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THE DUAL SPACES OF THE SETS OF Λ–STRONGLY CONVERGENT AND BOUNDED SEQUENCES

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Abstract

We shall give the first and second $\alpha-$, $\beta-$, $\gamma-$ and $f-$duals of the sets $c_0(\Lambda)$, $c(\Lambda)$, and $c_\infty(\Lambda)$ of sequences that are Λ–strongly convergent to naught, convergent and bounded. Furthermore, we shall determine the first and second continuous dual spaces of the sets $c_0(\Lambda)$ and $c(\Lambda)$.

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1. Introduction and well–known results

We shall write ω for the set of all complex sequences $x = (x_k)_{k=0}^\infty$, ϕ, l_∞, c and c_0 for the sets of all finite, bounded, convergent sequences and sequences convergent to naught, respectively, further cs, bs and l_1 for the sets of all convergent, bounded and absolutely convergent series.

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By e and $e^{(n)}$ $(n \in \mathbb{N}_0)$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, \cdots$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. For any sequence $x = (x_k)_{k=0}^{\infty}$, let $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$ be its n–section.

Let $X, Y \subset \omega$ and $z \in \omega$. Then we shall write

$$z^{-1} * X = \{x \in \omega : xz = (x_k z_k)_{k=0}^{\infty} \in X\}$$

and

$$M(X,Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in \omega : ax \in Y \text{ for all } x \in X\}$$

for the *multiplier space of X and Y*. The sets $M(X, l_1)$, $M(X, cs)$ and $M(X, bs)$ are called the *α–, β– and γ–duals of X*.

A Banach subspace X of ω is called an *BK space* if it has continuous coordinates, that is if convergence in X implies coordinatewise convergence. A BK space $X \supset \phi$ is said to have AK if, for every sequence $x = (x_k)_{k=0}^{\infty} \in X$, $x^{[n]} \to x$ $(n \to \infty)$; it is said to have AD if ϕ is dense in X.

If X is a normed space then we shall write X^* for the set of all continuous linear functionals on X, the so–called *continuous dual of X*, with its norm $\|\cdot\|$ given by

$$\|f\| = \sup\{|f(x)| : \|x\| = 1\} \text{ for all } f \in X^*.$$

Let $X \supset \phi$ be a BK space. Then the set $X^f = \{(f(e^{(n)}))_{n=0}^{\infty} : f \in X^*\}$ is called the *f–dual of X*.

The sets $c_0(\Lambda)$, $c(\Lambda)$, and $c_\infty(\Lambda)$ of sequences that are Λ–strongly convergent to naught, Λ–strongly convergent and Λ–strongly bounded were introduced and studied by Móricz [7]. Their first β– and continuous duals were determined in [5] and [6]. In this paper, we shall give more elementary proofs for the results in [5, 6]. Furthermore, we shall determine the second β– and continuous duals of these spaces as well as their first and second α–, γ–, and f-duals.

2. Some notations and preliminary results

Given any infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$ of complex numbers and any sequence $x \in \omega$, we shall write $A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k$ $(n = 0, 1, \cdots)$, $A(x) = (A_n(x))_{n=0}^{\infty}$, provided the series converge, and $X_A = \{x \in \omega : A(x) \in X\}$.

We define the matrix Δ by $\Delta_{nk} = 1$ for $k = n$, $\Delta_{nk} = -1$ for $k = n-1$ and $\Delta_{nk} = 0$ otherwise $(n = 0, 1, \cdots)$, and use the convention that any symbol with a negative subscript has the value 0.

Let $\mu = (\mu_n)_{n=0}^{\infty}$ be a nondecreasing sequence of positive reals tending to infinity throughout. We shall consider the sets

$$w_0(\mu) = \left\{x \in \omega : \lim_{n\to\infty}\left(\frac{1}{\mu_n}\sum_{k=0}^{n}|x_k|\right) = 0\right\} \quad c_0(\mu) = \mu^{-1} * (w_0(\mu))_\Delta$$

$$w_\infty(\mu) = \left\{x \in \omega : \sup_n\left(\frac{1}{\mu_n}|x_k|\right) < \infty\right\} \quad c_\infty(\mu) = \mu^{-1} * (w_\infty(\mu))_\Delta$$

$$c(\mu) = \{x \in \omega : x - le \in c_0(\mu) \text{ for some } l \in \mathbf{C}\}.$$

