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SPACES OF TEMPERED ULTRADISTRIBUTIONS AND DIFFERENTIAL EQUATIONS

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Abstract

In the paper we considere the Laplace equation and apply the heat kernel method to obtain some properties of solutions.

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1. Preliminaries

We use the multiindex notation $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_d$, $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_d!$, $x^\alpha =$
$= x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$, $\quad |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$ and

$$\varphi^{(\alpha)}(x) = (\partial/\partial x)^\alpha = (\partial/\partial x_1)^{\alpha_1}(\partial/\partial x_2)^{\alpha_2}\cdots(\partial/\partial x_d)^{\alpha_d}\varphi(x), \quad x \in \mathbb{R}^d,$$

where $d \in \mathbb{N}$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{N}_0^d$, $\varphi \in C^\infty(\mathbb{R}^d)$ and $x = (x_1, x_2, ..., x_d) \in \mathbb{R}^d$.

Let $\{M_p,\ p \in \mathbb{N}_0\}$ and $\{N_p,\ p \in \mathbb{N}_0\}$ be the sequences of positive numbers. In the paper, the following conditions will be used. For their detailed analysis see [3].

(M.1)

$$M_p^2 \leq M_{p-1}M_{p+1}, \quad p = 1, 2, \ldots.$$

(M.2) There are constants A and H such that

$$M_p \leq AH^p \min_{0 \leq q \leq p} M_q M_{p-q}, \quad p = 0, 1, \ldots.$$

(M.3) There is a constants A such that

$$\sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \leq A \frac{pM_p}{M_{p+1}}, \quad p = 1, 2, \ldots.$$

The corresponding conditions for the sequence $\{N_p,\ p \in \mathbb{N}_0\}$ will be denoted by (N.1), (N.2) and (N.3).

Throughout the paper we assume that $M_0 = 1$ and $N_0 = 1$.

Note, the Gevrey sequence

$$p^{sp} \quad \text{or} \quad (p!)^s \quad \text{or} \quad \Gamma(1 + sp), \quad p \in \mathbb{N}_0,$$

$s > 1$ satisfies the above conditions.

The so-called associated functions for the sequence $\{M_p,\ p \in \mathbb{N}_0\}$ are

$$M(\rho) = \sup_{p \in \mathbb{N}_0} \log \frac{\rho^p}{M_p}, \quad \overline{M}(\rho) = \sup_{p \in \mathbb{N}_0} \log \frac{\rho^p p!}{M_p^2},$$

where $\rho > 0$.

The corresponding associated functions for the sequence $\{N_p,\ p \in \mathbb{N}_0\}$ will be denoted by $N(\cdot)$, and $\overline{N}(\cdot)$.

The spaces $\mathcal{S}_{(N_p)}^{(M_p)}$ and $\mathcal{S}_{\{N_p\}}^{\{M_p\}}$ are defined in [1] in the following way:

Definition 1. *Let $m, n > 0$. The space of smooth functions φ on $\mathbb{R}^d$, such that*

(1) $$|x^\beta \varphi^{(\alpha)}(x)| \leq C_\varphi \frac{M_{|\alpha|} N_{|\beta|}}{m^{|\alpha|} n^{|\beta|}}, \quad \text{for every } \alpha, \beta \in \mathbb{N}_0^d,$$

where the constant C_φ *depends only on* φ, *we denote* $\mathcal{S}_{N,n}^{M,m}$, *and equip with the norm*

$$s_{m,n}(\varphi) = \sup_{\alpha,\beta\in\mathbb{N}_0^d} \frac{m^{|\alpha|}n^{|\beta|}}{M_{|\alpha|}N_{|\beta|}} \|x^\beta\varphi^{(\alpha)}(x)\|_\infty. \tag{2}$$

The spaces $\mathcal{S}_{(N_p)}^{(M_p)}$ *and* $\mathcal{S}_{\{N_p\}}^{\{M_p\}}$ *are*

$$\mathcal{S}_{(N_p)}^{(M_p)} = \operatorname{proj}\lim_{\substack{m\to\infty\\ n\to\infty}} \mathcal{S}_{N,n}^{M,m}, \qquad \mathcal{S}_{\{N_p\}}^{\{M_p\}} = \operatorname{proj}\lim_{\substack{m\to 0\\ n\to 0}} \mathcal{S}_{N,n}^{M,m}.$$

We use the notation $\mathcal{S}_\dagger^*$ *to denote* $\mathcal{S}_{(N_p)}^{(M_p)}$ *or* $\mathcal{S}_{\{N_p\}}^{\{M_p\}}$.

For the properties of the spaces $\mathcal{S}_\dagger^*$ see [1].

Denote by $E(x,t)$ the heat kernel:

$$E(x,t) = \begin{cases} (4\pi t)^{-d/2}\exp[-|x|^2/4t], & t>0,\\ 0, & t<0. \end{cases} \tag{3}$$

We will use the following result

Theorem 1. *[1] 1. Let the conditions (M.1), (M.2), (M.3), (N.1), (N.2) and (N.3) be satisfied and* $f \in \mathcal{S}'^{(M_p)}_{(N_p)}$. *The function*

$$U(x,t) = < f(y), E(x-y,t) >$$

is well defined on $\mathbb{R}_+^{d+1} = \{(x,t) | x\in\mathbb{R}^d,\ t>0\}$, *belongs to* $C^\infty(\mathbb{R}_+^{d+1})$ *and satisfies the heat equation*

$$\Big(\frac{d}{dt} - \Delta\Big)U(x,t) = 0. \tag{4}$$

Furthermore, for some $m,n>0$, *and arbitrary* $T>0$, *there exists a positive constant* C *such that*

$$|U(x,t)| \le C\exp\Big[N(n|x|) + \frac{1}{2}\overline{M}\Big(\frac{m}{t}\Big)\Big],\ \ x\in\mathbb{R}_+^d,\ \ t\in(0,T). \tag{5}$$

