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# ON SPECTRAL COMPLEMENTARY GRAPHS

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**Abstract**

Let $G$ be an arbitrary simple graph of order $n$. We say that $G$ is a spectral complementary graph if $P_G(\lambda) - P_{\overline{G}}(\lambda) = (-1)^n \big(P_G(-\lambda - 1) - P_{\overline{G}}(-\lambda - 1)\big)$. In this paper we investigate some properties of such graphs. In particular, we prove that a graph $G$ is spectral complementary if and only if its Seidel spectrum $\sigma^*(G)$ is symmetric with respect to the zero point. Besides, we determine all connected spectral complementary graphs of order $n = 2, 3, \ldots, 8$.

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In this paper we consider only simple graphs. The vertex set of a graph $G$ is denoted by $V(G)$, and its edge set is denoted by $E(G)$. The spectrum $\sigma(G)$ of such a graph is the set $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ of eigenvalues of its (0,1) adjacency matrix. The Seidel spectrum $\sigma^*(G)$ of $G$ is the set $\lambda_1^* \geq \lambda_2^* \geq \ldots \geq \lambda_n^*$ of its Seidel $(-1, 1, 0)$ adjacency matrix. Besides, let $P_G(\lambda)$ and $P_G^*(\lambda)$ denote the characteristic polynomial and the Seidel characteristic polynomial of $G$, respectively.

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For a graph $G$ let $\overline{G}$ be its complementary graph, whose vertex set is also $V(G)$ and two distinct vertices $x, y \in V(G)$ are adjacent in $\overline{G}$ if and only if $x$ and $y$ are not adjacent in $G$. If $G$ and $H$ are two connected graphs without common vertices, then the union $G \cup H$ is the graph whose vertex set is $V(G) \cup V(H)$ and whose edge set is $E(G) \cup E(H)$.

**Definition 1.** *A graph $G$ of order $n$ is spectral complementary, if*

$$P_G(\lambda) - P_{\overline{G}}(\lambda) = (-1)^n \left(P_G(-\lambda-1) - P_{\overline{G}}(-\lambda-1)\right). \tag{1}$$

As is well-known, a graph $G$ is called self-complementary if it is isomorphic to its complementary graph $\overline{G}$. The following two results are immediate.

**Proposition 1.** *If $G$ is a spectral complementary graph, then so is $\overline{G}$.*

**Proposition 2.** *Every self-complementary graph is also a spectral complementary graph.*

By setting

$$P_G(\lambda) = \lambda^n + \sum_{k=2}^{n} a_k \lambda^{n-k} \quad \text{and} \quad P_{\overline{G}}(\lambda) = \lambda^n + \sum_{k=2}^{n} \overline{a}_k \lambda^{n-k},$$

we easily obtain the next result.

**Proposition 3.** *A graph $G$ of order $n$ is a spectral complementary graph if and only if*

$$a_k - \overline{a}_k = \sum_{i=2}^{k} (-1)^i \binom{n-i}{n-k} (a_i - \overline{a}_i),$$

*for any $k = 2, 3, \ldots, n$.*

According to [1], the generating function $H_G(t)$ of the numbers $N_k$ of walks of length $k$ in the graph $G$ reads $H_G(t) = \sum_{k=0}^{+\infty} N_k t^k$. Besides, it was proved in [1] that

$$H_G(t) = \frac{1}{t}\left[\frac{(-1)^n P_{\overline{G}}\left(-\frac{t+1}{t}\right)}{P_G\left(\frac{1}{t}\right)} - 1\right]. \tag{2}$$

**Proposition 4.** *Let $G_1$ be a self-complementary graph of order $n_1$, and let $G_2$ be any spectral complementary graph of order $n_2$. Then $G_1 \cup G_2$ is spectral complementary if and only if $P_{G_2}(\lambda) = P_{\overline{G_2}}(\lambda)$.*

