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SOME REMARKS ON INTEGRAL GRAPHS WITH MAXIMUM DEGREE FOUR

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Abstract. An integral graph is a graph whose spectrum (of its adjacency matrix) consists entirely of integers. Here we prove some results on bipartite, nonregular integral graphs with maximum degree four. In particular, trees, unicyclic graphs and graphs with some numbers excluded from their spectrum are considered.

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1. Introduction

For a simple graph $G = (V(G), E(G))$ with $n = |V(G)|$, the *spectrum* of $A(G)$, the adjacency matrix of G, is called the spectrum of G and denoted by $Sp(G)$ (see [3]). A graph is *integral* if its spectrum consists entirely of integers. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ be the eigenvalues of G given in non-increasing order ($\lambda_1(G)$ is usally called the *index* of G). $M_k(G) = \sum_{i=1}^{n} \lambda_i(G)^k$ $(k \geq 0)$ is the *spectral moment* of the k-th order. We use $ecc(v)$ for eccentricity of v $(\in V(G))$, and $rad(G)$ $(diam(G))$ for radius (resp. diameter) of G. In addition, $\delta(G)$ denotes the minimum (vertex) degree of G.

The following result is known as *the interlacing theorem* ([3] p. 19).

Theorem 1.1. *Let A be a real symmetric matrix of order n, and let B be one of its principal submatrices of order m. Then $\lambda_{n-m+i}(A) \leq \lambda_i(B) \leq \lambda_i(A)$, where $i = 1, \ldots, m$.*

Theorem 1.2. *Let G be a connected graph. Then the following holds:*

1^o $\lambda_1(G) > \lambda_2(G)$ *(cf. Theorem 0.3 [3] p. 18);*

2^o $\lambda_1(G) > \lambda_1(H)$*, for any H, the (proper) induced subgraph of G (cf. Theorem 0.6 [3] p. 19).*

The next result is due to J.H. Smith (see [3] p. 78); it completely characterizes the graphs whose index does not exceed 2.

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Theorem 1.3. *$\lambda_1(G) \leq 2$ ($\lambda_1(G) < 2$) if and only if each component of G is a subgraph (resp. proper subgraph) of one of the graphs shown in Fig. 1, each having the index equal to 2.*

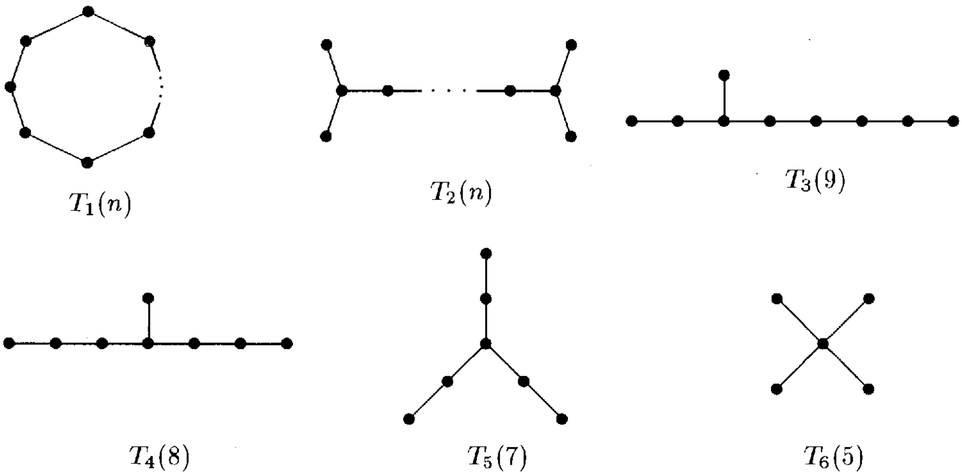


Fig. 1. Smith's graphs

The next result can be found in [3] (see also [2]).

Theorem 1.4. *If G is a bipartite graph, then $M_2(G) = 2m$ and $M_4(G) = 2m + 4f + 8q$, where m, f, and q is the number of subgraphs isomorphic to K_2, $K_{1,2}$ and C_4, respectively.*

Let $\mathcal{S}$ be the set of all bipartite, nonregular integral graphs with maximum degree four. Some basic properties of these graphs are summarized below (cf. [2]).

1) If $G \in \mathcal{S}$, then $Sp(G) \subseteq \{-3, -2, -1, 0, 1, 2, 3\}$. In addition, if $3 \notin Sp(G)$, then $G = K_{1,4}$.

2) $diam(G) \leq |Sp(G)| - 1 \leq 6$.

3) If $G \in \mathcal{S}$, and if H is a proper (induced) subgraph of G, then $\lambda_1(H) < 3$.

