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ABOUT SOME GENERALIZATIONS OF THE SQUARE ROOT FUNCTIONS USING A SIMPLE VECTOR CHAIN

Petar R. Lazov, Aneta L. Buchkovska
Faculty of Electrical Engineering
p.o.box 574, 91000 Skopje, Macedonia

Abstract

We report on some generalizations of the square root functions. We correspond the classical results for recurrence of vector chain with results obtained from the ergodic theory, to prove the main results of this paper. We will prove that the series $S(\frac{x}{4};y)$ converges in the region

$$0 < x \leq 4, \quad 0 < y \leq \left(\frac{2-\sqrt{x}}{\sqrt{x}}\right)^2.$$

Also we prove that the following holds

$$t^2 + (t^2 - 2t^3 + t^4) + (t^2 - 4t^3 + 7t^4 - 6t^5 + 2t^6) + \cdots = t, \quad 0 < t < 1.$$

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1. Introduction and main results

1. If for $a > 0, b > 0$, we denote:

$$(I) \qquad S(a;b) = ab + \sum_{n=2}^{\infty} \frac{1}{n} \cdot a^n \sum_{k=1}^{n-1} C_{n-2}^{k-1} \cdot C_n^k \cdot b^k,$$

then, using the identity

$$\sum_{k=1}^{n-1} C_{n-2}^{k-1} \cdot C_n^k = C_{2n-2}^{n-1} = -\frac{n}{2}(-4)^n \cdot \binom{\frac{1}{2}}{n},$$

for $b = 1$ we obtain

$$(1.1) \quad 1 - 2S(a;1) = 1 - 2 \cdot \left(a + \sum_{n=2}^{\infty} \frac{1}{n} \cdot a^n \sum_{k=1}^{n-1} C_{n-2}^{k-1} \cdot C_n^k \right) = 1 + \sum_{n=1}^{\infty} \binom{\frac{1}{2}}{n} (-4a)^n = \sqrt{1-4a}, \ 0 < a < \frac{1}{4}.$$

Thus, the series $1 - 2S(\frac{x}{4}; y)$, $x, y > 0$, can be considered as a generalization of the function $\sqrt{1-x}$, being equal to this function for $y = 1$, and $0 < x \leq 1$. In the following theorem we present some properties of this series.

Theorem 1. *The series $S(\frac{x}{4}; y)$ converges in the region*

$$(1.2) \qquad 0 < x \leq 4, \quad 0 < y \leq \left(\frac{2-\sqrt{x}}{\sqrt{x}}\right)^2$$

such that (Fig. 1)

$$1 - 2S(\frac{x}{4}; 1) = \sqrt{1-x}, \quad \textit{for} \quad 0 < x \leq 1,$$

$$(1.2) \qquad 1 - 2S\left(\frac{x}{4}; \left(\frac{2-\sqrt{x}}{\sqrt{x}}\right)^2\right) = \sqrt{x} - 1, \quad \textit{for} \quad 0 < x \leq 4,$$

and

$$(1.3) \qquad 1 - 2S\left(\frac{x}{4}; y\right) \geq -\frac{x}{4}(y-1) + \sqrt{\left[1 - \frac{x}{4}(y-1)\right]^2 - x},$$

for

$$0 < x < 4, \quad 0 < y < \left(\frac{2-\sqrt{x}}{\sqrt{x}}\right)^2.$$

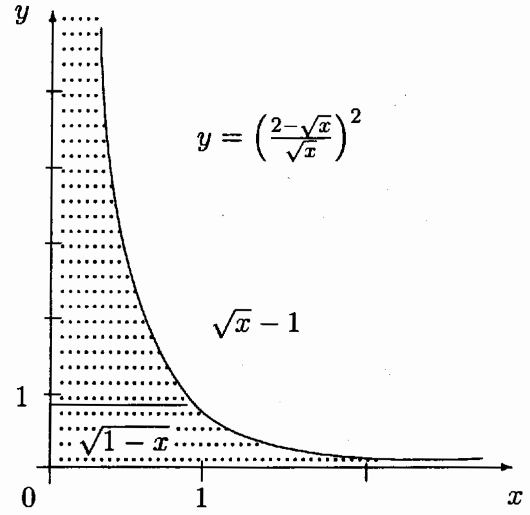

Fig. 1: The shaded region is the region (1.2) The bold curves denote the series, whose elementary functions $\sqrt{1-x}$ and $\sqrt{x}-1$ are its sum