*Proof.* Since $H_{G_1 \cup G_2}(t) = H_{G_1}(t) + H_{G_2}(t)$ and $P_{G_1 \cup G_2}(\lambda) = P_{G_1}(\lambda)\, P_{G_2}(\lambda)$, using (2) we obtain that

$$\begin{aligned} P_{\overline{G_1 \cup G_2}}(\lambda) &= (-1)^{n_2} P_{\overline{G_1}}(\lambda)\, P_{G_2}(-\lambda-1) + (-1)^{n_1} P_{\overline{G_2}}(\lambda)\, P_{G_1}(-\lambda-1) \\ &\quad -(-1)^{n_1+n_2} P_{G_1}(-\lambda-1)\, P_{G_2}(-\lambda-1)\,. \end{aligned}$$

First, let $P_{G_2}(\lambda) = P_{\overline{G_2}}(\lambda)$. Then, having in mind that $P_{G_1}(\lambda) = P_{\overline{G_1}}(\lambda)$, from the last relation we have

$$P_{G_1 \cup G_2}(\lambda) - P_{\overline{G_1 \cup G_2}}(\lambda) = \prod_{i=1}^{2} \left(P_{G_i}(\lambda) - (-1)^{n_i} P_{G_i}(-\lambda-1)\right),$$

wherefrom we get that $G_1 \cup G_2$ is spectral complementary.

Conversely, assume that $G_1 \cup G_2$ is a spectral complementary graph. Then, according to (1) and using (2), we find that

$$(-1)^{n_2} P_{G_1}(\lambda)(P_{G_2}(-\lambda-1) - P_{\overline{G_2}}(-\lambda-1)) =$$

$$(-1)^{n_1} P_{G_1}(-\lambda-1)(P_{G_2}(\lambda) - P_{\overline{G_2}}(\lambda)).$$

Suppose, contrary to the statement, that $P_{G_2}(\lambda) \neq P_{\overline{G_2}}(\lambda)$. Since by assumption $G_2$ is the spectral complementary graph, from the last relation we obtain $P_{G_1}(\lambda) = (-1)^{n_1} P_{G_1}(-\lambda-1)$. From this, if we set

$$P_{G_1}(\lambda) = \lambda^{n_1} + \sum_{i=2}^{n_1} a_i\, \lambda^{n_1-i} \quad \text{then} \quad 0 \cdot \lambda^{n_1-1} = n_1 \lambda^{n_1-1}\,,$$

which is a contradiction. Hence $P_{G_2}(\lambda) = P_{\overline{G_2}}(\lambda)$.

This completes the proof. □

**Corollary 1.** *Let $G$ be a spectral complementary graph and let $K_1$ denote the graph having just one vertex. Then $G \cup K_1$ is a spectral complementary graph if and only if $P_G(\lambda) = P_{\overline{G}}(\lambda)$.*

**Corollary 2.** *Let $G_1$ and $G_2$ be two self-complementary graphs. Then $G_1 \cup G_2$ is a spectral complementary graph.*

The next result is also trivial so its proof is omitted.

**Proposition 5.** *For any simple graph $G$, the graph $G \cup \overline{G}$ is spectral complementary.*

For any graph $G$, let $G^* = G \cup \overline{G}$. Using this notation, we obtain

**Proposition 6.** *Let $G$ be a regular graph of order $n$ and degree $r$. Then $\overline{G^*}$ is a connected spectral complementary graph and*

$$\sigma(\overline{G^*}) = \Big(\sigma(G^*) \setminus \{r, n-r-1\}\Big) \bigcup \left\{ \frac{n-1 \pm \sqrt{(n-2r-1)^2 + 4n^2}}{2} \right\}.$$

