NOVI SAD J. MATH.
VOL. 31, NO. 1, 2001, 39-45

LINEAR AND BILINEAR HILBERT TRANSFORM

Aneta Bučkovska[1]

Abstract. General definitions of the Hilbert transform and bilinear Hilbert transform in appropriate Colombeau algebras $\mathcal{G}_{L^2} \times \mathcal{G}_{L^\infty}$ are given. The relations of distribution of the Hilbert and bilinear Hilbert transforms with the corresponding definitions in Colombeau algebras are given through special regularizations.

AMS Mathematics Subject Classification (1991): 46F10, 46F12

Key words and phrases: Colombeau algebras, algebras of generalized functions, Hilbert transform, bilinear Hilbert transform

1. Introduction and preliminaries

There are many paper related to the Hilbert transform (HT) in various spaces of generalized functions. As a simplest singular integral and a prototype of a pseudodifferential operator, it is used in many branches of analysis. We refer to the monograph [11] for the study of HT of Schwartz distributions and its subspaces.

Recently, the bilinear Hilbert transform (BHT) on L^p spaces has attracted considerable attention. Important papers of Lacey and Thiele [6]-[8] has strongly stimulated the investigations in this direction. In [3] is given the extension of BHT onto distributions.

The problem of multiplication in distribution spaces and (because of that) the inconvenience of distribution spaces for nonlinear problems, motivated Colombeau to construct an algebra of generalized functions in which the multiplication of a smooth function is the usual multiplication and in which the distribution spaces can be embedded and then multiplied. For this theory we refer to [4,5], [2], [9,10] and [12]. Distributions are embedded into the Colombeau algebra under a process of regularizations using special delta nets.

In this paper we are concerned with the definition of HT and BHT in Colombeau type algebras. For this reason, we consider the weighted Colombeau algebras $\mathcal{G}_{L^p}$. With suitable regularizations we analyze the relations between the embedded HT (resp. BHT) of a distribution (resp. distributions) and the HT (resp. BHT) of embedded distribution (resp. distributions).

Breafly, the content of the paper is: In first section we give a general definition of HT for $f \in \mathcal{D}'$. When $f \in \mathcal{D}'_{L^p}$, $p > 1$ this general definition of HT

[1]Faculty of Electrical Engineering, pb. 574, Skopje, Macedonia

equals with Pandey's definition of HT. Then we define the HT in Colombeau's algebra and prove the association of $H(f * \delta_\varepsilon)$ and $H(f) * \delta_\varepsilon$.

In the second paragraph we define BHT in Colombeau's algebra $\mathcal{G}_{L^2} \times \mathcal{G}_{L^\infty}$. We use the results of Lacey and Thiele in [6]-[8] and Bučkovska, Pilipović in [3] for definition and properties of the bilinear HT. Then we prove the association of $H_\alpha(f * \delta_\varepsilon, a * \delta_\varepsilon)$ and $H^*_\alpha(f, a) * \delta_\varepsilon$.

In the sequel for $1 < p < \infty$ we will write $q = \frac{p}{p-1}$ and denote by $\mathcal{D}'_{L^p}$ the dual of the space $\mathcal{D}_{L^q}$. We will not recall the definitions (cf. [14], [11]).

At the end of introduction, as a note, we will relate the distributional HT and the most general one in the spirit of sequential approach [1].

More precise: For $f \in \mathcal{D}'$, we define

$$Hf = \lim_{\varepsilon \to 0} (f * \phi_\varepsilon) * p.v. \frac{1}{x} \tag{1}$$

if this limit exist for every delta net ϕ_ε [2] and prove that when $f \in \mathcal{D}'_{L^p}$, $p > 1$, then (1) exist and equals with the Pandey's definition of Hilbert transform [11].

2. Hilbert transform in Colombeau algebra $\mathcal{G}_{L^p}$

We refer to the monographs given in the literature for the definitions and properties of general Colombeau type algebras. Here we will recall definitions of those algebras needed in the sequal. They are introduced by Biagioni in [2] and Oberguggenberger in [9].

Let Ω be an open set of $\mathbb{R}^n$. Denote by $\mathcal{E}[\Omega]$ the set of nets $(F_\varepsilon)_\varepsilon$, $\varepsilon \in (0,1)$, of smooth functions on Ω.

