) & x \in [0, c\varepsilon] \\ z(x) & x \in [c\varepsilon, 1-c\varepsilon] \\ u_R(x) & x \in [1-c\varepsilon, 1] \end{cases}, \tag{5}$$

where $u_L(x)$ is the approximation to the left layer solution and it satisfies

$$L_\varepsilon u_L(x) = 0\ ,\ u_L(0) = 0\ ,\ u_L(c\varepsilon) = z(c\varepsilon) \tag{6}$$

and $u_R(x)$ is the approximation to the right layer solution and it satisfies

$$L_\varepsilon u_R(x) = 0\ ,\ u_R(1-c\varepsilon) = z(1-c\varepsilon) \tag{7}$$

$$c_0 u_R(1) = \sum_{i=1}^{r} c_i u(s_i) + c_{r+1} \int_0^1 u(x)dx + d \tag{8}$$

2. The division point

By the procedure described in [1] the division point $x_0 = c\varepsilon$ is determined in such a way that $c = c(m)$, where m is the degree of the spectral approximation for the layer functions. That procedure is based on the introduction of the *resemblance function* for the left layer solution, given by the following definition

Definition 1. *A polynomial $p_m(x)$, $x \in [0, c\varepsilon]$, $m \geq 2$, is called resemblance function for the problem (6) if*

1. $p_m(0) = 0, \quad p_m(c\varepsilon) = z(0),$
2. *$x = c\varepsilon$ is the only stationary point for $p_m(x)$,*
3. *$p_m(x)$ is concave for $z(0) < 0$ and convex for $z(0) > 0$.*

Verifying the conditions from Definition 1 it can be easily proved that the following lemma holds:

Lemma 1. *The polynomial*

$$p_m(x) = -z(0)\left(\frac{c\varepsilon - x}{c\varepsilon}\right)^m + z(0)\,,\quad m \geq 2 \tag{9}$$

is a resemblance function for the problem (6).

In order to determine the division point we ask that the resemblance function satisfies the differential equation at the layer point $x = 0$. This gives us

$$c = c(m) = \sqrt{\frac{-z(0)m(m-1)}{b(0,0)}}.$$

3. Approximation of the left layer solution

We shall approximate the solution of the problem (6) by monotone iteration sequence $\{u_n(x)\}$ represented in the form of the truncated Chebyshev series of degree m

$$u_n(x) = \sum_{k=0}^{m}{}' a_{n,k} T_k\left(\tfrac{2x}{c\epsilon} - 1\right),\ n \in \mathbf{N}, \tag{10}$$

where $T_k(t) = \cos(k \cdot \arccos t),\ \ t \in [-1,1],\ \ k = 0,1,\ldots.$

If we transform $[0, c\varepsilon]$ into $[-1,1]$ using stretching variable $t = \frac{2x}{c\epsilon} - 1$, the truncated series (10) becomes

$$v_n(t) = \sum_{k=0}^{m}{}' a_{n,k} T_k(t), \tag{11}$$

and the problem (6) becomes

$$w''(t) + g(t,w) = 0,\ \ w(-1) = 0,\ \ w(1) = z(c\varepsilon)\,, \tag{12}$$

$$g(t,w) = -\frac{c^2}{4} \cdot b\left(\tfrac{c\varepsilon}{2}(t+1), w\right).$$

It was proved in [3] that if $g(t,w)$ is continuous on $[-1,1] \times \mathbf{R}$ and satisfies

$$K_1(v-\omega) \leq g(t,v) - g(t,\omega) \leq K_2(v-\omega),\ v-\omega \geq 0,\ K_1, K_2 \in \mathbf{R},\ K_2 \leq \tfrac{\pi^2}{4},$$

the iteration sequence $w_n(t)$, defined by

$$w_n''(t) + K_1 w_n(t) = K_1 w_{n-1}(t) - g(t, w_{n-1}(t)), \tag{13}$$

$$w_n(-1) = 0,\ w_n(1) = z(c\varepsilon),$$

converges to the solution $w(t)$ of the problem (12), starting from an arbitrary function $w_0(t)$.

Applying *Mean value theorem*, it can be easily seen that in our case

$$K_1 = -\frac{c^2}{4} \max_{x\in[0,c\varepsilon]} b_y(x,y) \ , \quad K_2 = -\frac{c^2}{4} \cdot B^2 \ ,$$

so if we start from $w_0(t) = \frac{z(0)(t+1)}{2}$ the iterations sequence determined by (13) will converge to $w(t)$ in such a way that

a) if $w_0''(t) + g(t, w_0(t)) \le 0$, then $w_{n-1}(t) \ge w_n(t) \ge w(t)$, and

b) if $w_0''(t) + g(t, w_0(t)) \ge 0$, then $w_{n-1}(t) \le w_n(t) \le w(t)$.