*Proof.* It is known that $n-1-r \geq -\lambda_n - 1 \geq \ldots \geq -\lambda_2 - 1$ are eigenvalues of the regular graph $\overline{G}$. From this fact and using (1) we easily obtain that

$$P_{\overline{G^*}}(\lambda) = \frac{\lambda^2 - (n-1)\lambda - (n^2 - nr + r^2 + r)}{(\lambda - r)(\lambda - n + r + 1)} P_G(\lambda) P_{\overline{G}}(\lambda),$$

wherefrom we get the statement. *Box*

**Proposition 7.** *Let $G$ be a regular graph of order $n$ and degree $r$. Then $G$ is spectral complementary if*

$$\sigma(\overline{G}) = \Big(\sigma(G) \setminus \Big\{r, \frac{n}{2} - r - 1\Big\}\Big) \bigcup \Big\{n - r - 1, -\frac{n}{2} + r\Big\} \quad \textit{for } n \textit{ even, or}$$

$$\sigma(\overline{G}) = \sigma(G) \quad \textit{for } n \textit{ odd.}$$

*Proof.* According to (1), the characteristic polynomial of $\overline{G}$ may be written in the form

$$P_{\overline{G}}(\lambda) = \frac{(\lambda - n + r + 1)(2\lambda + n - 2r)}{(\lambda - r)(2\lambda - n + 2r + 2)} P_G(\lambda).$$

Obviously, the last relation provides the proof if $n$ is an even number. If $n$ is an odd number, due to the fact that $\lambda \in \sigma(G)$ cannot be a non-integer rational number, we obtain from the last relation that $G$ is spectral complementary if and only if $\sigma(\overline{G}) = \sigma(G)$. □

**Corollary 3.** *Let $G$ be a regular graph of order $2n$ and degree $r$. If $n-r-1$ does not belong to $\sigma(G)$, then $G$ is not spectral complementary.*

The next result is based on the following theorem from [1].

**Theorem 1.** *If $P_G(\lambda)$ is the characteristic polynomial of a graph $G$ and $P^*_G(\lambda)$ is the Seidel characteristic polynomial of $G$, then:*

$$P_G(\lambda) = \frac{(-1)^n}{2^n} \; \frac{P^*_G(-2\lambda-1)}{1+\frac{1}{2\lambda} H_G(\frac{1}{\lambda})}. \tag{3}$$

**Proposition 8.** *A graph $G$ of order $n$ is a spectral complementary graph if and only if $P^*_G(\lambda) = P^*_{\overline{G}}(\lambda)$.*

*Proof.* Using (2) and (3), we can see that

$$P^*_G(-2\lambda-1) = 2^{n-1}(P_{\overline{G}}(-\lambda-1) + (-1)^n P_G(\lambda)). \tag{4}$$

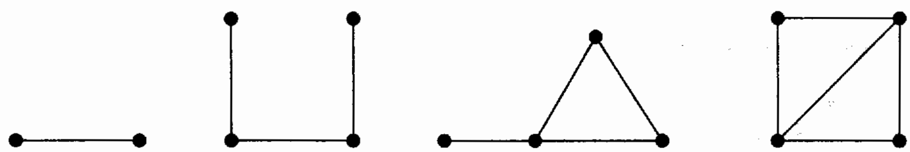
Making use of relations (1) and (4), an easy calculation gives the statement. □

Since $P^*_{\overline{G}}(\lambda) = (-1)^n\, P^*_G(-\lambda)$ for any graph $G$ of order $n$, using Proposition 8 we obtain the next result.

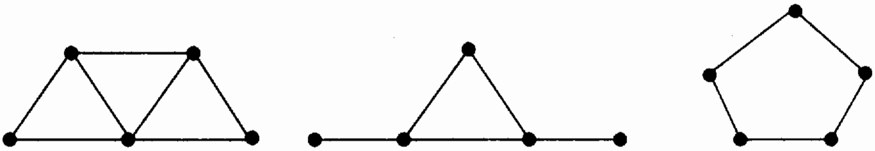
**Corollary 4.** *A graph $G$ is spectral complementary if and only if its Seidel spectrum $\sigma^*(G)$ is symmetric with respect to the zero point.*

In the following theorems we describe all spectral complementary graphs of order $n \leq 8$. For this purpose we use tables from [1] (for $n \leq 5$) and from [2] for $n = 6$; for $n = 7$ and 8 we carry on a computer search (note there are 853 and 11117 nonisomorphic connected graphs on 7 and 8 vertices, respectively).