Using some elementary transformations, from (1.3) we obtain:

Corollary. *If we denote*

$$\alpha_{n,2n-k} = \frac{1}{n} \sum_{i=\left[\frac{k+1}{2}\right]}^{n-1} C_{n-2}^{i-1} C_n^i C_{2i}^k, \quad k = 0, 1, ..., 2n-2,$$

then

$$\sum_{n=2}^{\infty} (\alpha_{n,2}t^2 - \alpha_{n,3}t^3 + \alpha_{n,4}t^4 - \cdots + \alpha_{n,2n}t^{2n}) = t - t^2, \quad 0 < t < 1,$$

i.e. it holds that

$$t^2 + (t^2 - 2t^3 + t^4) + (t^2 - 4t^3 + 7t^4 - 6t^5 + 2t^6) + \cdots = t, \quad 0 < t < 1.$$

It is clear that if the parentheses in the last equation are dropped, the obtained series diverges for every $t \in (0, 1)$.

2. If for $a > 0, b > 0$, we denote

$$(II) \qquad Q(a;b) = \sum_{n=1}^{\infty} a^n \sum_{k=1}^{n} C_{n-1}^{k-1} C_n^k \cdot b^k,$$

we conclude that

$$1+2Q\left(\frac{x}{4};1\right)=\frac{1}{\sqrt{1-x}},\quad 0<x<1. \tag{1.5}$$

Thus, the series $1+2Q(\frac{x}{4};y)$ for $x>0, y>0$, can be considered as a generalization of the function $\frac{1}{\sqrt{1-x}}$, being equal to this function for $y=1$, and $0<x<1$. In the following theorem we treat the problem of convergence of this series.

Theorem 2. *The series $Q(\frac{x}{4};y)$ converges for*

$$0<x<1,\quad 0<y<\left(\frac{2-\sqrt{x}}{\sqrt{x}}\right)^2, \tag{1.6}$$

and diverges for

$$0<x<4,\quad y=\left(\frac{2-\sqrt{x}}{\sqrt{x}}\right)^2. \tag{1.7}$$

It is clear that the series $Q(\frac{x}{4};y)$ also diverges for $y>\left(\frac{2-\sqrt{x}}{\sqrt{x}}\right)^2$.

3. Using (1.1) and (1.5), it follows that

$$\left[1+2Q\left(\frac{x}{4};1\right)\right]\cdot\left[1-2S\left(\frac{x}{4};1\right)\right]=1,\quad \textit{for}\quad 0<x<1.$$

The following theorem extends this result in the region (1.6).

Theorem 3.

(a) For every point (x,y) in the open region (1.6) it holds that:

$$\left[\frac{x}{4}y-S^2\left(\frac{x}{4};y\right)\right]\cdot\left[1+2Q\left(\frac{x}{4};y\right)\right]=\frac{x}{4}y+S^2\left(\frac{x}{4};y\right). \tag{1.8}$$

(b) If for $a>0, b>0$, we use the denotation

$$C(a;b)=\sum_{n=1}^{\infty}a^n\sum_{k=1}^{n}(C_{n-1}^{k-1})^2b^k, \tag{1.9}$$

then the series $C(\frac{x}{4};y)$ also converges in the region (1.6), so that for every point (x,y) in this region, the following equality holds:

$$\left[1+2Q\left(\frac{x}{4};y\right)\right]\cdot\left[1-2S\left(\frac{x}{4};y\right)\right]=1+Q\left(\frac{x}{4};y\right)-2C\left(\frac{x}{4};y\right). \tag{1.10}$$

From (1.9) it follows that:

$$C\Big(\frac{x}{4};1\Big) = \frac{x}{4\sqrt{1-x}}, \quad 0 < x < 1.$$

Thus, the series $C(\frac{x}{4};y)$ can be considered as a generalization of the function $\frac{x}{4\sqrt{1-x}}$ being equal to this function for $y = 1$, and $0 < x < 1$. Note that this generalization is different from the generalization that can be obtained from (1.5): $\frac{x}{4}\Big[1 + 2Q(\frac{x}{4};y)\Big]$.