Definition 2.1. *The set $\mathcal{E}_{L^p}[\Omega]$ consists of all nets $(F_\varepsilon)_\varepsilon \in \mathcal{E}[\Omega]$ with the following property:*
For every $\alpha \in \mathbb{N}_0^n$ there exist $a > 0$ and $c > 0$ such that

$$||F_\varepsilon{}^{(\alpha)}||_p \le c \cdot \varepsilon^{-a}.$$

The set of all L^p - null functions $\mathcal{N}_{L^p}[\Omega]$ consists of all nets $(F_\varepsilon)_\varepsilon \in \mathcal{E}_{L^p}[\Omega]$ such that for every $\alpha \in \mathbb{N}_0^n$ and every $a > 0$ there exist $c > 0$ such that

$$||F_\varepsilon{}^{(\alpha)}||_p \le c \cdot \varepsilon^{a}.$$

For the sake of simplicity, we will write F_ε instead of $(F_\varepsilon)_\varepsilon$.

Theorems 2.1 and 2.2 which are to follow, are proved by Biagioni and Oberguggenberger.

[2] If $\phi \in \mathcal{D}$, and $\int_{\mathbb{R}} \phi(x)dx = 1$, then $\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})$, $\varepsilon \in (0,1)$ is a delta net: $\lim_{\varepsilon \to 0} \phi_\varepsilon = \delta$.

Theorem 2.1. *$\mathcal{E}_{L^p}[\Omega]$ and $\mathcal{N}_{L^p}[\Omega]$ are vector spaces and algebras under the pointwise multiplication of representatives. Moreover, with the differentiation defined by the differentiation of the representatives:*

$$[F_\varepsilon]^{(\alpha)} = [F_\varepsilon^{(\alpha)}],\ \alpha \in \mathbb{N}_0^n,$$

they are differential algebras. $\mathcal{N}_{L^p}[\Omega]$ is an ideal of $\mathcal{E}_{L^p}[\Omega]$.

Definition 2.2. *The space of generalized functions on Ω, $\mathcal{G}_{L^p}(\Omega)$ is defined by*

$$\mathcal{G}_{L^p}(\Omega) = \mathcal{E}_{L^p}[\Omega]/\mathcal{N}_{L^p}[\Omega].$$

Elements of $\mathcal{G}_{L^p}$ will be denoted by $f = [F_\varepsilon],\ g = [G_\varepsilon], \ldots$

Theorem 2.2. *$\mathcal{G}_{L^p}(\Omega)$ is an algebra under multiplication $[F_\varepsilon] \cdot [G_\varepsilon] = [F_\varepsilon \cdot G_\varepsilon]$.*

Definition 2.3. *Let $f = [F_\varepsilon],\ g = [G_\varepsilon] \in \mathcal{D}'_{L^p}$. It is said that they are associated if*

$$\int (F_\varepsilon - G_\varepsilon)\psi \to 0$$

as $\varepsilon \to 0,\ \psi \in \mathcal{D}_{L^q}$.

It is well known that if $f \in \mathcal{D}'_{L^p}$, then $F_\varepsilon = f * \phi_\varepsilon \in \mathcal{D}_{L^p}$, and (cf. [11]) if $\psi \in \mathcal{D}_{L^p}$ then

$$||(H\psi)^{(k)}||_p \le C_p \cdot ||\psi^{(k)}||_p,\ \forall k \in \mathbb{N}. \tag{2}$$

Theorem 2.3. *If $F_\varepsilon \in \mathcal{E}_{L^p}$ then $H(F_\varepsilon) \in \mathcal{E}_{L^p}$.*

Proof. Let $F_\varepsilon \in \mathcal{E}_{L^p}$. Then (2) implies

$$||(H(F_\varepsilon))^{(\alpha)}||_p \le C_p \cdot ||(F_\varepsilon)^{(\alpha)}||_p \le c \cdot \varepsilon^{-a}, \quad \varepsilon \in (0, \varepsilon_0).$$

It follows now that $H(F_\varepsilon) \in \mathcal{E}_{L^p}$. □

Theorem 2.4. *If F_ε and $\widetilde{F}_\varepsilon$ are the representatives of $f \in \mathcal{G}_{L^p}$ then $H(F_\varepsilon - \widetilde{F}_\varepsilon) \in \mathcal{N}_{L^p}$.*

