NOVI SAD J. MATH.
VOL. 30, NO. 3, 2000, 41-46

QUASI-CONTRACTIVE NONSELF MAPPINGS ON TAKAHASHI CONVEX METRIC SPACES

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Abstract

The purpose of this paper is to generalize the fixed point theorem proved by Lj. Cirić [1] for the class of Takahashi convex metric spaces.

AMS Mathematical Subject Classification (1991): 54C20, 47H10
Key words and phrases: Takahashi convex metric space, nonself mappings, fixed point.

1. Introduction

In 1970 Takahashi [2] introduced the definition of convexity in metric space and generalized some important fixed point theorems previously proved for Banach spaces. On the other hand, the problem of fixed point for some classes of nonself mapping is very actual today. In this paper we shall prove the fixed point theorem previously proved by Lj. Ćirić for a class of nonself mappings in Takahashi convex metric spaces.

2. Preliminaries

Definition 1. *Let X be a metric space and I be the close unit interval. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for* *all* $x, y, u \in X,\ \lambda \in I$

$$d(u, W(x, y, \lambda)) \leq \lambda\, d(u, x) + (1 - \lambda) d(u, y).$$

X together with a convex structure is called a Takahashi convex metric space.

Remark. Clearly, any convex subset of normed space is a convex metric space with $W(x, y, \lambda) = \lambda\, x + (1 - \lambda) y$.

Let (M, d) be a Takahashi convex metric space. For $x, y \in M$ set:

$$seg[x, y] = \{W(x, y, \lambda) |\ \lambda \in [0, 1]\}.$$

Proposition 1. *[2] If (M, d) is a Takahashi convex metric space with convex structure W, then for every $x, y \in M$ and every $\lambda \in [0, 1]$*

$$d(x, y) = d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y).$$

Proposition 2. *(O. Hadžić) Let K be a non-empty closed subset of the Takahashi convex metric space with convex structure W continuous on third variable. Let $x \in K$ and $y \notin K$. Then there exists a $\lambda^* \in [0, 1]$ such that*

$$W(x, y, \lambda^*) \in seg[x, y] \cap \partial K.$$

Proof. Let

$$A = \{q : q \geq 0,\ \ W(x, y, \eta) \in K \quad \text{for all} \ \ q \leq \eta \leq 1\}.$$

Since $W(x, y, 1) = x \in K$, A is a nonempty set. Let $\lambda = \inf_{q \in A} q$ and let $\{q_n\}_{n \in \mathbf{N}} \subset A$ be a sequence such that $\lim_{n \to \infty} q_n = \lambda$. But $q_n \in A,\ n \in \mathbf{N}$, implies that for $n \in \mathbf{N}$

$$W(x, y, q_n) \in K.$$

The function W is continuous on the third variable and K is closed, so

$$W(x, y, \lambda) = \lim_{n \to \infty} W(x, y, q_n) \in K.$$

It is easy to see that $\lambda > 0$ since $W(x, y, 0) = y \notin K$.

Let us prove that $W(x,y,\lambda) \in \partial K$. It is obvious that for any $\varepsilon > 0$, $L(W(x,y,\lambda),\varepsilon) \cap K \neq \emptyset$ so we have to check that for any $\varepsilon > 0$

$$L(W(x,y,\lambda),\varepsilon) \cap (M \setminus K) \neq \emptyset.$$

By definition of λ and continuity of W we can choose $\theta \in (0,1)$ such that $\theta < \lambda$, $W(x,y,\theta) \notin K$ and

$$d(W(x,y,\theta),\ W(x,y,\lambda)) < \varepsilon.$$

Then $W(x,y,\theta) \in L(W(x,y,\lambda),\varepsilon) \cap (M \setminus K)$, so the proof is completed.

3. Main result

Theorem. *Let (M,d) be a complete convex metric space with convex structure W which is continuous on the third variable, C be a nonempty closed subset of M and $T : C \to M$ be a nonself mapping satisfying the contractive type condition $(*)$, that is: there exists $q \in (0,1)$ such that for every $x,y \in C$*

$$(*) \quad d(Tx,Ty) \leq q \cdot \max\{d(x,y), d(x,Tx), d(y,Ty),\ \ d(x,Ty), d(y,Tx)\}.$$

If T has the additional property

$$T(\partial C) \subset C$$

then T has a unique fixed point in C.

