Novi Sad J. Math.
Vol. 31, No. 1, 2001, 1-7

SPECIAL MESHES AND HIGHER–ORDER SCHEMES FOR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS

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Abstract. Bakhvalov (B) and Shishkin (S) meshes are used very often to discretize singular perturbation problems. The smoother B meshes are more complicated than the piecewise equidistant S meshes, but their considerably better accuracy usually outweighs this. In this paper, we point out that the real advantage of S meshes comes to light when constructing higher–order discretizations. We show this by considering an almost third–order finite–difference scheme for a semilinear problem with two small parameters.

AMS Mathematics Subject Classification (1991): 65L10

Key words and phrases: singular perturbation, two small parametres, Bakhvalov and Shishkin mesh

1. Introduction

Let us consider the following singularly perturbed boundary value problem:

$$(1)\quad -\varepsilon^2 u'' - \mu u' + c(x,u) = 0, \quad x \in X = [0,1], \quad u(0) = U_0, \quad u(1) = U_1,$$

where

$$(2)\qquad 0 < \varepsilon << 1, \quad \mu = \varepsilon^{1+p}, \quad p \geq \frac{1}{2},$$

c is a sufficiently smooth function and U_0 and U_1 are real numbers. For $x \in X$ and $u \in \mathbb{R}$, we also assume

$$(3)\qquad c_u(x,u) > m^2 > 0, \quad m > 0.$$

This problem is used as a suitable problem to illustrate our point that the only advantage of the Shishkin [13], or S, meshes over the Bakhvalov [2], or B, meshes is that higher–order discretizations are much simpler on S meshes, since too complicated nonequidistant schemes can be avoided. Problem (1) is not artificially constructed for this purpose: it also models transport phenomena arising in chemistry or biology, [3]. It belongs to the class of singularly perturbed boundary value problems with two small parameters, which have been analyzed

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asymptotically in [9] and numerically in [15], [16], and most recently in two dimensions in [5]. On numerical methods for singular perturbation problems in general, one can find in two 1996 books, [8] and [12], and on S and B meshes in particular, in [14], [17], [10], [11], [18], and [6], for instance.

Space limitations prevent us from presenting here some generalizations; they will appear elsewhere. One of them is straightforward, viz. replacing the $-\mu u'$–term in (1) with $-\mu b(x)u'$. Another, the inclusion of the case $0 < p < \frac{1}{2}$, requires some modifications of the numerical method. It is also possible to construct a similar scheme for the case $p = 0$ and to prove its stability, but the proof of ε-uniform convergence is still open.

2. The discretizations

Let X^h denote any mesh with the points $0 = x_0 < x_1 < \cdots < x_N = 1$. Problem (1) requires a mesh which is dense near both $x = 0$ and $x = 1$. This is because the unique solution, u_ε, of (1) has in general two exponential boundary layers at the endpoints of X. Moreover, the following estimates hold for $x \in X$ and $k = 0, 1, 2, \ldots$, see [15] and [16]:

$$|u_\varepsilon^{(k)}| \le M[1 + \varepsilon^{-k} v_0(x) + \varepsilon^{-k} v_1(x)], \tag{4}$$

where $v_t(x) = \exp(-m|x - t|/\varepsilon)$, $t = 0, 1$, and M is used throughout the paper as a generic constant which is independent of both ε and N.

For simplicity, let N be even and let both B and S meshes be symmetric with respect to $x_{N/2} = \frac{1}{2}$. The meshes are described below on $[0, \frac{1}{2}]$. A B mesh introduced in [14] is used in this paper as a comparison to the standard S mesh. It is generated by $x_i = \lambda(i/N)$, where

$$\lambda(t) = \begin{cases} \varphi(t) := \varepsilon \frac{t}{q-t} & \text{if } t \in [0, \alpha], \\ \tau(t) := \varphi'(\alpha)(t - \alpha) + \varphi(\alpha) & \text{if } t \in [\alpha, \frac{1}{2}], \end{cases}$$

with $0 < \alpha < q < \frac{1}{2}$ and α solving the equation $\tau(\alpha) = \frac{1}{2}$.

The S mesh is piecewise equidistant. It is formed by using a fine mesh on the interval $[0, \sigma := a\varepsilon \ln N]$ and a coarse mesh on $[\sigma, \frac{1}{2}]$ (it is assumed that $a > 0$ and $\sigma < \frac{1}{2}$). Let the index J be defined by $x_J = \sigma$ and let $N \le MJ$.

Let $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$, and let $w^h = [w_1, w_2, \ldots, w_{N-1}]^T$ be the vector corresponding to a mesh function on $X^h \setminus \{0, 1\}$. We formally set $w_0 := U_0$ and $w_N := U_1$.

The following nonequidistant central scheme can be used on both meshes:

$$T_C w_i = -\varepsilon^2 D_C'' w_i - \mu D_C' w_i + c_i, \quad i = 1, 2, \ldots, N-1,$$

with

$$D_C'' w_i = \frac{2}{h_i + h_{i+1}} \left(\frac{w_{i-1} - w_i}{h_i} + \frac{w_{i+1} - w_i}{h_{i+1}} \right),$$

$$D'_C w_i = \frac{1}{h_i + h_{i+1}} \left[\frac{h_{i+1}}{h_i}(w_i - w_{i-1}) + \frac{h_i}{h_{i+1}}(w_{i+1} - w_i) \right],$$

and $c_i = c(x_i, w_i)$.