Proof: We will denote $R_\varepsilon = F_\varepsilon - \widetilde{F}_\varepsilon$ and let $R_\varepsilon \in \mathcal{N}_{L^p}$. Then

$$||(H(R_\varepsilon))^{(\alpha)}||_p \le C_p \cdot ||(R_\varepsilon)^{(\alpha)}||_p \le \cdot \varepsilon^{a}, \quad \varepsilon \in (0, \varepsilon_0)$$

so we get that $H(R_\varepsilon) \in \mathcal{N}_{L^p}$. □

This theorem directly implies the following definition:

Definition 2.4. *Let $f \in \mathcal{G}_{L^p}$. Then*

$$Hf = [HF_\varepsilon].$$

2.1 Embeddings

The space $\mathcal{D}'_{L^p}$ is embedded into $\mathcal{G}_{L^p}$ by the use of a special net of mollifiers, which is a delta net, but constructed via a function with special properties. We will denote it by δ_ε.

Let $\phi \in \mathcal{S}$, $\int \phi = 1$, $\int \phi x^k = 0$, for $k \geq 1$. In the sequel we will put

$$\delta_\varepsilon = \frac{1}{\varepsilon}\phi(\frac{\cdot}{\varepsilon}),\ \varepsilon \in (0,1).$$

Regularizations of elements in $\mathcal{D}'_{L^p}$ by this delta net will lead to the embeddings.

Let $f \in \mathcal{D}'_{L^p}$. Then the corresponding element in $\mathcal{G}_{L^p}$ is denoted by $Cd\ f$ and it is defined as

$$Cd\ f = [f * \delta_\varepsilon].$$

Then

$$H(Cd\ f) = [H(f * \delta_\varepsilon)] = \left[\left(f * \delta_\varepsilon\right) * p.v.\frac{1}{x}\right].$$

Theorem 2.5. *Let $f \in \mathcal{D}'_{L^p}$. Then $Cd\,Hf$ is associated with $H(Cd\ f)$.*

Roughly, "the Hilbert transform of a regularization and regularized Hilbert transform are associated".

Proof. Let $f \in \mathcal{D}'_{L^p}$. Then, for every $\psi \in \mathcal{D}_{L^q}$ we have

$$\langle H^* f, \psi\rangle = \langle f, -H\psi\rangle = \lim_{\varepsilon\to 0}\langle f, -(H\psi) * \delta_\varepsilon\rangle$$

$$= \lim_{\varepsilon\to 0}\langle f * \delta_\varepsilon, -H\psi\rangle = \lim_{\varepsilon\to 0}\langle H^*(f * \delta_\varepsilon), \psi\rangle.$$

On the other hand

$$\langle H^* f, \psi\rangle = \lim_{\varepsilon\to 0}\langle H^* f, \psi * \delta_\varepsilon\rangle = \lim_{\varepsilon\to 0}\langle (H^* f) * \delta_\varepsilon, \psi\rangle.$$

Finally we get

$$H^* f = \lim_{\varepsilon\to 0} H^*(f * \delta_\varepsilon) = \lim_{\varepsilon\to 0}(H^* f) * \delta_\varepsilon,$$

which completes the proof. □

3. Bilinear Hilbert transform in $\mathcal{G}_{L^2} \times \mathcal{G}_{L^\infty}$

Definition 3.1. *Let $F_\varepsilon \in \mathcal{E}_{L^2}$ and $A_\varepsilon \in \mathcal{E}_{L^\infty}$. Then*

$$H_{\alpha,A_\varepsilon}(F_\varepsilon) = p.v.\int F_\varepsilon(x-t)A_\varepsilon(x+\alpha t)\frac{dt}{t},\quad \alpha \in \mathbb{R}\setminus\{-1,0\}.$$

It was proved in [3] that if $a \in \mathcal{D}_{L^\infty}$ and $\varphi \in \mathcal{D}_{L^2}$ then,

$$||(H_{\alpha,a}\varphi)^{(m)}||_2 \leq C\sum_{k=0}^{m}\binom{m}{k}||\varphi^{(m-k)}||_2 \cdot ||a^{(k)}||_\infty \quad \forall m \in \mathbb{N}. \tag{3}$$

Theorem 3.1.

(i) Let $F_\varepsilon \in \mathcal{E}_{L^2}$ *and* $A_\varepsilon \in \mathcal{E}_{L^\infty}$. *Then* $H_{\alpha,A_\varepsilon}(F_\varepsilon) \in \mathcal{E}_{L^2}$.

(ii) Let $R_\varepsilon \in \mathcal{N}_{L^2}$ *and* $A_\varepsilon \in \mathcal{E}_{L^\infty}$. *Then* $H_{\alpha,A_\varepsilon}(R_\varepsilon) \in \mathcal{N}_{L^2}$.

