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NOTES ON FRACTAL INTERPOLATION

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Abstract

It is known that a class of fractal functions $f : I(\subset \mathbf{R}) \to \mathbf{R}$ can be defined using an IFS (Iterated Function System) code. Moreover, these functions can be constructed to interpolate the data set $Y = \{(x_i, y_i)\}_{i=0}^{n} \subset \mathbf{R}^2$. Here, affine transformations (in $\mathbf{R}^2$) of such functions (defined as affine mappings of their graphs) are examined. Particularly, it is shown that fractal interpolating functions are affine invariant only upon the class of affine mappings whose linear part is given by a lower-triangular matrix. Also, it is proved that the fractal interpolation scheme is a linear operator and can be written in the *Lagrange form*.

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1. Introduction

Interpolation of the $\mathbf{R}^2$-data by means of smooth functions (polynomials, splines) is a widely studied and well settled topic. On the other hand, some applications require nonsmooth and irregular interpolants. Fractal functions, namely the functions whose graphs are fractal sets [4], offer an adequate tool in such cases. Furthermore, they can be easily constructed using Iterated Function Systems [1], although their properties cannot be estabilished as easily. Some shape preserving properties of fractal interpolating schemes are investigated in [2]. The aim of this paper is to estabilish further properties, such as linearity and affine invariance, the latter being especially important for graphical applications and CAGD [3].

A fractal interpolatory scheme can be introduced in the following way.

Let $Y = \{(x_i, y_i)\}_{i=0}^{n}$, $n \geq 2$, $x_0 \neq x_n$, be a set of points in $\mathbf{R}^2$, and let $\Delta x_i = x_{i+1} - x_i$.

Definition 1. *We call Y a proper set of interpolating data if the sequence $\{\Delta x_i\}_{i=0}^{n-1}$ does not change the sign in the strong sense and has at least two nonzero elements.*

With the proper set of data Y and the vector $\mathbf{d} = [d_1 \ldots d_n]^T$, $|d_i| < 1$, one can associate the hyperbolic IFS $\sigma(Y, \mathbf{d}) = \{\mathbf{R}^2; w_1, \ldots, w_n\}$, where each w_i is the affine transformation $\mathbf{R}^2 \to \mathbf{R}^2$ given by

$$w_i(\mathbf{x}) = \begin{bmatrix} a_i & 0 \\ c_i & d_i \end{bmatrix} \mathbf{x} + \begin{bmatrix} e_i \\ f_i \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \tag{1}$$

with

$$a_i = \frac{\Delta x_{i-1}}{x_n - x_0}, \qquad c_i = \frac{\Delta y_{i-1}}{x_n - x_0} - d_i \frac{y_n - y_0}{x_n - x_0},$$

$$e_i = x_i - a_i x_n, \qquad f_i = y_i - c_i x_n - d_i y_n.$$

The vector $\mathbf{d}$ is called *vertical scaling vector*.

If θ is chosen such that $0 < \theta < \min_i((1 - |a_i|)/(1 + |c_i|)$, w_i is a contraction in the norm $\|\mathbf{x}\|_\theta = |\mathbf{x}| + \theta|\mathbf{y}|$ since, obviously, $0 \leq a_i < 1$, and therefore $\sigma(Y, \mathbf{d})$ is hyperbolic.

Let $\mathcal{H}(\mathbf{R}^2)$ be the set of nonempty compact subsets of $\mathbf{R}^2$ and h_θ the Hausdorff metric on $\mathcal{H}(\mathbf{R}^2)$ generated by $\|\cdot\|_\theta$. Let $W : \mathcal{H}(\mathbf{R}^2) \mapsto \mathcal{H}(\mathbf{R}^2)$ (so called Hutchinson operator) be defined by

$$W(\cdot) = \cup_1^n w_i(\cdot). \tag{2}$$

Then W is a contraction of the complete metric space $(\mathcal{H}(\mathbf{R}^2), h_\theta)$. Its fixed point $F_Y \subset \mathbf{R}^2$, called the *attractor* of the IFS $\sigma(Y, \mathbf{d})$, is the graph of a continuous function $f : [x_0, x_n] \to \mathbf{R}$ that interpolates the data set Y ([1], [4]), so we can refer to $\sigma(Y, \mathbf{d})$ as an interpolatory scheme.