If $\mu_n = \frac{1}{n+1}$ for $n = 0, 1, \cdots$ then the sets $c_0(\mu)$, $c(\mu)$ and $c_\infty(\mu)$ reduce to the sets $[c_0]_1$, $[c]_1$ and $[c_\infty]_1$ introduced and studied by Hyslop, Kuttner and Thorpe [1, 2]. Following the notations introduced in [5], we say that a nondecreasing sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ of positive reals tending to infinity is *exponentially bounded* if there are reals s and t with $0 < s \le t < 1$ such that for some subsequence $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ of Λ, we have

$$\text{(2.1)} \qquad s \le \frac{\lambda_{n(\nu)}}{\lambda_{n(\nu+1)}} \le t \text{ for all } \nu = 0, 1, \cdots;$$

such a subsequence $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ will be called an *associated subsequence.* If $(n(\nu))_{\nu=0}^{\infty}$ is a strictly increasing sequence of non-negative integers then we shall write $K^{<\nu>}$ for the set of all integers k with $n(\nu) \le k \le n(\nu+1) - 1$, and $\sum_\nu$ and $\max_\nu$ for the sum and maximum taken over all k in $K^{<\nu>}$.

If X is a normed sequence space and $a \in \omega$, then we shall write

$$\|a\|_X^* = \sup\left\{\left|\sum_{k=0}^{\infty} a_k x_k\right| : \|x\| = 1\right\}$$

provided the term on the right exists and is finite. This is the case whenever $X \supset \phi$ is a BK space and $a \in X^\beta$ by [8, Theorem 7.2.9, p. 107].

Let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be a nondecreasing exponentially bounded sequence of positive reals and $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ an associated subsequence throughout.

If $X(\Lambda)$ denotes any of the sets $w_0(\Lambda)$, $w_\infty(\Lambda)$, $c_0(\Lambda)$, $c(\Lambda)$ or $c_\infty(\Lambda)$ then we shall write $\tilde{X}(\Lambda)$ for the respective space with the sections $1/\lambda_n \sum_{k=0}^{n} \cdots$

replaced by the blocks $1/\lambda_{n(\nu+1)}\sum_\nu \cdots$. Further, we define

$$2\|x\|_{w_\infty(\Lambda)} = \sup_n \left(\frac{1}{\lambda_n}\sum_{k=0}^{n}|x_k|\right) \qquad \|x\|_{\tilde{w}_\infty(\Lambda)} = \sup_\nu \left(\frac{1}{\lambda_{n(\nu+1)}}\sum\nolimits_\nu |x_k|\right)$$

$$\|x\|_{c_\infty(\Lambda)} = \|\Delta(\mu x)\|_{w_\infty(\Lambda)} \qquad \|x\|_{\tilde{c}_\infty(\Lambda)} = \|\Delta(\mu x)\|_{\tilde{w}_\infty(\mu)}.$$

Then we have (cf. [5, Theorem 2]) $X(\Lambda) = \tilde{X}(\Lambda)$ in each case, the norms $\|\cdot\|_{X(\Lambda)}$ and $\|\cdot\|_{\tilde{X}(\Lambda)}$ are equivalent on $X(\Lambda)$, each space $X(\Lambda)$ is a BK space, $c_0(\Lambda)$ and $c(\Lambda)$ are closed subspaces of $c_\infty(\Lambda)$, $c_0(\Lambda)$ has AK and every sequence $x = (x_k)_{k=0}^\infty \in c(\Lambda)$ has a unique representation

$$(2.2) \quad x = le + \sum_{k=0}^{\infty}(x_k - l)e^{(k)} \text{ where } l \in \mathbf{C} \text{ is such that } x - le \in c_0(\Lambda).$$

3. The duals of the sets $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$

We need the following lemma for the determination of the $\alpha-$ duals of $c_0(\mu)$, $c(\mu)$ and $c_\infty(\mu)$.

Lemma 1. *Let $X \subset l_\infty$ be a BK space such that $\sup_n \|e^{[n]}\| < \infty$. Then $X^\alpha = l_1$.*

Proof. First we observe that $X \subset l_\infty$ implies $l_\infty^\alpha = l_1 \subset X^\alpha$.
Conversely, let $a \in X^\alpha$. For each $m \in \mathbb{N}_0$, we define the map $f_a^{(m)} : X \to \mathbb{R}$ by $f_a^{(m)}(x) = \sum_{k=0}^m |a_k x_k|$ $(x \in X)$. Then $(f_a^{(m)})_{m=0}^\infty$ is a sequence of seminorms on X which are continuous, since X is a BK space. Further $f_a^{(m)}(x) \le \sum_{k=0}^\infty |a_k x_k| = M(x) < \infty$ for all $m \in \mathbb{N}_0$ and all $x \in X$. By the uniform boundedness principle, there is a constant M_1 such that $\|f^{(m)}\| \le M_1$ for all $m \in \mathbb{N}_0$. ¿From this and $\sup_n \|e^{[n]}\| < \infty$, we conclude $a \in l_1$. □

