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ON CONVERGENCE OF BÖRSCH-SUPAN'S METHOD WITH WEIERSTRASS' CORRECTIONS

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Abstract. We consider Börsch-Supan's method with Weierstrass' corrections for determining polynomial zeros. Special attention is devoted to the determination of initial approximations which provide a safe convergence of the considered method. Using correction approach initial conditions were improved.

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1. Introduction

Let us consider the monic polynomial

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

where $n \geq 3$ and $a_j \in \mathbb{C}$, for $j = 0, \ldots, n-1$. Let the polynomial $P(z)$ have the mutually different zeros $\zeta_1, \zeta_2, ..., \zeta_n$, then

$$P(z) = \prod_{j=1}^{n} (z - \zeta_j).$$

Börsch-Supan's method with Weierstrass' corrections is one of the simultaneous methods for determining polynomial zeros. Before we proceed to the mentioned method, let us introduce the definition of general simultaneous method.

1.1 Simultaneous methods

Simultaneous method for finding polynomial zeros is the one in which all zeros are determined at the same time. Here, we will consider simultaneous methods of the form

(1) $$z^{(m+1)} = z^{(m)} - C\left(z^{(m)}\right), \quad m = 0, 1, \ldots,$$

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where $z^{(m)} = \left[z_1^{(m)}, \dots, z_n^{(m)}\right]^{\mathsf{T}}$, $m = 0, 1, \dots$. Mapping $C(z)$, called correction of the method, for $z = [z_1, \dots, z_n]^{\mathsf{T}}$ is defined with

$$C(z) = (C_1(z), \dots, C_n(z)), \quad C_i(z) = C_i(z_1, \dots, z_n), \quad i \in I_n$$

where $I_n = \{1, 2, \dots, n\}$. Method (1), viewed by components, has the form

$$z_i^{(m+1)} = z_i^{(m)} - C_i\left(z_1^{(m)}, \dots, z_n^{(m)}\right), \quad i \in I_n, \quad m = 0, 1, \dots \tag{2}$$

Under certain conditions (see [4], [2]) the approximations $z_i^{(m+1)}$ obtained by (2) will converge to the zero ζ_i of the polynomial $P(z)$, i.e..

$$\lim_{m \to \infty} z_i^{(m)} = \zeta_i, \quad i \in I_n.$$

Let $d^{(m)}$ denote minimal distance between the approximations of polynomial zeros in the m-th iteration

$$d^{(m)} = \min_{\substack{1 \le i, j \le n \\ i \ne j}} \left| z_i^{(m)} - z_j^{(m)} \right|, \quad m = 0, 1, \dots.$$

1.2 Weierstrass' corrections

Weierstrass' corrections for $z_j \in \mathbb{C}$, $j \in I_n$ and $i \in I_n$ are defined by

$$W_i(z_i) = \frac{P(z_i)}{\prod_{j \ne i}(z_i - z_j)}.$$

Sometimes, instead of $W_i(z_i)$ we write W_i, and in the m-th iteration we have

$$W_i^{(m)} = \frac{P\left(z_i^{(m)}\right)}{\prod\limits_{j \ne i}\left(z_i^{(m)} - z_j^{(m)}\right)},$$

where $z_j \in \mathbb{C}$ for $j \in I_n$, $i \in I_n$ and $m = 0, 1, \dots$. Maximal module of Weierstrass' corrections in the m-th iteration is

$$w^{(m)} = \max_{1 \le i \le n} \left| W_i^{(m)} \right|, \quad m = 0, 1, \dots.$$

1.3 Börsch-Supan's method

Börsch-Supan's (BS) method a is simultaneous method of the form (1). The order of convergence of this method is 3. For $\zeta_i \notin \{z_1, \ldots, z_n\}$, $i \in I_n$, let us consider representation of the polynomial $P(z)$ using Langrange's interpolation polynomial at the points $z_1, z_2, \ldots, z_n$ with the corresponding error

$$P(z) = \left(\sum_{j=1}^{n} \frac{W_j(z_j)}{z - z_j} + 1 \right) \prod_{j=1}^{n} (z - z_j).$$