We would also like to use an equidistant four–point third–order scheme, D'' to approximate u''. Let h denote the mesh step and let $s = (3-\sqrt{15})/6 \approx -.145$. Then,

$$D'' w_i = h^{-2}[(1-s)w_{i-1} + (3s-2)w_i + (1-3s)w_{i+1} + sw_{i+2}], \tag{5}$$

is a $O(h^3)$ scheme for $u''(x_i + sh)$. It is interesting to compare this scheme to that in [4], which is also a four–point third–order scheme for u'' and even makes use of the same quantity s. However, the latter, which is optimal in the sense of minimizing the truncation error, uses special nonequidistant points and therefore cannot be applied here. It is too complicated to construct a nonequidistant generalization of (5). Besides, that can be done in several different ways and it is hard to tell in advance which one will produce the most suitable scheme, cf. [17]. Because of all those complications, (5) will be used here only on a portion of the fine parts of the S mesh. It will be combined with two other third–order schemes,

$$D' w_i = (12h)^{-1}[(6s-5)w_{i-1} - 3(2s+1)w_i - 3(2s-3)w_{i+1} + (6s-1)w_{i+2}]$$

and

$$D w_i = \frac{1}{12} w_{i-1} + \left(\frac{5}{6} - s\right) w_i + \left(\frac{1}{12} + s\right) w_{i+1},$$

to give the following discretization scheme:

$$T w_i = -\varepsilon^2 D'' w_i - \mu D' w_i + c(x_i + sh, D w_i).$$

Then, T and T_C are used to form a hybrid scheme T_H,

$$T_H w_i = \begin{cases} T w_i \text{ for } 1 \le i \le J-2, \\ T_C w_i \text{ for } J-1 \le i \le N/2, \\ \text{symmetrical scheme w.r.t. } x_{N/2} = \frac{1}{2} \text{ for } N/2+1 \le i \le N-1. \end{cases}$$

Thus, we are going to consider two discretizations of problem (1), both of the form

$$R w_i = 0, \quad i = 1, 2, \ldots, N-1, \tag{6}$$

where either $R \equiv T_C$ or $R \equiv T_H$. This is an $(N-1) \times (N-1)$ nonlinear system. Our numerical results will show that the special meshes stabilize T_C, which is not surprising having the result in [1] in mind. On the B mesh, we can expect second–order ε–uniform accuracy, whereas on the S mesh, the second order is diminished by logarithmic factors. As for T_H, it is analyzed in the next section.

3. The error estimate for T_H

The key assumption in the following analysis of the scheme T_H is

$$\varepsilon \le MN^{-1}(\ln N)^{3/2}. \tag{7}$$

Even though this is certainly a theoretical restriction, it is practically quite acceptable, since the relationship between ε and N is usually such that no mesh point lies inside the layer when the mesh is equidistant. This can be expressed by the inequality

$$\varepsilon \ln \frac{1}{\varepsilon} \le MN^{-1},$$

which implies (7).

Let F be an $(N-1)\times(N-1)$ matrix denoting the Fréchet derivative of the operator T_H on the S mesh, $F = T'_H(w^h)$ for an arbitrary vector w^h. Let also $\|w^h\| = \max_{1\le i\le N-1}|w_i|$ and let the corresponding subordinate matrix norm be denoted in the same way. Moreover, let N_0 denote a sufficiently large positive integer independent of ε. Then we can prove the following stability result which is crucial for our main result.

Theorem 1. *Let (2), (3), and (7) hold and let $N \ge N_0$. Then F is a nonsingular matrix and $\|F^{-1}\| \le M$. Thus, the discrete problem (6) with $R \equiv T_H$ on the S mesh has a unique solution.*

Proof. This is a nonstandard stability proof, since $F = [f_{ij}]$ is not an L–matrix, nor can we fully apply to F Lorenz's standard decomposition (SD), [7]. We consider several cases.

1. $p \ge 1$. In this case, the nonzero elements of F are $f_{ii} > 0$, $f_{i,i\pm1} \le 0$, $i = 1,2,\dots,N-1$, (setting formally $f_{10} = f_{N-1,N} = 0$), and because of the four–point scheme D'', $f_{i,i+2} > 0$, $i = 1,2,\dots,J-2$, and symmetrically $f_{i,i-2} > 0$, $i = N-J+2,\dots,N-1$. By looking at the coefficients of the schemes D'' and D''_C which dominate the elements of F, we can prove that

$$4f_{i,i+2}f_{i+1,i+1} \le f_{i,i+1}f_{i+1,i+2}, \quad i = 1,2,\dots,J-2.$$

The last $J-2$ rows of F satisfy an analogous inequality. This is equivalent to Lorenz's SD and implies that F is an inverse–monotone matrix. Then $\|F^{-1}\| \le m^{-2}$ follows easily.