2. Affine invariance of the scheme

Let ω be a nonsingular affine mapping $\mathbf{R}^2 \to \mathbf{R}^2$ given by

$$(3) \qquad \omega(\mathbf{x}) = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \mathbf{x} + \begin{bmatrix} g \\ h \end{bmatrix}, \qquad p, q, r, s, g, h \in \mathbf{R}, \quad \begin{vmatrix} p & q \\ r & s \end{vmatrix} \neq 0.$$

Recall that a regular linear transformation ($g = h = 0$) or a translation ($q = r = 0, p = s = 1, |g| + |h| > 0$) are special cases of (3), and, in turn, any affine transformation can be seen as a composition of a linear transformation and a translation.

Definition 2. *The mapping ω given by (3) is feasible for a given proper data set Y, if $\omega(Y) = \{(x_i', y_i')\}_{i=0}^n$ is also a proper data set.*

Consider the sequence $\{L_i\}_{i=0}^{+\infty}$ of polygonal lines such that L_0 is the polygonal interpolation of Y, $L_1 = W(L_0)$,..., $L_k = W(L_{k-1})$,... , where W is given by (2). By definition of attractor ([4]), $F_Y = \lim_{k\to\infty} L_k$ and, by continuity of ω, $\omega(F_Y) = \lim_{k\to\infty} \omega(L_k)$, where the limiting process is taken in the Hausdorff metric.

Since a feasible mapping ω transforms the given data set $Y = \{(x_i, y_i)\}_{i=0}^n$ into the new data set $\omega(Y) = \{\omega([x_i\, y_i]^T)\}_{i=0}^n = \{(px_i + qy_i + g,\, rx_i + sy_i + h)\}_{i=0}^n$, a new IFS, corresponding to the transformed data set $\omega(Y)$ and the same scaling vector, is given by $\hat{\sigma} = \sigma(\omega(Y), \mathbf{d}) = \{\mathbf{R}^2; \hat{w}_1, \ldots, \hat{w}_n\}$ where

$$(4) \qquad \hat{w}_i(\mathbf{x}) = \begin{bmatrix} \hat{a}_i & 0 \\ \hat{c}_i & d_i \end{bmatrix} \mathbf{x} + \begin{bmatrix} \hat{e}_i \\ \hat{f}_i \end{bmatrix},$$

with

$$\hat{a}_i = \frac{p\Delta x_{i-1} + q\Delta y_{i-1}}{p(x_n - x_0) + q(y_n - y_0)},$$

$$(5)\qquad \hat{c}_i = \frac{r\Delta x_{i-1} + s\Delta y_{i-1}}{p(x_n - x_0) + q(y_n - y_0)} - d_i \frac{r(x_n - x_0) + s(y_n - y_0)}{p(x_n - x_0) + q(y_n - y_0)},$$

$$\hat{e}_i = px_i + qy_i + g - \hat{a}_i(px_n + qy_n + g),$$

$$\hat{f}_i = rx_i + sy_i + h - \hat{c}_i(px_n + qy_n + h) - d_i(rx_n + sy_n + h).$$

Note that $\hat{\sigma}$ is also a hyperbolic IFS since the $\mathbf{d}$ vector is unchanged and the denominators in (5) cannot vanish because of the regularity of ω.

If $L'_0 = \omega(L_0)$ and $L'_k = W'(L'_{k-1}) = \cup_{i=1}^{n} \hat{w}_i(L'_{k-1})$, $k = 1, 2, \dots$, then $F_{\omega(Y)} = \lim_{k\to\infty} L'_k$ is the attractor of $\hat{\sigma}$. The relationship between $F_{\omega(Y)}$ and F_Y is highlighted by the next theorem.

Consider the following diagram

$$\begin{array}{ccccc} L_k & \xrightarrow{\omega} & \omega(L_k) & = & L'_k \\ W \downarrow & & & & \downarrow W' \\ L_{k+1} & \xrightarrow{\omega} & \omega(L_{k+1}) & = & L'_{k+1} \end{array}$$

If, for all k, it is commutative, namely if $w_i \circ \omega = \omega \circ \hat{w}_i$, $\forall i$, then $L'_1 = \omega(L_1)$, and at every next step $L'_k = \omega(L_k)$, so that $F_{\omega(Y)} = \lim_{k\to\infty} L'_k = \lim_{k\to\infty} \omega(L_k) = \omega(F_Y)$.

The commutativity condition is given by the following theorem.

Theorem 1. *If $\omega(Y)$ denotes the image of the data set Y under a feasible transformation ω given by (3), then*

$$(6)\qquad \omega(F_Y) = F_{\omega(Y)}$$

if and only if $q = 0$.

Proof. First consider the regular linear case of (3).