4. As far as we know, the method we used to prove the results presented in Theorems 1-3 and Corollary are not used before. Namely, first we calculate the basic probabilities of a simple vector chain, and utilize the correspondence between the vector chain and the random walk of a second order on a line. Then, we correspond the classical results for recurrence of vector chain [1] with results obtained from ergodic theory [2], to prove the main results of this paper.

2. A two dimensional vector chain

1. Let the sequence of random vectors $\{Z_s, s \geq 0\}$, with possible values

$$\vec{i} = (i, i+1) \quad or \quad \vec{i} = (i, i-1), \quad i = 0, \pm 1, \pm 2, ..., \tag{2.1}$$

be a two dimensional Markov chain (Fig. 2), with one-step transition probabilities:

$$r_{i-1,i;i,j} = P\{Z_s = (i,j)|Z_{s-1} = (i-1,i)\} = \begin{cases} p_1, & j = i+1 \\ q_1, & j = i-1 \end{cases}$$

$$r_{i+1,i;i,j} = P\{Z_s = (i,j)|Z_{s-1} = (i+1,i)\} = \begin{cases} p_2, & j = i+1 \\ q_2, & j = i-1 \end{cases} \tag{2.2}$$

$$r_{i+1,i;\vec{j}} = 0 \quad if \quad \vec{j} \neq (i, i-1), (i, i+1)$$

where

$$0 < p_1, p_2 < 1, \quad p_1 + q_1 = 1, \quad p_2 + q_2 = 1. \tag{2.3}$$

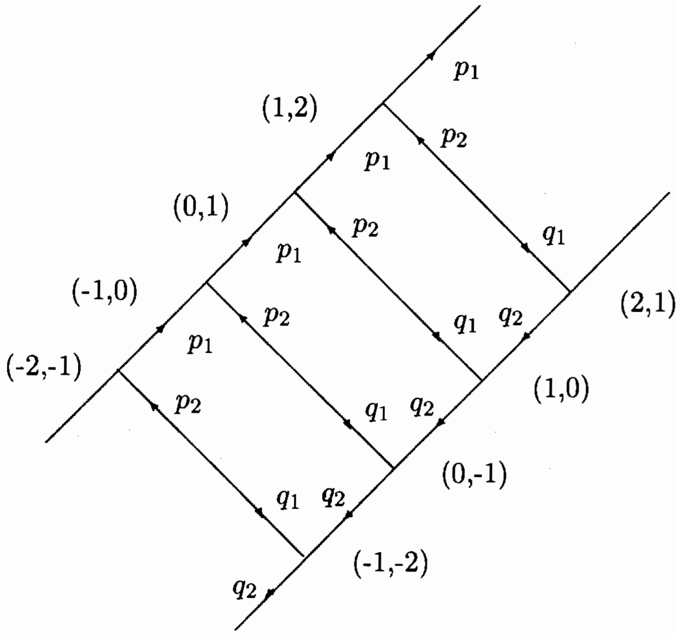

Fig. 2: The graph of vector-chain $\{Z_s, s \geq 0\}$

We denote $r^{(n)}_{\vec{i};\vec{j}}$ for the n-th step transition probabilities (where $\vec{i}$ and $\vec{j}$ have the form (2.1)). It is clear that the chain $\{Z_s, s \geq 0\}$ forms a class of essential states, with the period two, and

$$r^{(2n-1)}_{\vec{i};\vec{i}} = 0, \quad r^{(2n)}_{i,i+1;i+1,i} = 0, \quad n \geq 1.$$