Also, for any $\psi\in\mathcal{S}_{(N_p)}^{(M_p)}$, *we have*

$$\int_{\mathbb{R}^d} U(x,t)\psi(x)dx \to < f,\psi >,\ \ t\to 0. \tag{6}$$

2. If the conditions (M.1), (M.2), (M.3), (N.1), (N.2) and (N.3) are satisfied, the converse is also true: for every smooth function $U(x,t)$ defined on $\mathbb{R}_+^{d+1}$, satisfying the conditions 4 and 5, for some $m, n > 0$, there exists unique $f \in \mathcal{S}'^{(M_p)}_{(N_p)}$, such that

$$U(x,t) = < f(y), E(x-y,t) > . \tag{7}$$

We call $U(x,t) = < f(y), E(x-y,t) >$ the definding function for the function $f \in \mathcal{S}'^{(M_p)}_{(N_p)}$.

We note that the analogous result holds also for the Roumieu type spaces of tempered ultradistributions $\mathcal{S}'^{\{M_p\}}_{\{N_p\}}$.

2. Main Result

In the following theorem we will consider only the Beurling type tempered ultradistributions, but analogous results hold also for the Roumieu type tempered ultradistributions.

Theorem 2. *Let conditions (M.1), (M.2), (M.3), (N.1), (N.2) and (N.3) be satisfied. For any $f \in \mathcal{S}'^{(M_p)}_{(N_p)}$, there exists a solution u of the equation*

$$\Delta u(x) = f(x), \qquad x \in \mathbb{R}^d. \tag{8}$$

Furthermore, if f is an analytic function which satisfies that for some $n > 0$ there exists a positive constant C such that

$$|f(x)| \le C \exp\Big[N(n|x|)\Big], \quad x \in \mathbb{R}^d \tag{9}$$

then any solution u of the equation 8 is in the space $\mathcal{S}'^{(M_p)}_{(N_p)}$, and is an analytic function.

Proof. 1. First, we consider equation $\Delta u = \varphi$, where $\varphi \in \mathcal{S}^{(M_p)}_{(N_p)}$. Let $\omega_n = 1 - \varphi_n$, where $\varphi_n \in \mathcal{D}^{(M_p)}$ and $\varphi_n(x) = 1$, for $|x| > 1/n$, and $\varphi_n(x) = 0$ for $|x| < 1/2n$. It is easy to verify that $\theta_n(x) = |x|^{-2}\omega_n(x)$ belongs to the space of multipliers of the space $\mathcal{S}^{(M_p)}_{(N_p)}$, (i.e. $\theta_n \in \mathcal{E}^{(M_p)}$, and $\theta_n \cdot \psi \in \mathcal{S}^{(M_p)}_{(N_p)}$, for each $\psi \in \mathcal{S}^{(M_p)}_{(N_p)}$). Put $u_n(x) = \mathcal{F}^{-1}(\theta_n \cdot \hat{\varphi})$. Since $\theta_n \cdot \hat{\varphi} \in \mathcal{S}^{(M_p)}_{(N_p)}$, we have that u_n is an element of $\mathcal{S}^{(M_p)}_{(N_p)}$. One can verify that the sequence u_n is a Cauchy sequence in the complete space $\mathcal{S}^{(M_p)}_{(N_p)}$ and therefore

$$u(x) = \lim_{n\to\infty} u_n(x)$$

is well defined element of $\mathcal{S}^{(M_p)}_{(N_p)}$. Note

$$u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \mathcal{F}^{-1}(\theta_n \cdot \hat{\varphi}) =$$
$$= - \lim_{n\to\infty} (2\pi)^{-d} \int e^{i<x,\xi>} |\xi|^{-2} \omega_n(\xi)\hat{\varphi}(\xi)d\xi.$$

It follows from above that u is a solution of the equation $\Delta u = \varphi$.

2. Let us consider equation

$$\Delta u = f, \tag{10}$$

where $f \in \mathcal{S}'^{(M_p)}_{(N_p)}$. This equation can be solved by using the duality. Define $u \in \mathcal{S}'^{(M_p)}_{(N_p)}$, by

$$< u, \varphi > = < f, \Delta^{-1}\varphi >, \quad \varphi \in \mathcal{S}^{(M_p)}_{(N_p)}.$$

Then we have that

$$< u, \Delta\varphi > = < f, \varphi >, \quad \varphi \in \mathcal{S}^{(M_p)}_{(N_p)}.$$

and therefore u is a solution of the equation 10 and $u \in \mathcal{S}'^{(M_p)}_{(N_p)}$.

3. Let f be an analytic function which satisfies estimate 9, and let $U(x,t)$ and $F(x,t)$ be the defining functions of u and f. Note that, since f is an analytic funcyion, it follows that $F(\cdot,t)$ ia an analytic function too. We have

$$\Delta U(x,t) = \frac{d}{dt} U(x,t) = F(x,t), \quad (x,t) \quad \in \mathbb{R}^{d+1}_{+}.$$

It follows that $U(\cdot,t)$ is an analytic function.

$$U(x,t) = U(x,1) + \int_1^t F(x,\tau)d\tau.$$

Using Theorem 1 and passing $t \to 0$ we have the following equality in the space $\mathcal{S}'^{(M_p)}_{(N_p)}$:

$$u(x) = U(x,1) + \int_0^1 F(x,\tau)d\tau$$

It is easy to see that u is an analytic function. □

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