From the previous formula it is easy to obtain a fixed point relation of Börsch-Supan's type

$$\zeta_i = z_i - \frac{W_i(z_i)}{1 + \sum\limits_{j \neq i} \frac{W_j(z_j)}{\zeta_i - z_j}}, \quad i \in I_n, \tag{3}$$

from which we get the BS method

$$z_i^{(m+1)} = z_i^{(m)} - \frac{W_i\left(z_i^{(m)}\right)}{1 + \sum\limits_{j \neq i} \frac{W_j\left(z_j^{(m)}\right)}{z_i^{(m)} - z_j^{(m)}}}, \quad i \in I_n, \quad m = 0, 1, \ldots \tag{4}$$

For more about this method see [1], [3], [5].

1.4 Börsch–Supan's method with Weierstrass' corrections

Börsch–Supan's method with Weierstrass' corrections (BSW method) was introduced as an accelerated BS method. It has order of convergence 4 and can be obtained from (3) when ζ_i is substituted with $z_i - W_i(z_i)$. In this way we get the BSW method

$$z_i^{(m+1)} = z_i^{(m)} - \frac{W_i(z_i^{(m)})}{1 + \sum\limits_{j \neq i} \frac{W_j(z_j^{(m)})}{z_i^{(m)} - W_i(z_i^{(m)}) - z_j^{(m)}}}, \quad i \in I_n \quad m = 0, 1, \ldots. \tag{5}$$

It has been proved that the BSW method is a secant method for Weierstrass' function $W_i(z)$.

2. Correction approach

2.1 Convergence of simultaneous iterative method

Let us introduce the basic assumptions (BA) for the correction $C_i(z_1, \ldots, z_n)$, $i \in I_n$, of the method (2) by

1. $C_i(z_1, \ldots, z_n) = \frac{P(z_i)}{F_i(z_1, \ldots, z_n)}, \quad i \in I_n,$

2. $F_i(\zeta_1,\ldots,\zeta_n) \neq 0, \quad i \in I_n,$

3. $F_i(z_1,\ldots,z_n), \quad i \in I_n$, is continuous on $\mathbb{C}^K$

If we can prove that the limits

$$\zeta_i = \lim_{m\to\infty} z_i^{(m)}, \quad i \in I_n,$$

exist, and that they are mutually different, then ζ_i, $i \in I_n$, are simple zeros of the polynomial P. That is because for every $i \in I_n$

$$\begin{aligned} 0 &= \lim_{m\to\infty}\left(z_i^{(m)} - z_i^{(m+1)}\right) = \lim_{m\to\infty} C_i\left(z_1^{(m)},\ldots,z_n^{(m)}\right) \\ &= \lim_{m\to\infty}\frac{P\left(z_i^{(m)}\right)}{F_i\left(z_1^{(m)},\ldots,z_n^{(m)}\right)} = \frac{P(\zeta_i)}{F_i(\zeta_1,\ldots,\zeta_n)} \end{aligned}$$

holds, which implies that $P(\zeta_i) = 0$.

The following lemma was proved in [2]

Lemma 1. *Let*

$$s_m(t) = \sum_{i=0}^{m} t^i + t^m, \quad t \in (0,1),\ m = 1,2,\ldots.$$

Then $s_m(t) \le g(t)$, *where*

$$g(t) = \begin{cases} 1+2t, & 0 < t \le \frac{1}{2}, \\ \frac{1}{1-t}, & \frac{1}{2} < t < 1. \end{cases}$$

In [4], the following theorem was proved.

Theorem 1. *Suppose that the BAs are satisfied and let the mutually different initial approximations* $z_1^{(0)},\ldots,z_n^{(0)}$ *be given. If for some* $\gamma \in (0,1)$ *iterative method* (1) *satisfies*

(*i*) $\quad \left|C_i^{(m+1)}\right| \le \gamma\left|C_i^{(m)}\right|, \qquad m = 0,1,\ldots,$

(*ii*) $\quad \left|z_i^{(0)} - z_j^{(0)}\right| > g(\gamma)\left(\left|C_i^{(0)}\right| + \left|C_j^{(0)}\right|\right), \quad i \neq j, \quad i,j \in I_n,$

then the method converges.

Using properties of the module function, it is easy to verify the following lemma.