Theorem 1. *Let $\mu = (\mu_n)_{n=0}^\infty$ be a nondecreasing sequence of positive reals tending to infinity. Then*

(a) $$c_0^\alpha(\mu) = c^\alpha(\mu) = c_\infty^\alpha(\mu) = l_1;$$

(b) $$c_0^{\alpha\alpha}(\mu) = c^{\alpha\alpha}(\mu) = c_\infty^{\alpha\alpha}(\mu) = l_\infty.$$

Proof. (a) The sets $c_0(\mu)$, $c(\mu)$ and $c_\infty(\mu)$ are BK spaces with $c_0(\mu) \subset c(\mu) \subset c_\infty(\mu) \subset l_\infty$. For each $m \in \mathbb{N}_0$ and for all $n \geq m$,

$$\begin{aligned} \frac{1}{\lambda_n}\sum_{k=0}^{n}|\lambda_k e_k^{[m]} - \lambda_{k-1}e_{k-1}^{[m]}| &= \frac{1}{\lambda_n}\left(\sum_{k=0}^{m}(\lambda_k - \lambda_{k-1}) + \lambda_m\right) \\ &= \frac{2\lambda_m}{\lambda_n} \to 0 \quad (n \to \infty), \end{aligned}$$

so $e^{[m]} \in c_0(\Lambda)$ and $\|e^{[m]}\|_{c_\infty(\Lambda)} = 2$ for all m. Thus part (a) follows from Lemma 1. (b) Part (b) is obvious from part (a) and the fact that $l_1^\alpha = l_\infty$. □

If $a \in cs$ then we shall write $R(a)$ for the sequence with $R_n(a) = \sum_{k=n}^\infty a_k$ $(n = 0, 1, \cdots)$. We shall frequently apply Abel's summation by parts

$$(3.1) \qquad \sum_{n=0}^{m-1} a_n y_n = \sum_{n=0}^{m} R_n(a)(\Delta y)_n - R_m(a)y_m \text{ for all } m = 0, 1, \cdots.$$

If u is a sequence with $u_k \neq 0$ for all $k = 0, 1, \cdots$ then we shall write $1/u$ for the sequence with $(1/u)_k = 1/u_k$ for all k.

We need the following result (cf. [4, Theorems 4 and 6]).

Lemma 2. *We put*

$$\mathcal{W}(\Lambda) = \left\{a \in \omega : \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)}\max_\nu |a_k| < \infty\right\}$$

and

$$\|a\|_{\mathcal{W}(\Lambda)} = \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)}\max_\nu |a_k|.$$

Then $w_0^\beta(\Lambda) = w_\infty^\beta(\Lambda) = \mathcal{W}(\Lambda)$ and $w_0^*(\Lambda)$ is norm isomorphic to $\mathcal{W}(\Lambda)$ when $w_0(\Lambda)$ has the norm $\|\cdot\|_{\tilde{w}_\infty(\Lambda)}$.

Theorem 2. *We put*

$$\mathcal{C}(\Lambda) = \left\{a \in \omega : \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)}\max_\nu \left|\sum_{k=n}^{\infty} \frac{a_k}{\lambda_k}\right| < \infty\right\}$$

and

$$\|a\|_{\mathcal{C}(\Lambda)} = \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max_{\nu} \left| \sum_{k=n}^{\infty} \frac{a_k}{\lambda_k} \right|.$$

(a) If X is any of the sets $c_0(\Lambda)$, $c(\Lambda)$, or $c_\infty(\Lambda)$ and $\dagger = \beta, \gamma, f$, then $X^\dagger = \mathcal{C}(\Lambda)$.
(b) The continuous dual $c_0^*(\Lambda)$ is norm isomorphic to $\mathcal{C}(\Lambda)$ when $c_0(\Lambda)$ has the norm $\|\cdot\|_{\bar{c}_\infty(\Lambda)}$. Further

$$\|a\|^*_{\bar{c}_\infty(\Lambda)} = \|a\|_{\mathcal{C}(\Lambda)} \text{ on } c(\Lambda) \text{ and } c_\infty(\Lambda). \tag{3.2}$$