(iii) Let $A_\varepsilon \in \mathcal{N}_{L^\infty}$ *and* $F_\varepsilon \in \mathcal{E}_{L^2}$. *Then* $H_{\alpha,A_\varepsilon}(F_\varepsilon) \in \mathcal{N}_{L^2}$.

Proof. *(i)* We have to prove that for every $m \in \mathbb{N}_0^n$, there exist $b \in \mathbb{R}$, $A > 0$ and ε_0 such that:

$$\left|\left|\left(H_{\alpha,A_\varepsilon}(F_\varepsilon)\right)^{(m)}\right|\right|_2 \leq A \cdot \varepsilon^{-b}, \quad \varepsilon \in (0, \varepsilon_0).$$

Using (3) we obtain

$$\begin{aligned} \left|\left|\left(H_{\alpha,A_\varepsilon}(F_\varepsilon)\right)^{(m)}\right|\right|_2 &\leq C\sum_{k=0}^{m}\binom{m}{k}||F_\varepsilon^{(m-k)}||_2 \cdot ||A_\varepsilon^{(k)}||_\infty \\ &\leq C\sum_{k=0}^{m}\binom{m}{k}c_1 \cdot \varepsilon^{-a_1} \cdot c_2 \cdot \varepsilon^{-a_2} \\ &\leq A \cdot \varepsilon^{-b},\ \varepsilon < \varepsilon_0. \end{aligned}$$

(ii) Similarly to the proof of (i), we have to prove that for every $m \in \mathbb{N}_0^n$, and for every $b \in \mathbb{R}$ there exist $B > 0$ and ε_0 such that:

$$\left|\left|\left(H_{\alpha,A_\varepsilon}(R_\varepsilon)\right)^{(m)}\right|\right|_2 \leq B \cdot \varepsilon^{b}, \quad \varepsilon \in (0, \varepsilon_0).$$

Using (3) we obtain

$$\begin{aligned} \left|\left|\left(H_{\alpha,A_\varepsilon}(R_\varepsilon)\right)^{(m)}\right|\right|_2 &\leq C\sum_{k=0}^{m}\binom{m}{k}||R_\varepsilon^{(m-k)}||_2 \cdot ||A_\varepsilon^{(k)}||_\infty \\ &\leq C\sum_{k=0}^{m}\binom{m}{k}c_1 \cdot \varepsilon^{a_1} \cdot c_2 \cdot \varepsilon^{-a_2} \\ &\leq B \cdot \varepsilon^{b},\ \varepsilon < \varepsilon_0. \end{aligned}$$

(iii) Now we have to prove that for every $m \in \mathbb{N}_0^n$, and for every $b \in \mathbb{R}$ there exist $B > 0$ and ε_0 such that:

$$\left|\left|\left(H_{\alpha,A_\varepsilon}(F_\varepsilon)\right)^{(m)}\right|\right|_2 \leq B \cdot \varepsilon^{b}, \quad \varepsilon \in (0, \varepsilon_0).$$

Using (3) we have

$$\left|\left|\left(H_{\alpha,A_\varepsilon}(F_\varepsilon)\right)^{(m)}\right|\right|_2 \leq C\sum_{k=0}^{m}\binom{m}{k}||F_\varepsilon^{(m-k)}||_2 \cdot ||A_\varepsilon^{(k)}||_\infty$$

$$\leq \quad C\sum_{k=0}^{m}\binom{m}{k}c_1\cdot\varepsilon^{-a_1}\cdot c_2\cdot\varepsilon^{a_2}$$
$$\leq \quad B\cdot\varepsilon^{b},\ \varepsilon<\varepsilon_0. \qquad \square$$

Now we can define the bilinear Hilbert transform in $\mathcal{G}_{L^2}\times\mathcal{G}_{L^\infty}$.

Definition 3.2. *Let $f\in\mathcal{G}_{L^2}$ and $a\in\mathcal{G}_{L^\infty}$. Then*

$$H_\alpha(f,a)=\big[H_\alpha(F_\varepsilon,A_\varepsilon)\big].$$

In the theorem that follows we will use the same notations as in section 2.1.