Lemma 2. *For the mutually different values $z_1, ..., z_n$ and the arbitrary values $\hat{z}_1, ..., \hat{z}_n$ let*

$$d = \min_{1 \leq i < j \leq n} |z_i - z_j|$$

and

$$|\hat{z}_i - z_i| \leq \lambda_n d, \; i \in I_n.$$

Then for every $i, j \in I_n$

$$|\hat{z}_i - z_j| \geq (1 - \lambda_n)\, d,$$

$$|\hat{z}_i - \hat{z}_j| \geq (1 - 2\lambda_n)\, d$$

and $\hat{z}_1, ..., \hat{z}_n$ are mutually different values. Moreover, if $\lambda_n < \frac{1}{2}$ then

$$\left| \prod_{j \neq i} \frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j} \right| \leq \left(1 + \frac{\lambda_n}{1 - 2\lambda_n}\right)^{n-1}.$$

2.2 Convergence of the BSW method

The BSW method given by (5) is a special case of the general method (1) where the correction is given by

$$C_i = \frac{W_i}{1 + \sum\limits_{j \neq i} \frac{W_j}{z_i - W_i - z_j}}.$$

Before we proceed to the convergence theorem for the BSW method, it is necessary to give the following result.

Lemma 3. *Let $n \geq 3$ and $m \geq 0$. Let the mutually different approximations $z_1^{(m)}, \ldots, z_n^{(m)}$ be given. Let*

$$c_n \in \left(0, \frac{n + 4 - \sqrt{n\,(n+8)}}{8}\right)$$

satisfies

$$w^{(m)} \leq c_n d^{(m)}$$

and let

$$\delta_n \equiv \frac{\lambda_n c_n^2\,(n-1)^2}{(1 - nc_n)\,(1 - \lambda_n)\,(1 - c_n)} \left(1 + \frac{\lambda_n}{1 - 2\lambda_n}\right)^{n-1} < (1 - 2\lambda_n)\,\frac{c_n}{(2\lambda_n - c_n)},$$

where

$$\lambda_n = \frac{c_n\,(1 - c_n)}{1 - nc_n}.$$

Than

$$w^{(m+1)} \le c_n d^{(m+1)}$$

and

$$\left|W_i^{(m+1)}\right| \le \delta_n \left|W_i^{(m)}\right|.$$

Theorem 2. *Let $n \ge 3$, let the mutually different initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ be given. Let*

$$c_n \in \left(0, \frac{n+4-\sqrt{n(n+8)}}{8}\right) = \left(0, \frac{2}{n+4+\sqrt{n(n+8)}}\right) \subset \left(0, \frac{1}{n}\right)$$

be such that

$$w^{(0)} \le c_n d^{(0)}$$

and let

$$\lambda_n = \frac{c_n(1-c_n)}{1-nc_n}.$$

If

$$\begin{aligned} \delta_n &\equiv \frac{\lambda_n c_n^2 (n-1)^2}{(1-nc_n)(1-\lambda_n)(1-c_n)} \left(1 + \frac{\lambda_n}{1-2\lambda_n}\right)^{n-1} \\ &< (1-2\lambda_n) \frac{c_n}{(2\lambda_n - c_n)}, \end{aligned}$$

then the BSW method converges.

Proof. Using mathematical induction we easily obtain that the BSW method is well defined in each iteration. The previous lemma gives us

$$w^{(m)} \le c_n d^{(m)} \quad \text{and} \quad \left|W_i^{(m+1)}\right| \le \delta_n \left|W_i^{(m)}\right|$$

in each iteration. We will prove that the assumptions of Theorem 1 are satisfied, which will give us convergence of the BSW method. Now, let

$$M_n = 2 - \frac{c_n}{\lambda_n}$$

and

$$\beta_n = \frac{M_n \lambda_n}{c_n} \delta_n.$$

Then

$$\beta_n < \frac{M_n \lambda_n}{c_n} \frac{1-2\lambda_n}{M\lambda_n} c_n = 1 - 2\lambda_n < 1$$

and

$$\frac{c_n}{\lambda_n} \le \frac{\left|W_i^{(m)}\right|}{\left|C_i^{(m)}\right|} \le M_n, \quad m = 0, 1, \ldots.$$