Lemma 1. *For every state $\vec{i}$ of the chain $\{Z_s, s \geq 0\}$ it holds that:*

$$r^{(2n)}_{\vec{i};\vec{i}} = (p_1q_2)^n \sum_{k=1}^{n} C^{k-1}_{n-1} C^{k}_{n} \Big(\frac{p_2q_1}{p_1q_2}\Big)^k, \tag{2.4}$$

$$r^{(2n-1)}_{i,i+1;i+1,i} = \frac{1}{p_2}(p_1q_2)^n \sum_{k=1}^{n} (C^{k-1}_{n-1})^2 \Big(\frac{p_2q_1}{p_1q_2}\Big)^k, \tag{2.5}$$

$$r^{(2n)}_{i-1,i;i+1,i} = \frac{p_1}{p_2}(p_1q_2)^n \sum_{k=1}^{n} C^{k-1}_{n-1} C^{k}_{n} \Big(\frac{p_2q_1}{p_1q_2}\Big)^k, \tag{2.6}$$

$$(2.7)\qquad r^{(2n)}_{i+1,i;i-1,i} = \frac{q_2}{q_1}(p_1q_2)^n \sum_{k=1}^{n} C^{k-1}_{n-1} C^k_n \left(\frac{p_2q_1}{p_1q_2}\right)^k.$$

2. If we denote

$$(2.8)\quad f^{(m)}_{\vec{i};\vec{j}} = P\{Z_{s+m} = \vec{j}, Z_{s+k} \neq \vec{j},\quad k = 1,2,...,m-1 | Z_s = \vec{i}\},\quad m \geq 1,$$

where $\vec{i}$ and $\vec{j}$ are arbitrary states of the chain $\{Z_s, s \geq 0\}$, then

$$(2.9)\qquad f_{\vec{i};\vec{j}} = P\{Z_{s+N} = \vec{j} \quad for \quad some \quad N \geq 1 | Z_s = \vec{i}\} = \sum_{m=1}^{\infty} f^{(m)}_{\vec{i};\vec{j}},$$

$$(2.10)\qquad f^{(2m)}_{i,i+1;i+1,i} = f^{(2m)}_{i+1,i;i,i+1} = 0, \quad m = 1, 2, ...$$

and it is easy to check that

$$(2.11)\qquad f_{i,i+1;i,i+1} = f_{i+1,i;i+1,i} = f_{i,i+1;i+1,i} \cdot f_{i+1,i;i,i+1}.$$

Lemma 2.
(2.12)

$$p_2 \cdot f^{(2n-1)}_{i,i+1;i+1,i} = q_1 \cdot f^{(2n-1)}_{i+1,i;i,i+1} = \frac{1}{n}(p_1q_2)^n \sum_{k=1}^{n-1} C^{k-1}_{n-2} \cdot C^k_n \cdot \left(\frac{p_2q_1}{p_1q_2}\right)^k, \quad n \geq 2,$$

$$(2.13)\qquad f_{i,i+1;i+1,i} = \frac{1}{p_2}\sigma(p_1, p_2), \quad f_{i+1,i;i,i+1} = \frac{1}{q_1}\sigma(p_1, p_2),$$

where

$$(2.14)\qquad \sigma(p_1, p_2) = p_2q_1 + \sum_{n=2}^{\infty} \frac{1}{n}(p_1q_2)^n \sum_{k=1}^{n-1} C^{k-1}_{n-2} \cdot C^k_n \cdot \left(\frac{p_2q_1}{p_1q_2}\right)^k,$$

$$(2.15)\qquad f_{\vec{i};\vec{i}} = \frac{1}{p_2q_1}\sigma^2(p_1, p_2).$$

Proof. It is sufficient to prove that the product $p_2 \cdot f^{(2n-1)}_{i,i+1;i+1,i}$ is equal to the sum of the right-hand side of (2.12). Therefore, using symmetry properties, it follows that the product $q_1 \cdot f^{(2n-1)}_{i+1,i;i,i+1}$ is given by the same sum. Further on, the equalities (2.13) directly follow from (2.9), (2.10) and (2.12), i.e. the equality (2.15) from (2.13) and (2.11). For $n = 2$, we immediately conclude that

$$f^{(3)}_{i,i+1;i+1,i} = p_1 q_1 q_2 = \frac{1}{p_2}(p_1 q_2)(p_2 q_1). \tag{2.16}$$