4) If $G \in \mathcal{S}$, and if H is a proper (induced) subgraph of G, then $\lambda_2(H) \leq 2$.

Besides, graphs from $\mathcal{S}$ up to 16 vertices (just 20) are all generated. Here we study some subsets of $\mathcal{S}$: trees and unicyclic graphs (Section 2), and graphs without ± 2 and ± 1 in spectrum (Section 3).

2. Trees and unicyclic graphs

Theorem 2.1. *Except for $K_{1,4}$, there are no other trees in $\mathcal{S}$.*

Proof. For any tree T the following result of C. Jordan (see [1]) holds

$$rad(T) = \lfloor \frac{diam(T)+1}{2} \rfloor .$$

Since $T \in \mathcal{S}$, then $diam(T) \leq 6$ as already noted, and thus $rad(T) \leq 3$. So any tree $T \in \mathcal{S}$ is imbeddable in the symmetric tree $T^2_{4,3}$ (cf. [3], p. 130). [3] The index of $T^2_{4,3}$ is 3 (by calculations) and it is not integral. So $\lambda_1(T) < 3$, and thus $\lambda_1(T) = 2$, and consequently $T = K_{1,4}$ (cf. Theorem 1.3). □

Theorem 2.2. *There are no unicyclic graphs in $\mathcal{S}$.*

Proof. Suppose to the contrary, that G is unicyclic and belongs to $\mathcal{S}$. Then G consists of a (unique) even cycle C and some trees appended to its vertices. Notice that G cannot be a cycle.

Observe first that g, the length of C, is less than or equal to 10. Namely, if $g > 12$ then $diam(G) > 6$; for $g = 12$, it follows that $G = C_{12}$, again a contradiction.

Let $l = \max_u d(u, C)$ (here $d(u, C)$ is the distance between a vertex u and the cycle C). Then, clearly, $\frac{1}{2}g + l \leq 6$. Notice that all vertices at a distance l from C are of degree one.

Case (i): $g = 4$ ($l \leq 4$).

Assume first that $l = 4$. If all vertices of degree four are at distance at most two from C, then G is an induced subgraph of the graph shown in Fig. 2 (follows from the bound on diameter).

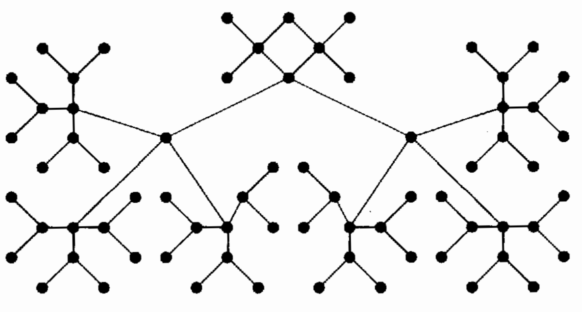


Fig. 2. A unicyclic graph with $g = 4$ and $l = 4$ (no vertex of degree 4 is at distance 3 from the cycle)

[3] In general, $T^q_{r,m}$ is a tree with q central vertices, all internal vertices of degree r, and leaves at distance m or $m+1$ to central vertices.

The index of the above graph is 2.9962, and thus $\lambda_1(G) < 3$, a contradition.

Assume now that there exists (in G) a vertex of degree four at distance three from C. Then, by Theorem 1.4 (part 1^o, with u as a cut-vertex), all vertices of C but one are of degree two. Now depending on the number of vertices of degree four and at the distance three from C we get three possible graphs (shown in Fig. 3) in which G can be imbedded.

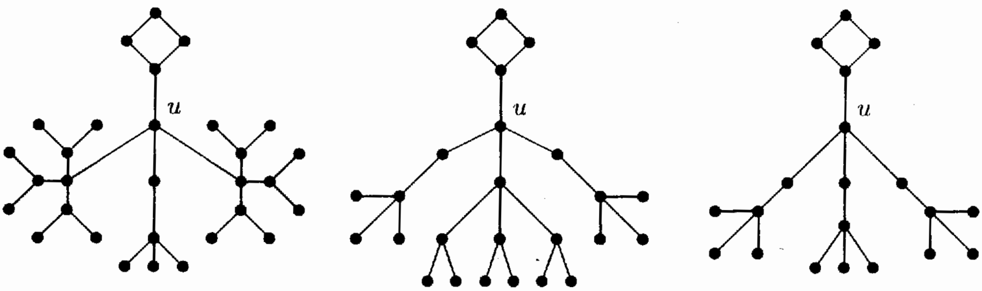


Fig. 3. Unicyclic graphs with $g = 4$ and $l = 4$ (a vertex of degree four is at distance 3 from the cycle)

The indices of these graphs are 2.7189, 2.6316 and 2.4972, respectively. Consequently, $\lambda_1(G) < 3$, a contradiction as above.