We have

$$\begin{aligned}\left|C_i^{(m+1)}\right| &\le \tfrac{\lambda_n}{c_n}\left|W_i^{(m+1)}\right| \le \tfrac{\lambda_n}{c_n}\delta_n\left|W_i^{(m)}\right| \\ &\le \tfrac{\lambda_n}{c_n}\delta_n\left|C_i^{(m)}\right| M_n = \tfrac{M_n\lambda_n}{c_n}\delta_n\left|C_i^{(m)}\right| = \beta_n\left|C_i^{(m)}\right|,\end{aligned}$$

so the first assumption of Theorem 1 holds. For the second assumption, it can be easily verified that under the conditions of the theorem

$$g(\beta_n) < \frac{1}{2\lambda_n}.$$

Then we have

$$\frac{1}{\lambda_n}\left|C_i^{(0)}\right| \le \frac{1}{c_n}\left|W_i^{(0)}\right| \le \frac{w^{(0)}}{c_n} \le d^{(0)},$$

so we obtain

$$\begin{aligned}\left|z_i^{(0)} - z_j^{(0)}\right| &\ge d^{(0)} \ge \frac{w^{(0)}}{c_n} \ge \frac{1}{2\lambda_n}\left(\left|C_i^{(0)}\right| + \left|C_j^{(0)}\right|\right) \\ &> g(\beta_n)\left(\left|C_i^{(0)}\right| + \left|C_j^{(0)}\right|\right).\end{aligned}$$

Now, as all the conditions are satisfied, we can apply Theorem 1, which implies convergence of the BSW method. □

Theorem 3. *Let for the fixed $n \ge 3$*

$$c_n = \begin{cases} \frac{1}{1.64n+1.944} & 3 \le n \le 23, \\ \frac{1}{1.42n+8.7} & n > 23. \end{cases}$$

If $z_1^{(0)}, \ldots, z_n^{(0)}$ are mutually different approximations that satisfy

$$w^{(0)} \le c_n d^{(0)},$$

then the BSW method converge.

3. Numerical results

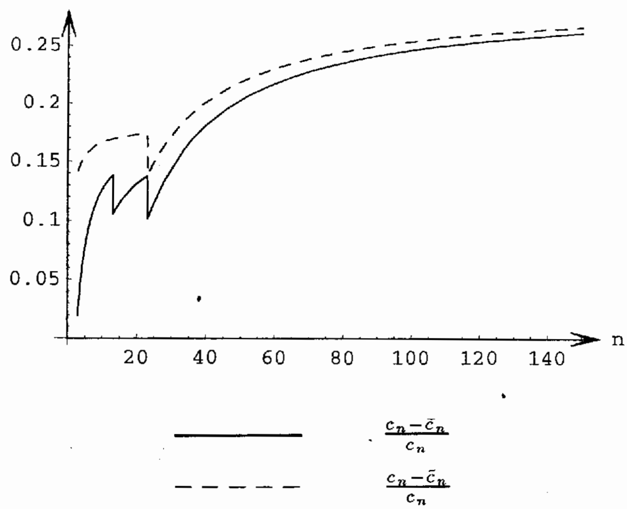
Constants c_n mentioned in the previous section have great influence on the determination of initial approximations. The greater they are, the greater is the choice of such initial approximations that provide safe convergence of the considered method. We compare the constants given in this paper with the ones given in ([2]) and ([4]). These constants have the form

$$c_n = \begin{cases} \frac{1}{1.64n+1.944} & 3 \le n \le 23, \\ \frac{1}{1.42n+8.7} & n > 23, \end{cases}$$

$$\bar{c}_n = \begin{cases} \frac{1}{2n+1} & 3 \le n \le 13, \\ \frac{1}{2n} & n > 13, \end{cases} \qquad ([2])$$

$$\tilde{c}_n = \frac{1}{2n+2} \quad ([4]).$$

The following figure shows the relative difference between the constants given in this paper and the ones previously mentioned for different n.



As we can see, constants c_n from this paper are for larger n about 25% better than the constants obtained in ([2]) and ([4]). This means that the initial conditions for the convergence of the BSW method are here significantly improved, which enables a greater application of this method.

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