2. $\frac{1}{2} \le p < 1$. We cannot prove now that F is inverse monotone, since the signs of the F–elements resulting from T_C are not fixed any longer. Instead, we decompose F appropriately, $F = A + B$. The scheme $-\mu D'_C w_i$, $i = K,\dots,N-K$, is separated from the rest of the discretization to form B and $A = F - B$. Here K is either J or $J+1$.

2.a $\varepsilon^{p-1} \le MN$. In this case we choose $K = J+1$, since $f_{JJ} > 0$ and $f_{J,J\pm1} \le 0$.

It holds that $||B|| \le M\mu N$.

2.b $\varepsilon^{1-p} \le M/N$. Now $K = J$ and

$$||B|| \le M\varepsilon^p \frac{N}{\ln N} \le M \frac{1}{N^{p/(1-p)}} \cdot \frac{N}{\ln N} \le M \frac{1}{\ln N}.$$

The last inequality holds because of $p/(1-p) \ge 1$.

In both subcases 2.a and 2.b, A is inverse monotone by SD and satisfies $||A^{-1}|| \le m^{-2}$. Also, note that because of (7) and $N \ge N_0$, $||B||$ can be made sufficiently small so that $||A^{-1}||||B|| \le M < 1$. Then we use

$$||F^{-1}|| = ||(I + A^{-1}B)^{-1}A^{-1}|| \le \frac{||A^{-1}||}{1 - ||A^{-1}B||} \le M$$

to conclude the proof. □

Let $u_\varepsilon^h = [u_\varepsilon(x_1), u_\varepsilon(x_2), \ldots, u_\varepsilon(x_{N-1})]^T$. In the next theorem, we prove an ε-uniform error estimate for T_H.

Theorem 2. *Let (2), (3), and (7) hold and let $N \ge N_0$. Then,*

$$||w^h - u_\varepsilon^h|| \le M \left(\frac{\ln N}{N}\right)^3,$$

where w^h is the solution of the system (6) with $R \equiv T_H$ on the S mesh with $am \ge 4$.

Proof. Using (4) and a fairly standard technique on the S mesh (see the relevant references mentioned in the introduction), we can prove the consistency error estimate

$$||T_H u_\varepsilon^h|| \le M \left[\frac{\varepsilon^2}{N} + \left(\frac{\ln N}{N}\right)^3\right] \le M \left(\frac{\ln N}{N}\right)^3,$$

where the last inequality follows from (7). Then Theorem 1 completes the proof. Note that the above term ε^2/N results from $T_C u_\varepsilon^h(x_J)$. □

4. Numerical results

Let us consider the following enzyme kinetics problem from [3],

$$Pu := -\varepsilon^2 u'' - \varepsilon^{3/2} u' + \frac{u}{1+u} = 0, \quad u(0) = u(1) = 1. \tag{8}$$

This problem satisfies (2) with $p = \frac{1}{2}$ and (3) holds only locally. Since the constant functions 1 and 0 are respectively the upper and lower solutions of (8), only the values $u \in [0,1]$ are of interest and then $c_u \geq \frac{1}{4}$, so that any $m \in (0, \frac{1}{2})$ can be used. The exact solution of problem (8) is not known but it behaves like

$$y_\varepsilon(x) = e^{-x/\varepsilon\sqrt{2}} + e^{(x-1)/\varepsilon\sqrt{2}}.$$

In order to run our numerical experiments more easily, we changed the differential equation in (8) to $Pu = f(x)$, where $f(x) = Py_\varepsilon(x)$. This means that we can take y_ε for the solution of the modified problem, since y_ε practically satisfies the boundary conditions.

In addition to $p = \frac{1}{2}$, we have tested other values of p and the results are similar, even for the theoretically unsafe values of $p \in (0, \frac{1}{2})$.

The table below shows a comparison between T_H and T_C. T_H is used on the S mesh with $J = .45N$ and $a = 8.2$, so that $am \geq 4$. The B mesh uses $q = .45$. Err stands for the maximum pointwise error and Ord is the numerically calculated order of convergence. All the methods represented in the table are uniform in ε, since the results are the same for $\varepsilon = 10^{-k}$, $k = 6, 8, 10, 12$. It may be disappointing that Ord for T_H is considerably less than 3, but the results are still significantly better than those obtained by T_C, even on the B mesh. We can conclude from the results for T_C on the S mesh that the reason for the lower Ord is not the scheme but the S mesh itself. Nevertheless, it is obvious that the use of S mesh pays off when it is combined with a higher–order scheme.

Err and Ord for T_H and T_C

N	T_H on S mesh		T_C on B mesh		T_C on S mesh	
	Err	Ord	Err	Ord	Err	Ord
100	2.8E–5	—	5.2E–5	—	8.9E–4	—
200	6.3E–6	2.2	1.2E–5	2.1	3.1E–4	1.5
400	1.3E–6	2.3	3.0E–6	2.0	1.0E–4	1.6
800	2.4E–7	2.4	7.5E–7	2.0	3.3E–5	1.6
1600	4.2E–8	2.5	1.9E–7	2.0	1.0E–5	1.7

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