The probability $f^{(2n-1)}_{i,i+1;i+1,i}$ is the sum of probabilities of every two disjoint events, corresponding to the certain path of the graph in Fig. 2. Each path, that according to (2.11) in the first $2n-2$ steps does not pass through $(i+1,i)$, contains $2n-1$ branches, whose beginning is in $(i,i+1)$, and end in $(i+1,i)$. The branches of the type $(j,j+1) \to (j+1,j)$ are called q_1-branches. We can also define p_2, q_2 and p_1-branches. Let us consider those possible paths with $2n-1$ branches from $(i,i+1)$ into $(i+1,i)$ (of the mentioned type) which contain k, q_1-branches. It is clear that $k \geq 1$. It is obvious that these paths contain $(k-1)$, p_2-branches. Consequently, we obtain that $k \leq n-1$. Let us assume that $n \geq 3$ and $k \geq 2$, then the probability of a such possible path has the following form:

$$(p_1^{r_1} q_1 q_2^{s_1-1} p_2)(p_1^{r_2-1} q_1 q_2^{s_2-1} p_2) \cdots (p_1^{r_{k-1}-1} q_1 q_2^{s_{k-1}-1} p_2)(p_1^{r_k-1} q_1 q_2^{s_k}) =$$

$$= \frac{1}{p_2}(p_2 q_1)^k p_1^{(r_1+\cdots+r_k)-(k-1)} q_2^{(s_1+\cdots+s_k)-(k-1)}, \tag{2.17}$$

where

$$r_i, s_i \geq 1, \quad i = 1, 2, \ldots \tag{2.18}$$

and

$$\begin{aligned} & r_1 + 1 > s_1, \\ & (r_1+1) + r_2 > s_1 + s_2, \\ & \cdots\cdots\cdots \\ & (r_1+1) + r_2 + \cdots + r_{k-1} > s_1 + s_2 + \cdots + s_{k-1}, \end{aligned} \tag{2.19}$$

and

$$\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} s_i.$$

Since

$$\Big(\sum_{i=1}^{k} r_i - (k-1)\Big) + k + \Big(\sum_{i=1}^{k} s_i - (k-1)\Big) + (k-1) = 2n-1,$$

it follows that

$$\sum_{i=1}^{k} r_i = \sum_{i=1}^{k} s_i = n - 1. \tag{2.20}$$

Thus, for the probability in (2.17) we obtain:

$$\frac{1}{p_2}(p_1 q_2)^{n-k}(p_2 q_1)^k \tag{2.21}$$

which does not depend on the numbers r_i and s_i. Considering the number - sequences $(r_1+1, r_1+1+r_2, \dots, r_1+1+r_2+\cdots+r_{k-1})$ and $(s_1, s_1+s_2, \dots, s_1+s_2+\cdots+s_{k-1})$, it is not difficult to notice that the total number of ways one can choose $2k$ numbers $r_1, r_2, \dots, r_k, s_1, s_2, \dots, s_k$, satisfying the conditions (2.18), (2.19) and (2.20), is $\Phi_{k-1}^{(n-2;n-1)}$, where the number $\Phi_k^{(N;M)}$ can be easily computed. Therefore, and according to (2.21), the set of all possible paths of that type, which contains k, q_1-branches for $k \geq 2$, has the following probability

$$\Phi_{k-1}^{(n-2;n-1)} \frac{1}{p_2}(p_1 q_2)^{n-k}(p_2 q_1)^k = \frac{1}{np_2} C_{n-2}^{k-1} C_n^k (p_1 q_2)^{n-k}(p_2 q_1)^k. \tag{2.22}$$