(c) We have $f \in c^*(\Lambda)$ if and only if

$$\begin{aligned} &f(x) = l\chi_f + \sum_{n=0}^{\infty} a_n x_n \text{ for all } x \in \mathcal{C}(\Lambda) \\ &\text{where } a \in \mathcal{C}(\Lambda),\ l \in \mathbf{C} \text{ with } x - le \in c_0(\Lambda) \text{ and} \\ &\chi_f = f(e) - \sum_{n=0}^{\infty} a_n. \end{aligned} \tag{3.3}$$

Further, $\|f\|$ is equivalent to

$$|\chi_f| + \|a\|_{\mathcal{C}(\Lambda)}. \tag{3.4}$$

Proof. (a) First let $a \in \mathcal{C}(\Lambda)$. We write $b = a/\lambda$. Obviously, $a \in \mathcal{C}(\Lambda)$ implies $\Lambda R(b) \in c_0$. Let $x \in c_\infty(\Lambda)$ be given. Since $c_\infty(\Lambda) \subset l_\infty$, we conclude $R(b)\Lambda x \in c_0$. Further

$$\sum_{n=0}^{\infty} |R_n(b)|\,|(\Delta(\Lambda x))_n| \le \|a\|_{\mathcal{C}(\Lambda)} \|x\|_{\bar{c}_\infty(\Lambda)} < \infty,$$

hence $R(b)\Delta(\Lambda x) \in l_1$. Now, this $R(b)\Lambda x \in c_0$ and (3.1), with a and y replaced by b and Λx, together yield $ax \in cs$. Since $x \in c_\infty(\Lambda)$ was arbitrary,

$$\mathcal{C}(\Lambda) \subset c_\infty^\beta(\Lambda). \tag{3.5}$$

Now we assume $a \in c_0^\beta(\Lambda)$. Then $ax \in cs$ for all $x \in c_0(\Lambda)$. First $1/\Lambda \in c_0(\Lambda)$ implies $R_n(b) \in cs$ for all n. Further, since $c_0(\Lambda)$ is a BK space, the map $f_a : c_0(\Lambda) \to \mathbf{C}$ defined by $f_a(x) = \sum_{k=0}^\infty a_k x_k$ for all $x \in c_0(\Lambda)$ is an element of $c_0^*(\Lambda)$ by [8, Theorem 7.2.9, p. 107]. Therefore

$$|f_a(x)| \le \|f_a\| < \infty \text{ for all } x \in c_0(\Lambda) \text{ with } \|x\|_{c_\infty(\Lambda)} = 1. \tag{3.6}$$

Since, for all $m \in \mathbb{N}_0$, $e^{[m]} \in c_0(\Lambda)$, $\|e^{[m]}\|_{c_\infty(\Lambda)} = 2$ and, by (3.6),

$$|f_a(e^{[m]})| = \left|\sum_{k=0}^{m} a_k\right| \leq 2\|f_a\| < \infty,$$

we conclude $a \in bs$. We define the matrix B by $b_{mk} = \lambda_m/\lambda_k$ for $k \geq m$ and $b_{mk} = 0$ for $0 \leq k \leq m-1$ $(m = 0, 1, \cdots)$. If we fix $m \in \mathbb{N}_0$, then for all $n \geq m$

$$0 \leq \sum_{k=0}^{n} |b_{mk} - b_{m,k-1}| = 1 + \lambda_m \sum_{k=m+1}^{n} \left(\frac{1}{\lambda_{k-1}} - \frac{1}{\lambda_k}\right) = 2 - \frac{\lambda_m}{\lambda_n} \leq 2,$$

and so

$$\sup_m \sum_{k=0}^{\infty} |b_{mk} - b_{m,k-1}| < \infty. \tag{3.7}$$

Further $b_{mk} = \lambda_m/\lambda_k \to 0$ $(k \to \infty)$ for each fixed $m \in \mathbb{N}_0$. This and the condition (3.7) together imply $\Lambda R(b) = B(a) \in l_\infty$ by [8, Example 8.4.5C, p. 129]. Since $c_0(\Lambda) \subset c_0$, we conclude $R(b)\Lambda x \in c_0$ for all $x \in c_0(\Lambda)$. Further, $ax \in cs$, $R(b)\Lambda x \in c_0$, and (3.1) together imply $R(b)\Delta(\Lambda x) \in cs$ for all $x \in c_0(\Lambda)$. Since $x \in c_0(\Lambda)$ if and only if $\Delta(\Lambda x) \in w_0(\Lambda)$, this implies $R(b) \in \mathcal{W}(\Lambda)$ by Lemma 2, and so $a \in \mathcal{C}(\Lambda)$. Thus we have proved

$$c_0^\beta(\Lambda) \subset \mathcal{C}(\Lambda). \tag{3.8}$$

Since $c_0(\Lambda) \subset c_\infty(\Lambda)$ implies $c_\infty^\beta(\Lambda) \subset c_0^\beta(\Lambda)$, (3.8) and (3.5) together yield $c_0^\beta(\Lambda) = c_\infty^\beta(\Lambda) = \mathcal{C}(\Lambda)$. Finally, from this and $c_0(\Lambda) \subset c(\Lambda) \subset c_\infty(\Lambda)$, we obtain $c^\beta(\Lambda) = \mathcal{C}(\Lambda)$.