Since w_i maps L_k into the i-th piece of L_{k+1}, then $\omega \circ w_i$ maps L_k into the i-th piece of $\omega(L_{k+1})$. By (1), this mapping is given by

$$(\omega \circ w_i)(\mathbf{x}) = \omega(w_i(\mathbf{x})) = \begin{bmatrix} pa_i + qc_i & qd_i \\ ra_i + sc_i & sd_i \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} e_i \\ f_i \end{bmatrix}$$

which yields

$$(\omega \circ w_i)(\mathbf{x}) = \begin{bmatrix} (pa_i + qc_i)(x - x_n) + qd_i(y - y_n) + px_i + qy_i \\ (ra_i + sc_i)(x - x_n) + sd_i(y - y_n) + rx_i + sy_i \end{bmatrix}$$

and by (4)

$$(\hat{w}_i \circ \omega)(\mathbf{x}) =$$

$$\begin{bmatrix} \hat{a}_i[p(x-x_n)+q(y-y_n)]+px_i+qy_i \\ \hat{c}_i[p(x-x_n)+q(y-y_n)]+d_i[r(x-x_n)+s(y-y_n)]+rx_i+sy_i \end{bmatrix}.$$

Now, $\hat{w}_i$ maps $\omega(L_k)$ into $\omega(L_{k+1})$ if and only if $\omega \circ w_i$ and $\hat{w}_i \circ \omega$ are identical mappings.

For any $i = 1, \dots, n$, denote by δ_i the difference

(7)

$$\delta_i = (\omega \circ w_i - \hat{w}_i \circ \omega)(\mathbf{x}) = \begin{bmatrix} [p(a_i-\hat{a}_i)+qc_i]\,(x-x_n)+q(d_i-\hat{a}_i)(y-y_n) \\ [r(a_i-d_i)+sc_i-p\hat{c}_i]\,(x-x_n)-q\hat{c}_i(y-y_n) \end{bmatrix},$$

and note that (6) is equivalent to $\delta_i = [0\ 0]^T$, $\forall i$, which holds if and only if

(8) $$p(a_i-\hat{a}_i)+qc_i=0\,, \quad q(d_i-\hat{a}_i)=0\,, \quad \forall i$$

and

(9) $$r(a_i-d_i)+sc_i-p\hat{c}_i=0\,, \quad q\hat{c}_i=0\,, \quad \forall i.$$

It is evident from (7) that $q=0$ implies $\delta_i = [0\ 0]^T$, $\forall i$, since $\hat{a}_i|_{q=0} = a_i$ and $\hat{c}_i|_{q=0} = r(a_i-d_i)/p + sc_i/p$. Suppose now $q \neq 0$. Then, by the second equation in (8) and (9), $\hat{c}_i = 0$, $\hat{a}_i = d_i$ so that (8) and (9) reduce to

$$p(a_i-d_i)+qc_i=0\,, \quad r(a_i-d_i)+sc_i=0\,.$$

The above system has nontrivial solution if and only if

$$\begin{vmatrix} p & q \\ r & s \end{vmatrix} = 0\,,$$

which, by supposition, never occurs. So, it must be $q = 0$.

Consider now the translation case, i.e.

$$\omega(\mathbf{x}) = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} g \\ h \end{bmatrix}, \quad |g|+|h|>0.$$

In this case $\hat{a}_i = a_i$ and $\hat{c}_i = c_i$, while

$$\hat{e}_i = e_i + (1-a_i)g\,, \quad \hat{f}_i = f_i + (1-d_i)h - c_ig\,,$$

which yields

$$\hat{w}_i(\mathbf{x}) = w_i(\mathbf{x}) + \begin{bmatrix} (1-a_i)g \\ (1-d_i)h - c_i g \end{bmatrix} = \begin{bmatrix} a_i(x-g) \\ c_i(x-g)+d_i(y-h) \end{bmatrix} + \begin{bmatrix} g \\ h \end{bmatrix}.$$

Therefore

$$\hat{w}_i(\mathbf{x}) = (w_i \circ \omega^{-1})(\mathbf{x}) + \begin{bmatrix} g \\ h \end{bmatrix} = (\omega \circ w_i \circ \omega^{-1})(\mathbf{x}),$$

which again leads to $\hat{w}_i \circ \omega = \omega \circ w_i$ and therefore to $\omega(F_Y) = F_{\omega(Y)}$. □

Remark. It should be noted that if ω is nonsingular, then $q = 0$ implies $p \neq 0$, which, in turn, guarantees feasibility of ω.

An important consequence of Theorem 1 is the *symmetry property* of the scheme $\sigma(Y, \mathbf{d})$.