For $k = 1$ (then $r_1 = s_1 = n-1$), there is a unique possible path with the probability $(p_1 q_2)^{n-1} q_1$, and so (2.22) holds for $k = 1$, too. Thus, summing along all the possible values for k, for $n \geq 3$, we conclude that

$$f_{i,i+1;i+1,i}^{(2n-1)} = \frac{1}{np_2}(p_1 q_2)^n \sum_{k=1}^{n-1} C_{n-2}^{k-1} C_n^k \left(\frac{p_2 q_1}{p_1 q_2}\right)^k. \tag{2.23}$$

Using (2.16), (2.23) holds for $n = 2$ as well. This, completes the proof.

Remark. The proof of Lemma 1 is simpler than the proof of Lemma 2., since we do not have to use the combinatory result given in the appendix. Thus, for example, in a similar way as done in Lemma 2 one can find that the probability $r_{i+1,i;i-1,i}^{(2n)}$ is a sum of the probabilities:

$$(p_1^{r_1} q_1 q_2^{s_1-1} p_2)(p_1^{r_2-1} q_1 q_2^{s_2-1} p_2) \cdots (p_1^{r_{k-1}-1} q_1 q_2^{s_{k-1}-1} p_2)(p_1^{r_k-1} q_1 q_2^{s_k} p_2) p_1^{r_{k+1}-1} =$$
$$= (p_1 q_2)^{n-k}(p_2 q_1)^k.$$

Since the number n can be represented as a sum of μ numbers, $1 \leq \mu \leq m$, in a $C_{m-2}^{\mu-1}$ ways, we easily obtain the expression (2.4) for the considered probabilities.

3. Let us consider the discrete second-order random walk along a straight line, that is an integer valued sequence $\{X_s, s \geq 0\}$ of a random variable for which:

$$P\{X_{s+1} = j | X_s = i, X_{s-1} = i \mp 1, ...\} =$$

$$= P\{X_{s+1} = j | X_s = i, X_{s-1} = i \mp 1\} = \begin{cases} p_1, (p_2), & j = i+1 \\ q_1, (q_2), & j = i-1 \\ 0, & j \neq i \pm 1, \end{cases}$$

$$P\{X_s = l, X_{s+1} = k\} = 0, \quad \text{for } k \neq l \pm 1.$$

This walk is a special homogeneous chain of the second order. It is easy to see that the considered vector chain $\{Z_s, s \geq 0\}$ is corresponding to this second order walk. Therefore, from [2], Theorem 3 it follows that the condition for recurrence of the vector chain $\{Z_s, s \geq 0\}$ can be expressed as in the following lemma.

Lemma 3. *The vector chain* $\{Z_s, s \geq 0\}$ *is recurrent iff*

$$p_1 + p_2 = 1. \tag{2.24}$$

3. The proof of the main results

1. First of all, we will prove the Theorem 2.. Using the well known theorems [1,pgs.262-267], the vector chain is recurrent iff the series

$$\sum_{\nu=1}^{\infty} r_{\bar{i};\bar{i}}^{(\nu)} = \sum_{n=1}^{\infty} (p_1 q_2)^n \sum_{k=1}^{n} C_{n-1}^{k-1} C_n^k \left(\frac{p_2 q_1}{p_1 q_2}\right)^k \tag{3.1}$$

diverges ((2.4) is used). If we interchange the variables

$$a = p_1 q_2, \quad b = \frac{p_2 q_1}{p_1 q_2}, \tag{3.2}$$

the open square

$$0 < p_1 < 1, \quad 0 < p_2 < 1, \tag{3.3}$$

is mapped on the region

$$0 < a < 1, \quad 0 < b \leq \left(\frac{1-\sqrt{a}}{\sqrt{a}}\right)^2, \tag{3.4}$$

such that the segment (2.24) and only it, is mapped on the curve

$$0 < a < 1, \quad b = \left(\frac{1-\sqrt{a}}{\sqrt{a}}\right)^2. \tag{3.5}$$

In this case, the series (3.1) is transformed in the form (II). Now from Lemma 3 it follows that the series (II) converges in the region

$$0 < a < 1, \quad 0 < b < \left(\frac{1-\sqrt{a}}{\sqrt{a}}\right)^2, \tag{3.6}$$

while on the curve (3.5), the series (II) diverges.