Since $c_0(\Lambda)$ has AK and so AD, it follows from [8, Theorem 7.2.7 (ii) and (iii), p. 106] that

$$c_0^\beta(\Lambda) = c_0^\gamma(\Lambda) = c_0^f(\Lambda) = \mathcal{C}(\Lambda). \tag{3.9}$$

The fact that $c_0(\Lambda)$ and $c(\Lambda)$ are closed subspaces of $c_\infty(\Lambda)$ implies

$$c_0^f(\Lambda) = c^f(\Lambda) = c_\infty^f(\Lambda) = \mathcal{C}(\Lambda) \tag{3.10}$$

by [8, Theorem 7.2.6, p. 106]. From [8, Theorem 7.2.7 (i), p. 106], (3.10) and (3.9), we obtain $\mathcal{C}(\Lambda) \subset c^\beta(\Lambda) \subset c^\gamma(\Lambda) \subset c^f(\Lambda) = \mathcal{C}(\Lambda)$ and $\mathcal{C}(\Lambda) \subset c_\infty^\beta(\Lambda) \subset c_\infty^\gamma(\Lambda) \subset c_\infty^f(\Lambda) = \mathcal{C}(\Lambda)$.

(b) Since $c_0(\Lambda)$ has AK, $c_0^*(\Lambda)$ and $c_0^\beta(\Lambda) = \mathcal{C}(\Lambda)$ are isopmorphic by [8, Theorem 7.2.9, p. 107].
In the proof of part (a) we saw that if $a \in c_0^\beta(\Lambda)$ then

$$f(x) = \sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} R_k(b)\,(\Delta(\Lambda x))_k \text{ for all } x \in c_0(\Lambda). \tag{3.11}$$

Since $a \in \mathcal{C}(\Lambda)$ and $x \in c_0(\Lambda)$ if and only if $R(b) \in \mathcal{W}(\Lambda)$ and $\Delta(\Lambda x) \in w_0(\Lambda)$, and since $\|a\|_{\mathcal{C}(\Lambda)} = \|R(b)\|_{\mathcal{W}(\Lambda)}$ and $\|x\|_{\bar{c}_\infty(\Lambda)} = \|\Delta(\Lambda x)\|_{\bar{w}_\infty(\Lambda)}$, the norms $\|\cdot\|$ on $c_0^*(\Lambda)$ and $\|\cdot\|_{\mathcal{C}(\Lambda)}$ are the same, and (3.2) holds.

(c) First we assume $f \in c^*(\Lambda)$.
Then $f_1 = f|_{c_0(\Lambda)} \in c_0^*(\Lambda)$. Given $x \in c(\Lambda)$ there is a unique $l \in \mathbb{C}$ (cf. [5, Lemma 2]) such that $x - le \in c_0(\Lambda)$, hence $f(x) = f(e) + f_1(x - le)$. By part (b), there is a sequence $a \in \mathcal{C}(\Lambda)$ such that $f_1(x - le) = \sum_{k=0}^{\infty} a_k(x_k - l)$. Since $a \in \mathcal{C}(\Lambda) = c^\beta(\Lambda)$, we have $ax \in cs$ for all $x \in c(\Lambda)$, in particular, for $x = e \in c(\Lambda)$, this implies $ae = a \in cs$, and so we may write

$$f(x) = l\left(f(e) - \sum_{k=0}^{\infty} a_k\right) + \sum_{k=0}^{\infty} a_k x_k.$$