If $l = 3$, then G can be imbedded in the graph shown in Fig. 4(a) (follows from the diameter condition).

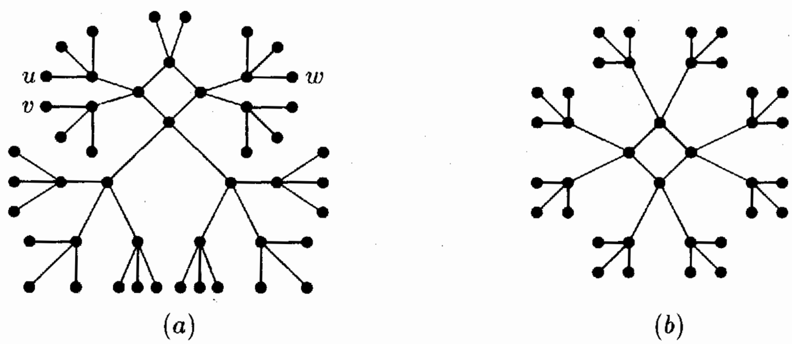


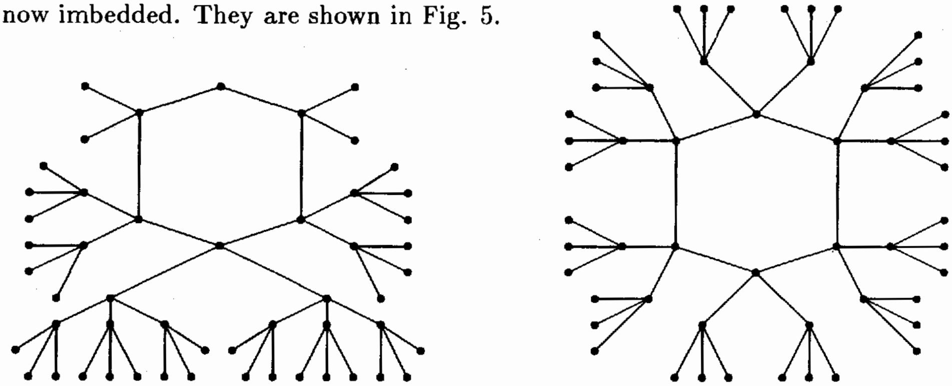
Fig. 4. Unicyclic graphs with $g = 4$ and (a) $l = 3$, (b) $l = 2$

The index of this graph exceeds three, and morover, its second largest eigenvalue exceeds two because some vertices are superfluos. The second largest eigenvalue can be reduced by deleting some vertices. By Theorems 1.1 and 1.3, it is enough to delete just two vertices out of u, v and w. This results in graphs with indices less than three. Namely, if u and v (u and w) are deleted, then, the corresponding index is 2.9971 (resp. 2.9965). So the same contradiction as above appears.

It remains to take that $l \leq 3$. Then G can be imbedded in the graph shown in Fig. 4(b). The index of this graph is three and it is not integral, a contradition.

Case (ii): $g = 6$ ($l \leq 3$)

By the same reasoning as above, there are two graphs in which G can be now imbedded. They are shown in Fig. 5.



(a) (b)

Fig. 5. Unicyclic graphs with $g = 6$ and (a) $l = 3$, (b) $l = 2$

The index of the first graph is 2.9557, and thus $\lambda_1(G) < 3$, a contradiction. The index of the second graph is three and it is not integral, a contradiction.

Case (iii): $g = 8$ ($l \leq 2$)

As in the previous two cases, we get either a graph shown in Fig. 6, or a graph $C_8 \circ 2K_1$ as graphs in which G can be imbedded ($\circ$ stands for corona of two graphs, see [5], p. 167; here, the new graph is obtained from a cycle by adding two pendant edges to each vertex of the cycle).

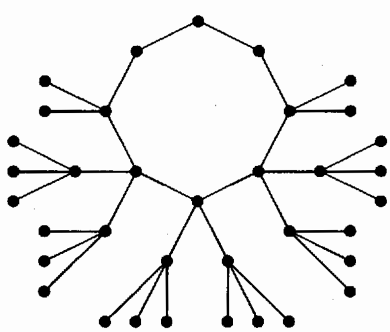


Fig. 6. A unicyclic graph with $g = 8$ and $l = 2$

The indices of both graphs in question are less than three (the index of the graph of Fig. 6 is $2.8523 < 3$; the index of $C_8 \circ 2K_1$ is $2.7321 < 3$). So, this yields contradictions.