Theorem 3.2. *Let $f\in\mathcal{D}'_{L^p}$, $a\in\mathcal{D}_{L^\infty}$. Then $Cd\ H^*_{\alpha,a}f$ is assoociated with $H_{\alpha,a}(Cd\ f)$.*

Proof. Let $f\in\mathcal{D}'_{L^2}$, and $a\in\mathcal{D}_{L^\infty}$. Then, for every $\psi\in\mathcal{D}_{L^2}$ using Th.2 (ii) in [1] we get

$$\begin{aligned}\langle(H^*_{\alpha,a}f)*\delta_\varepsilon,\psi\rangle &= \langle H^*_{\alpha,a}f,\psi*\delta_\varepsilon\rangle\\ &= \langle f,H_{\alpha,a}(\psi*\delta_\varepsilon)\rangle.\end{aligned}$$

Then

$$\lim_{\varepsilon\to0}\langle f,H_{\alpha,a}(\psi*\delta_\varepsilon\rangle=\langle H^*_{\alpha,a}f,\psi\rangle.$$

On the other hand we have:

$$\begin{aligned}\langle H_{\alpha,a}(f*\delta_\varepsilon),\psi\rangle &= \langle -H^*_{-1-\alpha,a}(f*\delta_\varepsilon),\psi\rangle\\ &= \langle f*\delta_\varepsilon,-H_{-1-\alpha,a}\psi\rangle=\langle f,\big(-H_{-1-\alpha}\psi\big)*\delta_\varepsilon\rangle,\end{aligned}$$

and if $\varepsilon\to0$ we have

$$\lim_{\varepsilon\to0}\langle f,-H_{-1-\alpha,a}\psi\rangle=\langle -H^*_{-1-\alpha,a}f,\psi\rangle.$$

So, we prove the above assertion. $\square$

In a similar way we can prove the next assertion.

Theorem 3.3. *Let $f\in\mathcal{D}'_{L^2}$ and $a\in\mathcal{D}'_{L^\infty}$, then $H_\alpha(Cd\ f,Cd\ a)$ is associated to $Cd\ H_\alpha(f,a)$.*

The generalized function $\left[\delta^2_\varepsilon\right]$ is an element of the algebra $\mathcal{G}_{L^p}$, $1\leq p\leq\infty$. This generalized function is not asso·iated with any Schwartz distribution, because if that is the case it must be δ^2, for which we know that is not in $\mathcal{D}'$. We will denote by $\delta^2=\left[\delta^2_\varepsilon\right]$.

Example: Let $a\in\mathcal{D}'_{L^\infty}$. Then

$$H_\alpha(\delta^2,a)=\left[p.v.\int\delta^2_\varepsilon(x-t)a(x+\alpha)\frac{dt}{t}\right].$$

References

[1] Antosik, P., Mikusinski, J., Sikorski, R., Theory of Distributions, The Sequential Approach, Elsevier, Warszawa, 1973.

[2] Biagoni, H.A., A Nonlinear Theory of Generalized Functions, Lecture Notes Math. 1421, Springer, Berlin 1990.

[3] Bučkovska, A., Pilipović, S., an extension of bilinear Hilbert transform to distributions, submited in Integr. Trans and Spec. Functions.

[4] Colombeau, J.F., New Generalized Functions and Multiplication of the Distribution, North Holland 1985.

[5] Colombeau, J.F., A multiplication of the distributions, Bull. Am. Math. Soc. 23(1990), 251-268.

[6] Lacey, M.T., Thiele, C.M., L^p estimates on the bilinear Hilbert transform for $2 < p < \infty$, Annals of Math. 146(1997), 693-724.

[7] Lacey, M.T., Thiele, C.M., On Calderon's conjecture for the bilinear Hilbert transform, Proc. Natl. Acad. Sci. USA, 95(1998), 4828-4830.

[8] Lacey, M.T., Thiele, C.M., On Calderon's conjecture, Annals of Math. 149(1999), 473-496.

[9] Oberguggenberger, M., Product of distributions, J. reine angew Math. 365(1986), 1-11.

[10] Oberguggenberger, M., Multiplication of distributions and applications to partial differential equations, Pitman Res. Not. Math. 259, Longman Sci. Techn., Essex 1992.

[11] Pandey, J.N., The Hilbert Transform of Schwartz Distributions and Applications, John Wiley & Sons, inc. New York, 1996.

[12] Pilipović, S., Colombeau's generalized functions and pseudo-differential operators, Lec. in Math. Sci. The University of Tokyo, 1994.

[13] Rudin, W., Real and Complex Analysis, McGraw-Hill, 1987.

[14] Schwartz, L., Théorie des distributions, I.2nd ed.,Hermann, Paris, 1957.