Putting $\chi_f = f(e) - \sum_{k=0}^{\infty} a_k$, we obtain the represenation in (3.3).
Conversely, if $a \in \mathcal{C}(\Lambda)$ then $ax \in cs$ for all $x \in c(\Lambda)$, so $a \in cs$, and (3.3) defines a linear functional on $c(\Lambda)$ for arbitrary $\chi_f \in \mathbb{C}$. In particular, for $x = e \in c(\Lambda)$, $f(e) = \chi_f + \sum_{k=0}^{\infty} a_k$, that is $\chi_f = f(e) - \sum_{k=0}^{\infty} a_k$. If $x \in c(\Lambda)$, then for the unique $l \in \mathbb{C}$ with $x - le \in c_0(\Lambda)$, we have

$$|l| \le \frac{1}{\lambda_m} \sum_{k=0}^{m} |(\Delta(\Lambda(x - le)))_k| + \|x\|_{c_\infty(\Lambda)} \text{ for all } m \in \mathbb{N}_0.$$

Letting $m \to \infty$, we conclude $|l| \le \|x\|_{c_\infty(\Lambda)}$. Further, by (3.11),

$$\left|\sum_{k=0}^{\infty} a_k x_k\right| \le \|a\|_{\mathcal{C}(\Lambda)} \|x\|_{\bar{c}_\infty(\Lambda)} \text{ for all } x \in c(\Lambda).$$

Since the norms $\|\cdot\|_{c_\infty(\Lambda)}$ and $\|\cdot\|_{\bar{c}_\infty(\Lambda)}$ are equivalent,

$$\|f\| \le K\left(|\chi_f| + \|a\|_{\mathcal{C}(\Lambda)}\right) < \infty \text{ for some constant } K. \tag{3.12}$$

This shows $f \in c^*(\Lambda)$.
Now let $\nu_m \in \mathbb{N}_0$ be arbitrary. For each $\nu \in \mathbb{N}_0$, let $k_\nu \in K^{<\nu>}$ be the

smallest integer such that $|R_{k_\nu}| = \max_\nu |R_k(b)|$. We define the sequence $y^{(m)}$ by

$$y_k^{(m)} = \begin{cases} \lambda_{n(\nu+1)}\operatorname{sgn}(R_{k_\nu}(b)) & \text{for } k = k_\nu \\ 0 & \text{for } k \neq k_\nu \quad (\nu = 0, 1, \cdots, \nu_m) \\ \operatorname{sgn}(\chi_f) & \text{for } k \geq n(\nu+1) \quad (\nu \geq \nu_m + 1) \end{cases}$$

and the sequence $x^{(m)}$ by $y^{(m)} = \Delta(\Lambda x^{(m)})$. Then $x^{(m)} \in c(\Lambda)$, $x^{(m)} - \operatorname{sgn}(\chi_k)e \in c_0(\Lambda)$, $\|x^{(m)}\|_{\tilde{c}_\infty(\Lambda)} \leq 1$ and

$$|f(x^{(m)})| = \left| |\chi_f| + \sum_{\nu=0}^{\nu_m} \lambda_{n(\nu+1)} \max_\nu |R_k(b)| + \sum_{k=n(\nu_m+1)}^{\infty} \operatorname{sgn}(\chi_f) R_k(b) \right| \leq \|f\|.$$

Since $a \in \mathcal{C}(\Lambda)$ obviously implies $R(b) \in l_1$, $\sum_{k=n(\nu_m+1)}^{\infty} \operatorname{sgn}(\chi_f)R_k(b) \to 0$ $(m \to \infty)$, and so $|\chi_f| + \|a\|_{\mathcal{C}(\Lambda)} \leq \|f\|$. Together with (3.12) this yields the equivalence of the norm $\|f\|$ and $|\chi_f| + \|a\|_{\mathcal{C}(\Lambda)}$. □

Theorem 3. *(a) The space $\mathcal{C}(\Lambda)$ with $\|\cdot\|_{\mathcal{C}(\Lambda)}$ is a BK space with AK.*
(b) The set $c_\infty(\Lambda)$ is β perfect *that is $c_\infty^{\beta\beta}(\Lambda) = c_\infty(\Lambda)$. Further $\|a\|^*_{\mathcal{C}(\Lambda)} = \|a\|_{\tilde{c}_\infty(\Lambda)}$ for all $a \in \mathcal{C}^\beta(\Lambda)$.*
(c) Finally, $\mathcal{C}^f(\Lambda) = \mathcal{C}^\gamma(\Lambda) = \mathcal{C}^\beta(\Lambda)$.