With the purpose of determining the Chebyshev coefficients in each iteration we can use either *direct method* or *collocation*.

Direct method

We shall represent $w_n''(t)$ and $g(t, w_{n-1}(t))$ by their truncated Chebyshev series

$$v_n''(t) = \sum_{k=0}^{m}{}' a_{n,k}^{(2)} T_k(t), \qquad G_n(t) = \sum_{k=0}^{m}{}' g_{n,k} T_k(t). \tag{14}$$

Introducing (11) and (14) into (13) we obtain

$$\sum_{k=0}^{m}{}' (a_{n,k}^{(2)} + K_1 a_{n,k}) T_k(t) = \sum_{k=0}^{m}{}' (g_{n,k} + K_1 a_{n-1,k}) T_k(t),$$

$$a_{n,k}^{(2)} + K_1 a_{n,k} = h_{n,k} \ , \quad h_{n,k} = g_{n,k} + K_1 a_{n-1,k} \ , \quad k = 0, \ldots, m. \tag{15}$$

Using recurrence relation $a_{n,k-1}^{(i)} - a_{n,k+1}^{(i)} = 2k a_{n,k}^{(i-1)}$, $i = 1,2$ we come to

$$K_1(k+1)a_{n,k-2} + 2k(2(k^2-1) - K_1)a_{n,k} + K_1(k-1)a_{n,k+2} = H_{n,k}, \tag{16}$$

$$a_{n,m+1} = a_{n,m+2} = h_{n,m+1} = h_{n,m+2} = 0$$

$$H_{n,k} = (k+1)h_{n,k-2} - 2kh_{n,k} + (k-1)h_{n,k+2},$$

$$k = 2, \ldots, m.$$

Equations (16), together with the equations obtained from the boundary conditions

$$\sum_{k=0}^{m}{}' (-1)^k a_{n,k} = 0 \ , \quad \sum_{k=0}^{m}{}' a_{n,k} = z(c\varepsilon) \tag{17}$$

represent the system that determines the coefficients of the truncated Chebyshev series (11).

Collocation method

We ask that the differential equation (13) is satisfied at the Gauss–Lobatto nodes $t_i = \cos\frac{i\pi}{m}, \quad i = 1, \ldots, m-1$. This will give us

$$\sum_{k=0}^{m}{}' \left(T_k''(t_i) + K_1 T_k(t_i)\right) a_{n,k} = K_1 v_{n-1}(t_i) - g(t_i, v_{n-1}(t_i)), \tag{18}$$

which, together with (17), represents the system that determines the coefficients of the truncated Chebyshev series (11).

4. Approximation to the right layer solution

We shall approximate the solution $u_R(x)$ of the problem (7),(8) by the solution of the problem

$$L\bar{u}_R(x) \equiv -\varepsilon^2 \bar{u}_R''(x) + b(x, \bar{u}_R) = 0, \quad x \in [1 - c\varepsilon, 1], \tag{19}$$

$$\bar{u}_R(1 - c\varepsilon) = z(1 - c\varepsilon), \tag{20}$$

$$c_0 \bar{u}_R(1) = \sum_{i=l+1}^{r} c_i \bar{u}_R(s_i) + c_{r+1} \int_{1-c\varepsilon}^{1} \bar{u}_R(x)dx + d_1, \tag{21}$$

with

$$d_1 = \sum_{i=1}^{j} c_i v_n(s_i) + \sum_{i=j+1}^{l} c_i z(s_i) + c_{r+1} \left(\int_0^{c\varepsilon} v_n(x)dx + \int_{c\varepsilon}^{1-c\varepsilon} z(x)dx \right) + d,$$

where

- $s_i \in (0, c\varepsilon)$ for $i \leq j$,
- $s_i \in (c\varepsilon, 1 - c\varepsilon)$ for $j < i \leq l$ and
- $s_i \in (1 - c\varepsilon, 1)$ for $i > l$.

The sequence of monotone iterations represented by the truncated Chebyshev series

$$\bar{v}_n(x) = \sum_{k=0}^{m}{}' b_{n,k} T_k\left(\frac{2(x-1)}{c\varepsilon} + 1\right), \tag{22}$$

is constructed in the same manner as in the case of left layer solution, which means that we can use either direct or collocation method to evaluate the coefficients $b_{n,k}$. Instead of the conventional boundary conditions (which we use in the case of left layer solution), in this case we use nonlocal boundary conditions (19),(20), which give us

$$\sum_{k=0}^{m}{}' (-1)^k b_{n,k} = z(1 - c\varepsilon),$$

$$c_0 \sum_{k=0}^{m}{}' b_{n,k} = \sum_{k=0}^{m}{}' \left(\sum_{i=l+1}^{r} c_i T_k(\xi_i) \right) b_{n,k} - 2c\varepsilon c_{r+1} \sum_{k=0}^{[\frac{m}{2}]}{}' \frac{b_{n,2k}}{4k^2-1} + d_1 ,$$