Corollary 1. *(Symmetry). Let S be any subset of $\mathbf{R}^2$. Define its symmetric image with respect to the fixed line $\{(c,y) \,|\, y \in \mathbf{R}\}$ as $S^* = \{(2c - x, y) \,|\, (x,y) \in S\}$. Let Y be a proper set of data. Then,*

$$F_{Y^*} = (F_Y)^*.$$

Proof. From Theorem 1, for $p = -1$, $r = 0$, $s = 1$, $g = 2c$ and $h = 0$. □

3. Linearity property

In $\mathcal{H}(\mathbf{R}^2)$, define scalar multiplication by

$$\lambda S = \{(x, \lambda y) \,|\, (x,y) \in S, \quad S \in \mathcal{H}(\mathbf{R}^2), \ \lambda \in \mathbf{R}\}, \tag{10}$$

and addition by

$$S_1 + S_2 = \{(x, y_1 + y_2) \,|\, (x, y_1) \in S_1, \ (x, y_2) \in S_2, \quad S_1, S_2 \in \mathcal{H}(\mathbf{R}^2)\}. \tag{11}$$

Note that the affine transformation ω given by (3), that satisfies conditions from Theorem 1, can be put in the form

$$(x, y) \mapsto (u(x), v(x,y)), \tag{12}$$

where $u(x) = px + g$ and $v(x,y) = rx + sy + h$. Taking $p = 1$, $r = g = h = 0$ and $s \neq 0$, the affine transformation (12) turns to be a scalar multiplication $Y \mapsto sY$, $s \in \mathbf{R}$ which is a feasible transformation, so one has

Corollary 2. *(Homogeneity). If Y is a proper set of data, then*

$$F_{\lambda Y} = \lambda F_Y \quad \lambda \in \mathbf{R},$$

where the scalar multiplication is given by (10). *In other words, the interpolating scheme given by $\sigma(Y, \mathbf{d})$ is homogeneous.*

Now it will be shown that the same scheme has also additivity property.

Theorem 2. *(Additivity). Let Y_1 and Y_2 be two proper data sets defined on the same mesh $\{x_i\}_{i=0}^n$. Suppose that the corresponding IFS's, $\sigma(Y_1, \mathbf{d})$ and $\sigma(Y_2, \mathbf{d})$ have the same vertical scaling vector $\mathbf{d}$. Then,*

$$F_{Y_1+Y_2} = F_{Y_1} + F_{Y_2},$$

where addition is taken in the sense of relation (11).

Proof. Let $\sigma(Y_j, \mathbf{d}) = \{\mathbf{R}^2; w_1^{(j)}, \ldots, w_n^{(j)}\}$ be the IFS associated with the data sets $Y_j = \{(x_i, y_i^{(j)})\}_{i=0}^n$ for $j = 1, 2$. In the componentwise notation, the mappings $w_i^{(j)}$ have the form $(x, y) \mapsto (u_i^{(j)}(x), v_i^{(j)}(x, y))$.

It follows from Theorem 1 that the interpolation scheme $\sigma(Y, \mathbf{d})$ defined for any set of proper data Y is affine invariant with respect to x-axis mapping by any affine map $x \mapsto px + g$, $p, g \in \mathbf{R}$. In fact, this is a special case of Theorem 1, for $r = 0$, $s = 1$ and $h = 0$. Accordingly, we can consider only the y-data on the normalized mesh $|x_0 - x_n| = 1$. Also, we can restrict ourselves to the nondecreasing meshes (by the symmetry property, Corollary 1, the consideration can be extended to nonincreasing meshes). In this case, $u_i^{(1)}(x) = u_i^{(2)}(x) = (x_i - x_{i-1})(x - x_n) + x_i$. Suppose $Y = Y_1 + Y_2$, and let $\sigma(Y, \mathbf{d})$ be the associated IFS, containing affine mappings w_i having decomposition (u_i, v_i). Then, $u_i = u_i^{(1)} = u_i^{(2)}$. Also, it is easy to see that $v_i = v_i^{(1)} + v_i^{(2)}$. So, by the same arguments as in the proof of Theorem 1, we can claim that the sum of attractors produced by $\sigma(Y_1, \mathbf{d})$ and $\sigma(Y_2, \mathbf{d})$ is the attractor generated by $\sigma(Y, \mathbf{d})$. □

Corollary 3. *(Linearity). For any pair of proper data sets Y_1 and Y_2 and any real constants λ and μ*

$$F(\lambda Y_1 + \mu Y_2) = \lambda F(Y_1) + \mu F(Y_2).$$

The important consequence of linearity property is that the interpolation scheme $\sigma(Y, \mathbf{d})$ for any set of proper data can be written in the *Lagrange form*.