Proof. (a) The space $\mathcal{W}(\Lambda)$ is a BK space with $\|\cdot\|_{\mathcal{W}(\Lambda)}$ (cf. [4, Theorem 2]). Further, the matrix A defined by $a_{nk} = 1/\lambda_k$ for $k \geq n$ and $a_{nk} = 0$ for $0 \leq n-1$ $(n = 0, 1, \cdots)$ is one–to–one, and $x = A(y) \in \mathcal{C}(\Lambda)$ if and only if $y \in \mathcal{W}(\Lambda)$. So, by [8, Theorem 4.3.2, p. 61], $\mathcal{C}(\Lambda)$ is a BK space with $\|x\|_{\mathcal{C}(\Lambda)} = \|A(y)\|_{\mathcal{W}(\Lambda)}$.
Now we show that $\mathcal{C}(\Lambda)$ has AK.
First we observe $\phi \subset \mathcal{C}(\Lambda)$, since $\mathcal{C}(\Lambda)$ is the β–dual of a sequence space.
Now let $x \in \mathcal{C}(\Lambda)$ and $\varepsilon > 0$ be given. For each $m \in \mathbb{N}_0$, let ν_m denote the uniquely determined integer for which $m \in N^{<\nu_m>}$. We choose $m_0 \in \mathbb{N}_0$ such that

$$\sum_{\nu=\nu_m}^{\infty} \lambda_{n(\nu+1)} \max_\nu |R_n(x/\Lambda)| < \varepsilon \text{ for all } m \geq m_0.$$

Let $m \geq m_0$ be given. Since the sequence $\Lambda = (\lambda_n)_{n=0}^\infty$ is exponentially bounded, there is $t \in (0,1)$ such that by (2.1)

$$\|x - x^{[m]}\|_{\mathcal{C}(\Lambda)} = \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max_\nu \left| R_n((x - x^{[m]})/\Lambda) \right|$$

$$\begin{aligned}
&\leq \sum_{\nu=0}^{\nu_m-1} \lambda_{n(\nu+1)} |R_{m+1}(x/\Lambda)| + \sum_{\nu=\nu_m}^{\infty} \lambda_{n(\nu+1)} \max_\nu |R_n(x/\Lambda)| \\
&< \varepsilon + \sum_{\nu=0}^{\nu_m-1} \frac{\lambda_{n(\nu+1)}}{\lambda_{n(\nu_m+1)}} \lambda_{n(\nu_m+1)} \max_{\nu_m} |R_n(x/\Lambda)| \\
&< \varepsilon + \varepsilon \sum_{\nu=0}^{\nu_m-1} t^{\nu_m - v} < \varepsilon \frac{1}{1-t}.
\end{aligned}$$

This shows that $\mathcal{C}(\Lambda)$ has AK. (b) First we show $\mathcal{C}^\beta(\Lambda) = c_\infty(\Lambda)$.
For any $X \subset \omega$, $X \subset X^{\beta\beta}$ by [8, Theorem 7.1.2, p. 105]. So we have to show $c_\infty(\Lambda) \subset \mathcal{C}^\beta(\Lambda)$ according to Theorem 2 (a).
Let $a \in \mathcal{C}^\beta(\Lambda)$. We define $f_a : \mathcal{C}(\Lambda) \to \mathbb{C}$ by $f_a(x) = \sum_{k=0}^\infty a_k x_k$ for all $x \in \mathcal{C}(\Lambda)$. Then $f_a \in \mathcal{C}^*(\Lambda)$ by [8, Theorem 7.2.9, p. 107], and so

(3.13) $$|f_a(x)| \leq \|f_a\| \|x\|_{\mathcal{C}(\Lambda)} < \infty \text{ for all } x \in \mathcal{C}(\Lambda).$$

Let $m \in \mathbb{N}_0$ be given and ν_m the uniquely determined integer such that $m \in N^{<\nu_m>}$. Since Λ is exponentially bounded, there are $s, t \in (0,1)$ such that by (2.1)

$$\begin{aligned}
\|e^{(m)}\|_{\mathcal{C}(\Lambda)} &= \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max_\nu \left| R_n(e^{(m)}/\Lambda_k) \right| = \sum_{\nu=0}^{\nu_m} \frac{\lambda_{n(\nu+1)}}{\lambda_m} \\
&\leq \sum_{\nu=0}^{\nu_m} \frac{\lambda_{n(\nu+1)}}{\lambda_{n(\nu_m+1)}} \frac{\lambda_{n(\nu_m+1)}}{\lambda_{n(\nu_m)}} \leq \frac{1}{s} \sum_{\nu=0}^{\nu_m} t^{\nu_m-\nu} \leq \frac{1}{s(1-t)} < \infty.
\end{aligned}$$