$$\xi_i = \frac{2(s_i - 1)}{c\varepsilon} + 1, \quad i = l+1, \ldots, r \, .$$

5. Numerical example

As a test model we have considered the problem

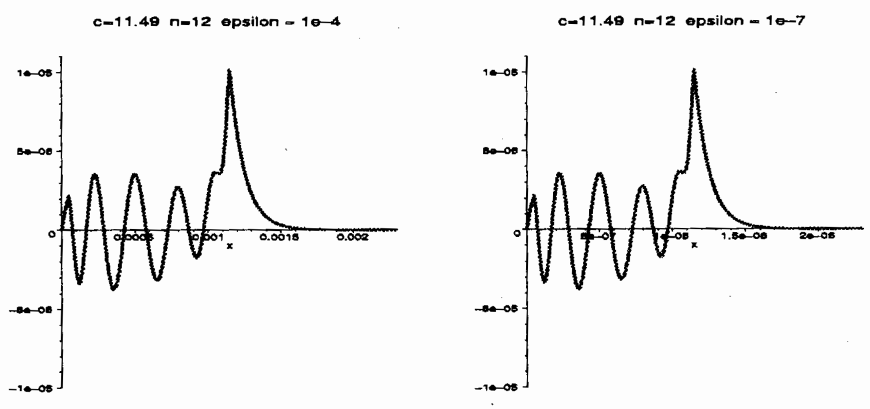
$$-\varepsilon^2 y''(x) + y^2(x) - f(x) = 0, \;\; y(0) = 0, \;\; y(1) = y(0.5) + d$$

where $f(x)$ is such a function that

$$y(x) = \frac{(2x-1)(e^{\frac{1-x}{\varepsilon}} - e^{\frac{x}{\varepsilon}})}{e^{\frac{1}{\varepsilon}} - 1} + 1$$

represents the exact solution. We have evaluated the approximate solution which corresponds to the reduced solution $z(x) \equiv 1$.

In Figure 1 which spreads through this and the next page, we can see plots of error $u_m(x) - y(x)$ for $x \in [0, 2c\varepsilon]$ obtained using the Chebyshev truncated series (10) of degree m for two values of ε. The number of iterations was $n = 10$. We can see that the error is uniform in ε and that its value becomes very small for degree m large enough.



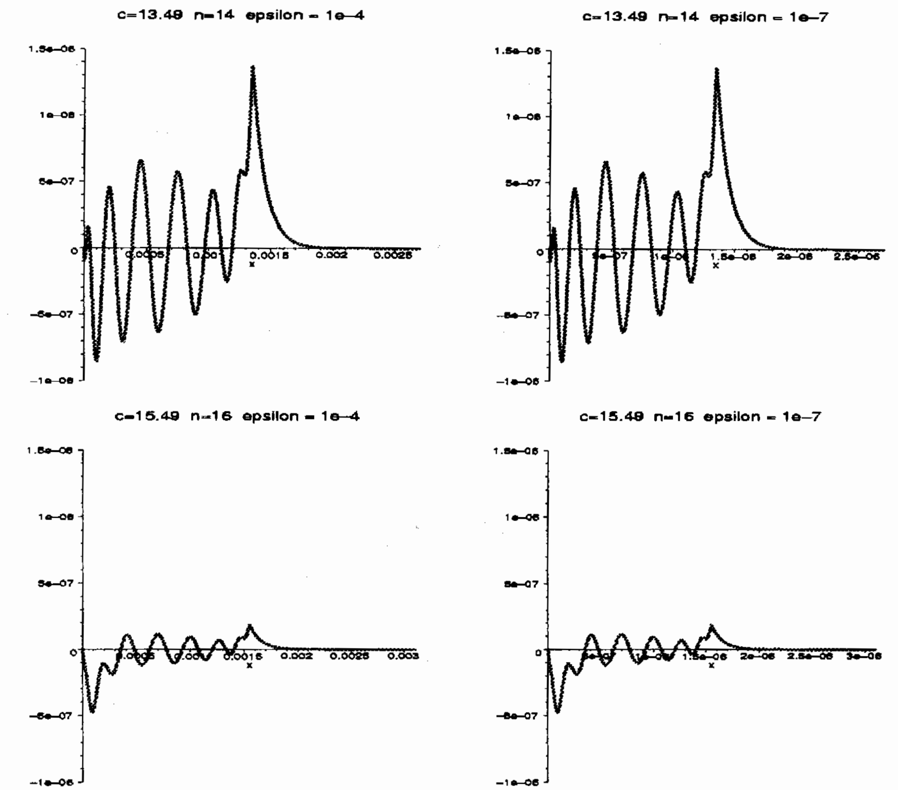


Figure 1: The error $u_m(x) - y(x)$ for $x \in [0, 2c\varepsilon]$

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