2. To prove Theorem 1, let us note first that for each point (a,b) of the region (3.4) there exist two pairs (p'_1, p'_2) and (p''_1, p''_2) for each of them (3.3) and (3.2) are true:

$$p'_1 = x_1, \quad p'_2 = 1 - \frac{a}{x_1} = 1 - x_2,$$

$$p''_1 = x_2, \quad p''_2 = 1 - \frac{a}{x_2} = 1 - x_1$$

where x_1 is the smaller, and the x_2 is the greater root of the equation

$$x^2 - [1 - a(b-1)]x + a = 0 \tag{3.7}$$

($x_1 = x_2$ iff (a,b) is on the curve (3.5)). In that case, according to (I) and (2.14) we get

$$S(a,b) = \sigma(p'_1, p'_2) = \sigma(p''_1, p''_2).$$

On the other hand, since $f_{i+1,i;i,i+1} \leq 1$, using the second relation in (2.13), it follows that

$$\sigma(p''_1, p''_2) \leq q''_1 = 1 - p''_1,$$

so that

$$S(a,b) \leq 1 - \frac{x}{2} \tag{3.8}$$

(since $p'_2 = 1 - p''_1$, using the first relation in (2.13), we obtain the same estimation (3.8)). If the point (a,b) is on the curve (3.5), then $x_1 = x_2$, $p'_1 + p'_2 = 1$ and using Lemma 3, it follows that:

$$S\left(a; \left(\frac{1-\sqrt{a}}{\sqrt{a}}\right)^2\right) = 1 - \sqrt{a}, \quad 0 < a < 1.$$

To finish the proof, we use (1.1) and, according to (3.7), we obtain the following relation:

$$2x_2 - 1 = -a(b-1) + \sqrt{(1-ab+a)^2 - 4a}.$$

3. Finally, we prove Theorem 3. We have already seen that if the point (a, b) belongs to (3.6), then there exist two numbers p_1 and p_2 in the open square (3.3), such that (3.2) is satisfied; in that case

$$p_1 + p_2 \neq 1. \tag{3.9}$$

Lemma 3 implies that the chain $\{Z_s, s \geq 0\}$ is transient. Since the relationship between the generic functions for the probabilities $r^{(l)}_{\vec{i};\vec{i}}$ and $f^{(l)}_{\vec{i};\vec{i}}$ are known, using the general recurrent theorem, it follows that for every state $\vec{i}$:

$$\sum_{l=0}^{\infty} r^{(l)}_{\vec{i};\vec{i}} = 1 + f_{\vec{i};\vec{i}} \cdot \sum_{l=0}^{\infty} r^{(l)}_{\vec{i};\vec{i}},$$

that is, according to (2.4), (2.15) and (3.2),

$$1 + Q(a;b) = 1 + F(a;b)(1 + Q(a;b))$$

or

$$Q(a;b) = F(a;b) \cdot Q(a;b) + F(a;b), \tag{3.10}$$

where

$$F(a;b) = \frac{1}{ab} S^2(a;b). \tag{3.11}$$

From (3.10), we easily obtain that

$$[1 - F(a;b)] \cdot [1 + 2Q(a;b)] = 1 + F(a;b).$$

Considering (3.11) and the latest equality, it follows that:

$$[ab - S^2(a;b)] \cdot [1 + 2Q(a;b)] = ab + S^2(a;b).$$

Further on, using (2.5), (3.2) and (1.9), we have:

$$\sum_{l=1}^{\infty} r^{(l)}_{i,i+1;i+1,i} = f_{i,i+1;i+1,i} \cdot \sum_{l=1}^{\infty} r^{(l)}_{i+1,i;i+1,i}. \tag{3.12}$$

Using this relation, and considering (3.12), (2.13), (2.4) and (3.2) we obtain

$$C(a;b) = S(a;b)(1 + Q(a;b)),$$

that is

$$[1 + Q(a;b)] \cdot [1 - 2S(a;b)] = 1 + Q(a;b) - 2C(a;b).$$

Thus, the proof of the theorem is complete.