**Theorem 2.** *There exist exactly 7 spectral complementary graphs of order $n$, for $n = 2, 3, 4, 5$. They are displayed in Fig.* 1.



$G-1$ $\quad$ $G_2$ $\quad$ $G_3$ $\quad$ $G_4$



$G_5$ $G_6$ $G_7$

Fig. 1.

**Theorem 3.** *There exist exactly* 33 *spectral complementary graphs of order* 6. *Their ordering numbers (according to [2]) are* 6, 8, 11, 14, 15, 22, 26, 27, 28, 33, 39, 42, 43, 47, 48, 53, 57, 59, 61, 65, 67, 68, 77, 78, 86, 87, 88, 96, 97, 104, 105, 110 *and* 111.

The proofs of Theorems 2 and 3 are obtained by a straightforward use of computer. Also by a computer search, we get that there are no spectral complementary graphs on 3 and 7 vertices. More generally we have:

**Theorem 4.** *There exists no connected or disconnected spectral complementary graph of order $n = 4m + 3$, where $m$ is any non-negative integer.*

*Proof.* On the contrary, assume that for some $m \in N$ there exists at least one spectral complementary graph of order $n = 4m + 3$. Then, according to Proposition 3 we have

$$2\,(a_3 - \overline{a}_3) = ((4m+3) - 2)\,(a_2 - \overline{a}_2)\,.$$

Since $|a_2|$ and $|\overline{a}_2|$ are the numbers of edges in the graph $G$ and $\overline{G}$ respectively, we get $|a_2| + |\overline{a}_2| = -\,(a_2 + \overline{a}_2) = \binom{4m+3}{2}$. From this, we obtain that

$$2\,(a_3 - \overline{a}_3) = \underbrace{(4m+1)\left[2\,a_2 + \binom{4m+3}{2}\right]}_{m_0},$$

which is a contradiction since $m_0$ is an odd number.

This completes the proof. □

Further, let $S$ be any subset of the vertex set $V(G)$. To switch $G$ with respect to $S$ means to remove all edges connecting $S$ with $\overline{S} = V(G)\setminus S$, and to introduce an edge between all nonadjacent vertices in $G$ which connect $S$ with $\overline{S}$. Two graphs $G$ and $H$ are switching (Seidel switching) equivalent if one of them is obtained from the other by switching. It is known that switching equivalent graphs have the same Seidel spectrum. Therefore, using Corollary 4, we obtain the following result.

**Corollary 5.** *A graph $G$ which is switching equivalent to some graph $H$ is spectral complementary if and only if the same holds for $H$.*

**Theorem 5.** *There exist exactly* 1142 *nonisomorphic spectral complementary graphs of order* 8.

Since it is almost impossible to display the whole list, we make a condensation of this result. Namely, the Seidel switching generates an equivalence relation in the set of graphs, where two graphs $G$ and $H$ are equivalent if and only if $G$ can be switched into $H$ with respect to some subset $S \subseteq V(G)$. Using this fact, in this paper we give the equivalence classes under Seidel switching of all spectral complementary graphs of order 8, where each equivalence class is given by a representative graph.

**Theorem 6.** *There exist exactly* 21 *equivalence classes under Seidel switching of all nonisomorphic spectral complementary graphs of order* 8. *All these classes (graphs) are represented in List* 1.

We notice that all classes (graphs) in the following lists are represented in the form:

$$n_1 \;\; n_2 \qquad a_{12} \;\; a_{13}\, a_{23} \quad \ldots \quad a_{18}\, a_{28} \ldots a_{78}\,,$$

where $n_1$ is the ordering number of the corresponding graph, $n_2$ the number of its edges and $a_{12}$ $a_{13}\, a_{23}$ $\ldots$ $a_{18}\, a_{28} \ldots a_{78}$ is the upper diagonal part of its adjacency matrix.