Case (iv): $g = 10$ $(l \leq 1)$

Now G is an induced subgraph of $C_{10} \circ 2K_1$, and since its index is $2.7321 < 3$, this yields a contradiction. □

3. Graphs without some numbers in spectrum

Let us consider the graphs from $\mathcal{S}$ without either ± 2 or ± 1 in the spectrum. These graphs have at most five distinct eigenvalues, and thus their diameter is at most four.

Theorem 3.1. *If ± 2 is not an eigenvalue of $G \in \mathcal{S}$, then G is isomorphic to S_2 or S_3 (see Fig. 7).*

Proof. The spectrum of G is $-3, -1^b, 0^c, 1^b, 3$ (here b and c stand for multiplicities of ± 1 and 0, respectively). Clearly, $n = 2 + 2b + c$, and $m = b + 9$ (by Theorem 1.4). Assuming that k $(= m - n + 1)$ is the number of independent cycles of G, it yields that $k = 8 - b - c$. By Theorems 2.1 and 2.2, since G is not a tree or a unicyclic graph, $k \geq 2$, and therefore $b + c \leq 6$. So $n \leq 14 - c$ (≤ 14). Since all integral graphs from $\mathcal{S}$ up to 16 vertices are known (as noted in Section 1), the prove follows (for more details see Fig. 4 from [2] – the names of graphs shown in Fig. 7 are the same as in [2]). □

In what follows assume that ± 1 is not an eigenvalue of graphs in question. In contrast to previous situation, it is not easy now to find all relevant graphs. So far, based on computer search, only the graphs S_1, S_9 and S_{17} (see Fig. 7) are known. Notice that S_9 and S_{17} have no vertices of degree one. In fact, the following holds in general.

Theorem 3.2. *If ± 1 is not an eigenvalue of $G \in \mathcal{S}$ and if $G \neq S_1$, then $\delta(G) = 2$.*

Proof. If $\delta(G) \geq 3$, then the index of G exceeds three, a contradiction. So assume that $\delta(G) = 1$, and let r be a vertex of degree one. Assume that u, its neighbour, is of degree d.

Now the spectrum of G is $-3, -2^a, 0^c, 2^a, 3$ (here a and c stand for multiplicities of ± 2 and 0, respectively). By the interlacing theorem, it follows that the spectrum of $G - r$ is of the form $-x, -2^{a'}, -y, 0^{c'}, y, 2^{a'}, x$ provided $\pm x$ and $\pm y$ do exist (the exponents have the same meaning as above). Moreover, if $\pm x$ or $\pm y$ exists, then $x \in (2, 3)$ and $y \in (0, 2)$. Notice first that $\pm x$ indeed exists; otherwise $G - r$ is a Smith graph (since it is a connected graph whose index equal to two – recall r is of degree one), and so G is a tree or a unicyclic graph, a contradition (by Theorems 2.1 and 2.2). On the other hand, the existence of $\pm y$

is not so important in what follows – if it does not exist, it can be considered to be equal to zero. By the interlacing theorem, $|a - a'| \leq 1$. Moreover, $a' = a - 1$ since otherwise, $ecc(r) \leq 2$ (see [4]) and then G becomes too small – a trivial situation.

Consider now the moments (of orders 2 and 4) for graphs G and $G - r$ (as in the proof of Proposition 2.4 in [2]). By calculating ΔM_2 $(= M_2(G) - M_2(G-r))$ and ΔM_4 $(= M_4(G) - M_4(G-r))$ in two ways (by definition and by Theorem 1.4) we get $x^2 = 6 + \sqrt{13 - d}$, and since $d \leq 4$, $x^2 \geq 9$, a contradition (cf. Theorem 1.2). □

More on these graphs (and on our main problem of constructing all graphs from $\mathcal{S}$) will be given in our fortcomming papers.

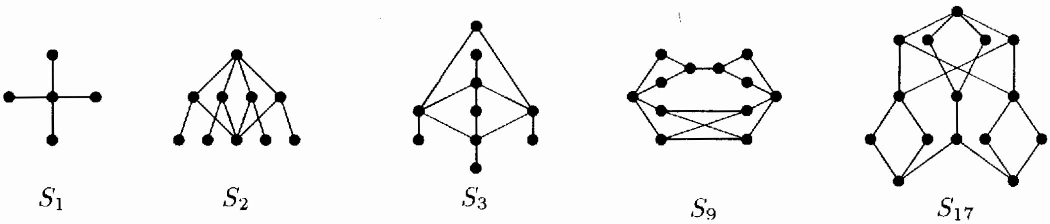


Fig. 7. Integral graphs without ±2 and ±1 in the spectrum

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