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APPENDIX

Let $\Phi_k^{(N;M)}$, $k \le N \le M$ and $M \ge 2$, be the number of pairs (ν, μ), where $\nu = (\nu_1, \dots, \nu_k)$ and $\mu = (\mu_1, \dots, \mu_k)$ are k - dimensional integer - valued vectors, such that:

$$1 \le \nu_1 < \dots < \nu_k \le N,$$

$$1 \le \mu_1 < \dots < \mu_k \le M,$$

$$\nu_1 < \mu_1, \quad \nu_2 < \mu_2, \dots, \nu_k < \mu_k.$$

Then

$$(A.1) \qquad \Phi_k^{(N;M)} = \frac{1}{k(k+1)} C_N^k \cdot C_{M-1}^{k-1} \cdot [(k+1)M - k(N+1)].$$

In particular,

$$(A.2) \qquad \Phi_k^{(N;N+1)} = \frac{1}{k+1} C_N^k \cdot C_{N+1}^k = \frac{1}{N+2} C_N^k \cdot C_{N+2}^{k+1}.$$

Proof. For $k = 1$, $(A.1)$ holds directly. The induction method will be applied using recurrent relation

$$(A.3) \quad \Phi_k^{(N;M)} = \sum_{k \le \nu_k \le N} \left\{ \Phi_{k-1}^{(\nu_k - 1;\nu_k)} + \Phi_{k-1}^{(\nu_k - 1;\nu_k+1)} + \dots + \Phi_{k-1}^{(\nu_k - 1;M-1)} \right\}$$

which can be easily verified.

Since

$$\xi_k \equiv \nu_k \left(C_{\nu_k-1}^{k-2} + \cdots + C_{M-2}^{k-2}\right) + k\left(C_{\nu_k}^{k-2} + 2C_{\nu_k+1}^{k-2} + \cdots + (M-1-\nu_k)C_{M-2}^{k-2}\right) =$$

$$= \nu_k \left(C_{M-1}^{k-1} - C_{\nu_k-1}^{k-1}\right) + k\left((M-1-\nu_k)C_{M-1}^{k-1} - C_{M-1}^{k} + C_{\nu_k}^{k}\right) =$$

$$= (k-1)\,(M-\nu_k)\,C_{M-1}^{k-1},$$

(where we use that $\nu_k C_{\nu_k-1}^{k-1} = kC_{\nu_k}^{k}$, $kC_{M-1}^{k} = (M-k)C_{M-1}^{k-1}$), then, assuming that $(A.1)$ holds for $k-1$, from $(A.3)$ we have:

$$\Phi_k^{(N;M)} = \frac{1}{(k-1)k} \sum_{k \le \nu_k \le N} C_{\nu_k-1}^{k-1} \Big\{ C_{\nu_k-1}^{k-2}\left[k\nu_k - (k-1)\nu_k\right] + \cdots +$$

$$+ C_{M-2}^{k-2}\left[k(M-1) - (k-1)\nu_k\right] \Big\} = \frac{1}{(k-1)k} \sum_{k \le \nu_k \le N} C_{\nu_k-1}^{k-1}\xi_k =$$

$$= \frac{1}{k} C_{M-1}^{k-1} \left(M \sum_{k \le \nu_k \le N} C_{\nu_k-1}^{k-1} - k \sum_{k \le \nu_k \le N} C_{\nu_k}^{k} \right) =$$

$$= \frac{1}{k} C_{M-1}^{k-1} \left(MC_N^k - kC_{N+1}^{k+1} \right) = \frac{1}{k(k+1)} C_{M-1}^{k-1} C_N^k \left[M(k+1) - k(N+1)\right].$$

This completes the proof.