LIST 1. THE EQUIVALENCE CLASSES OF SPECTRAL COMPLEMENTARY GRAPHS OF ORDER 8

01 07 1 10 010 0010 00010 000010 0100000
02 08 1 10 010 0010 00010 000010 0011000
03 08 1 10 010 0010 00010 100000 0100100

| | | |
|---|---|---|
| 04 | 08 | 1 10 010 0010 00010 100000 0010010 |
| 05 | 08 | 1 10 010 0010 00010 100000 1010000 |
| 06 | 08 | 1 10 010 0010 00010 001000 1010000 |
| 07 | 08 | 1 10 010 0010 10000 010000 1000010 |
| 08 | 08 | 1 10 010 0010 01000 001000 0101000 |
| 09 | 08 | 1 10 010 0010 01000 010000 0010100 |
| 10 | 08 | 1 10 010 0010 01000 101000 0000001 |
| 11 | 09 | 1 10 010 0010 00010 100000 0100101 |
| 12 | 09 | 1 10 010 0010 10000 000001 0001110 |
| 13 | 09 | 1 10 010 0010 00010 001000 1010010 |
| 14 | 09 | 1 10 010 0010 01000 010000 0011100 |
| 15 | 09 | 1 10 010 1000 01000 101000 1000100 |
| 16 | 09 | 1 10 010 1000 10000 010100 1010000 |
| 17 | 09 | 1 10 010 0010 11000 000001 0100010 |
| 18 | 10 | 1 10 010 0010 00010 100010 0110010 |
| 19 | 10 | 1 10 010 0010 00010 010010 1000011 |
| 20 | 11 | 1 10 010 0010 00010 011010 1001100 |
| 21 | 12 | 1 10 100 1000 11000 001110 1100010 |

It is clear that the complete list of all connected and/or disconnected nonisomorphic spectral complementary graphs of order 8 can be easily generated by switching the graphs (in all possible ways) in List 1. In particular, we get that the graphs from List 1 (01 through 21) generate 120, 120, 120, 68, 120, 65, 65, 66, 27, 27, 74, 20, 41, 41, 17, 35, 36, 28, 17, 26, and 9 connected nonisomorphic spectral complementary graphs of order 8, respectively.

Making use of the list of all spectral complementary graphs of order 8, we obtain the next result.

**Corollary 6.** *There exist exactly* 10 *self-complementary graphs of order* 8. *All these graphs are represented in List* 2.

LIST 2. THE SELF-COMPLEMENTARY GRAPHS OF ORDER 8

| | | |
|---|---|---|
| 01 | 14 | 1 10 010 0010 00011 110110 0110110 |
| 02 | 14 | 1 10 010 0010 00011 011110 1011010 |
| 03 | 14 | 1 10 010 0010 01010 101110 1001110 |
| 04 | 14 | 1 10 010 1010 01010 011010 1001011 |
| 05 | 14 | 1 10 010 0010 00010 101110 0111011 |
| 06 | 14 | 1 10 010 0010 00010 011110 1011011 |
| 07 | 14 | 1 10 010 1000 10100 101010 0011111 |

08 14 1 10 010 1000 01000 111100 1100111
09 14 1 10 010 1000 01000 111010 1101011
10 14 1 10 010 1100 11000 110010 1100110

Using the Seidel spectrum, it is also easy to check that the self-complementary graphs from List 2 (01 through 10) belong to the classes with ordering numbers 15, 12, 16, 21, 18, 19, 18, 12, 15 and 18, respectively.

## References

[1] Cvetković, D., Doob, M., Sachs, H., Spectra of graphs – Theory and applications, 3rd revised and enlarged edition, J.A. Barth Verlag, Heidelberg – Leipzig, 1995

[2] Cvetković, D., Petrić, M., A table of connected graphs on six vertices, Discrete Math. 50 (1984), 37-49