Now (3.13) implies

$$|a_m| = |f_a(e^{(m)})| \leq \|f_a\| \|e^{(m)}\|_{\mathcal{C}(\Lambda)} \leq \|f_a\| \frac{1}{s(1-t)}$$

for all $m \in \mathbb{N}_0$, and so $a \in l_\infty$. Further $x \in \mathcal{C}(\Lambda)$ implies that $R_n(x/\Lambda) \in cs$ all n, and $\Lambda R(x/\Lambda) \in c_0$. Therefore $a\Lambda R(x/\Lambda) \in c_0$. Now (3.1) yields

(3.14) $$\sum_{n=0}^\infty a_n x_n = \sum_{n=0}^\infty R_n(x/\Lambda)(\Delta(\Lambda a))_n \text{ for all } x \in \mathcal{C}(\Lambda).$$

Thus $R(x/\Lambda)\Delta(\Lambda a) \in cs$ for all $x \in \mathcal{C}(\Lambda)$. Now $x \in \mathcal{C}(\Lambda)$ if and only if $R(x/\Lambda) \in \mathcal{W}(\Lambda)$ and so $\Delta(\Lambda a) \in \mathcal{W}^\beta(\Lambda) = w_\infty(\Lambda)$ by [4, Theorem 4]. But this means $a \in c_\infty(\Lambda)$. Thus we have shown $\mathcal{C}^\beta(\Lambda) \subset c_\infty(\Lambda)$.
Now we show

(3.15) $$\|a\|^*_{\mathcal{C}(\Lambda)} = \|a\|_{\bar{c}_\infty(\Lambda)} \text{ for all } a \in \mathcal{C}^\beta(\Lambda).$$

Let $a \in \mathcal{C}^\beta(\Lambda) = c_\infty(\Lambda)$ by what we have just shown. Then by (3.14), for all $x \in \mathcal{C}(\Lambda)$,

$$\begin{aligned} \left|\sum_{n=0}^{\infty} a_n x_n\right| &\leq \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max\nolimits_\nu |R_n(x/\Lambda)| \frac{1}{\lambda_{n(\nu+1)}} \Sigma_\nu |(\Delta(\Lambda a))_n| \\ &\leq \|a\|_{\tilde{c}_\infty(\Lambda)} \|x\|_{\mathcal{C}(\Lambda)}, \end{aligned}$$

and so

$$\|a\|^*_{\mathcal{C}(\Lambda)} \leq \|a\|_{\tilde{c}_\infty(\Lambda)}. \tag{3.16}$$

Let $\nu_m \in \mathbb{N}_0$ be given. By $\nu_{0,m}$, we denote the smallest integer with $0 \leq \nu_{0,m} \leq \nu_m$ for which

$$\frac{1}{\lambda_{n(\nu_{0,m}+1)}} \Sigma_{\nu_{0,m}} |(\Delta(\Lambda a))_n| = \max_{0 \leq \nu \leq \nu_m} \left(\frac{1}{\lambda_{n(\nu+1)}} \Sigma_\nu |(\Delta(\Lambda a))_n| \right).$$

We define the sequences $R^{(m)}$ and $x^{(m)}$ by

$$R_n^{(m)} = \begin{cases} \frac{1}{\lambda_{n(\nu_{0,m}+1)}} \operatorname{sgn}((\Delta(\Lambda a))_n) & \text{for } n \in N^{<\nu_{0,m}>} \\ 0 & \text{for } n \notin N^{<\nu_{0,m}>}, \end{cases}$$

and $x_n^{(m)} = R_n^{(m)} - R_{n+1}^{(m)}$ $(n = 0, 1, \cdots)$. Then we have

$$\|x^{(m)}\|_{\mathcal{C}(\Lambda)} = \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max\nolimits_\nu \left|R_n^{(m)}\right| = \lambda_{n(\nu_{0,m}+1)} \max\nolimits_{\nu_{0,m}} \left|R_n^{(m)}\right| \leq 1$$

and

$$\left|\sum_{n=0}^{\infty} a_n x_n^{(m)}\right| = \max_{0 \leq \nu \leq \nu_m} \frac{1}{\lambda_{n(\nu+1)}} \Sigma_\nu |(\Delta(\Lambda a))_n| \leq \|a\|^*_{\mathcal{C}(\Lambda)} \|x\|_{\mathcal{C}(\Lambda)} \leq \|a\|^*_{\mathcal{C}(\Lambda)}.$$

Since ν_m was arbitrary, we obtain $\|a\|_{\tilde{c}_\infty(\Lambda)} \leq \|a\|^*_{\mathcal{C}(\Lambda)}$. Together with (3.16), this yields (3.15). (c) Since $\mathcal{C}(\Lambda)$ has AK by part (b) and so AD, part (c) follows from [8, Theorem 7.2.7 (ii) and (iii), p. 106]. □

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