Corollary 4. *(Lagrange form). Let* $Y = \{(x_i, y_i)\}_{i=0}^n$ *be a proper set of interpolating data. Let* F_j *be the attractor of the IFS* $\sigma(Y_j, \mathbf{d})$*, associated with the sequence of data sets* $Y_j = \{(x_i, \delta_{ij})\}_{i=0}^n$*,* $j = 0, \dots, n$ *where* δ_{ij} *is Kronecker's delta. Then,*

$$F_Y = \sum_{j=0}^{n} y_j F_j .$$

The set F_Y defines a fractal function $x \mapsto F(x)$. In the same manner, F_j defines, say, $\varphi_j(x)$. So, we can also write

$$F(x) = \sum_{j=0}^{n} y_j \varphi_j(x) , \quad x \in [x_0, x_n] .$$

In this sense, the set of functions $\{\varphi_j\}$ is a basis in the space of fractal functions interpolating the nodes (x_i, y_i), $i = 0, \dots, n$ and with a prescribed vertical scaling vector.

4. Examples

The following examples illustrate the theory.

Example 1. Consider the data $Y = \{(0,0),\ (3,4),\ (5,5),\ (7,1),\ (10,0)\}$ and choose $\mathbf{d} = [0.3\ \ -0.2\ \ 0.3\ \ 0.2]^T$ as the vertical scaling vector. The graph of the corresponding fractal interpolating function F_Y is shown in Figure 1-a. Also consider a linear transformation ω defined by

$$\omega(\mathbf{x}) = \begin{bmatrix} 0.9192 & 0.2257 \\ 0.7713 & 1.2803 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} .$$

Since this is a linear contraction, the first interpolation node $(0,0)$ is its fixed point. The graphs $\omega(F_Y)$ and $F_{\omega(Y)}$ are also shown (note that they are translated a little bit up for the sake of clarity). It is evident that these graphs differ from each other, which is to be expected because ω does not fulfil the conditions of Theorem 1.

Example 2. For the data $Y = \{(0,0), (3,4), (7,1), (10,0)\}$ and $\mathbf{d} = [0.5\ -0.28\ \ 0.3]^T$ and for the same transformation as above, the graphs of $\omega(F_Y)$ and $F_{\omega(Y)}$ are displayed together in Figure 1-b. The difference between them is also evident.

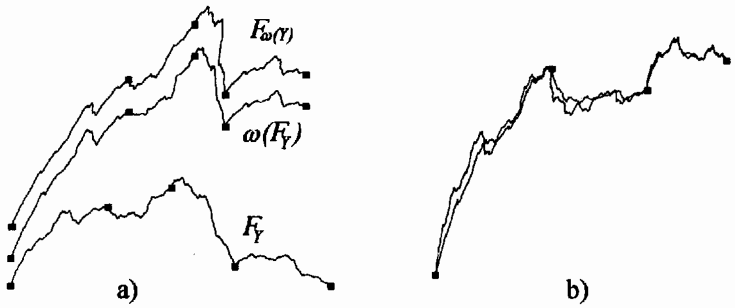

Figure 1.

Example 3. For the same data and the same vector **d** as in Ex. 1, but for

$$\omega(\mathbf{x}) = \begin{bmatrix} 1 & -0.2257 \\ 0 & 1.2803 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

the graphs of $\omega(F_Y)$ and $F_{\omega(Y)}$ again differ, as shown in Figure 2-a.

Example 4 (*Invariant case*)**.** Consider the same data and the same vector **d** as in Ex. 1, and the linear mapping

$$\omega(\mathbf{x}) = \begin{bmatrix} 0.866 & 0 \\ 0.5 & 1.2803 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

which satisfies conditions of Theorem 1. As expected, ω maps F_Y to the graph of the fractal function interpolating the transformed data $\omega(Y)$. This is shown in Figure 2-b.

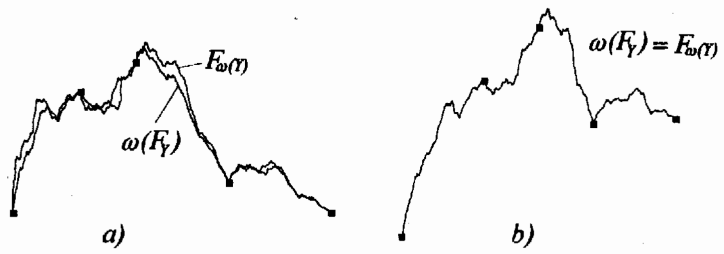

Figure 2.

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