Proof. Let us denote

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

For $x \in \partial C$ set $x_0 = x$. Since $x_0 \in \partial C$, $Tx_0 \in C$. Let $x_1 = Tx_0$. Define $y_2 = Tx_1$. If $y_2 \in C$ let be $x_2 = y_2$. If $y_2 \notin C$ there exists, by Proposition 2, $x_2 \in \partial C \cap seg[x_1, y_2]$. Continuing in this manner we shall obtain a sequence $\{x_n\}$ with the property that:

$$x_n = Tx_{n-1}, \quad \text{if } \ Tx_{n-1} \in C$$

$$x_n \in \partial C \cap [x_{n-1}, y_n] \ \text{ if } \ y_n = Tx_{n-1} \notin C.$$

At first, let us show that the sequences $\{x_n\}$ and $\{Tx_n\}$ are bounded.

For $n \in \mathbb{N}$ define

$$A_n = \{x_i\}_{i=0}^{n-1} U \{Tx_i\}_{i=0}^{n-1}$$

$$\alpha_n = diam A_n$$

and show that

$$\alpha_n = \max\{d(x_0, Tx_i);\ 0 \leq i \leq n-1\}.$$

If $\alpha_n = 0$, then x_0 is the fixed point of T, so we can suppose that $\alpha_n > 0$ for any $n \in \mathbb{N}$. Now we have to discuss several possible cases.

I case $\alpha_n = d(x_i, Tx_j)$ for some $0 \leq i, j \leq n-1$. For $i = 0$ the proof is over, so we can suppose $i \geq 1$.

(1) If $x_i = Tx_{i-1} \in C$ we have that

$$\alpha_n = d(Tx_{i-1}, Tx_j) \leq q \cdot M(x_{i-1}, x_j).$$

where

$$\begin{aligned} M(x_{i-1}, x_j) \quad &= \max\{d(x_{i-1}, x_j), d(x_{i-1}, Tx_{i-1}), \\ &d(x_j, Tx_j), d(x_j, Tx_{i-1}), d(x_{i-1}, Tx_j)\} \leq \alpha_n \end{aligned}$$

Since $q < 1$ we have

$$\alpha_n \leq qM(x_{i-1}, x_j) \leq q \cdot \alpha_n < \alpha_n,$$

a contradiction.

(2) Now, suppose that $x_i \neq Tx_{i-1}$, that is $Tx_{i-1} \notin C$. Then

$$x_i \in \partial C \cap seg[x_{i-1}, Tx_{i-1}], \quad \text{e.t.}$$

$$x_i = W(x_{i-1}, Tx_{i-1}, \lambda) \ \text{ for some } \ \lambda \in [0,1]$$

and $x_{i-1} = Tx_{i-2}$ so

$$\begin{aligned} \alpha_n = d(x_i, Tx_j) &= d(W(x_{i-1}, Tx_{i-1}, \lambda), Tx_j) \\ &\leq \lambda \cdot d(x_{i-1}, Tx_j) + (1-\lambda) d(Tx_{i-1}, Tx_j) \\ &\leq \max\{d(Tx_{i-2}, Tx_j), d(Tx_{i-1}, Tx_j)\} \end{aligned}$$

Then

$$\alpha_n \leq \max\{q \cdot M(x_{i-2}, x_j), q \cdot M(x_{i-1}, x_j)\}$$

$$\leq \max\{q \cdot \alpha_n, q \cdot \alpha_n\} < \alpha_n$$

which is a contradiction.

II case $\alpha_n = d(x_i, x_j)$ for some $0 \leq i, j \leq n-1$.

If $x_j = Tx_{j-1}$ we have case **I** again.

If $x_j \neq Tx_{j-1}$ then $x_j \in \partial C \cap seg[x_{j-1}, Tx_{j-1}]$, $j \geq 2$ and $x_{j-1} = Tx_{j-2}$ so one can see, similary as in **case I** (1), that

$$\alpha_n = d(x_j, x_i) \leq \max\{d(Tx_{j-2}, x_i),\ d(Tx_{j-1}, x_i)\}.$$

Therefore this case also reduces to case **I.**

Since the case $\alpha_n = d(Tx_i, Tx_j)$ is impossible we proved that $\alpha_n = d(x_0, Tx_i)$ for some $0 \leq i \leq n-1$ and further

$$\alpha_n = d(x_0, Tx_i) \leq d(x_0, Tx_0) + d(Tx_0, Tx_i) \leq$$

$$\leq d(x_0, Tx_0) + q \cdot M(x_0, x_i) \leq d(x_0, Tx_0) + q \cdot \alpha_n$$

so

$$\alpha_n \leq \frac{1}{1-q} d(x_0, Tx_0).$$

The sequence $\{\alpha_n\}$ is nondecreasing, so there exists $c \in \mathbb{R}$ so that $c = \lim_{n\to\infty} \alpha_n$ and

$$c \leq \frac{1}{1-q} d(x_0, Tx_0).$$

Let us show that the sequences $\{x_n\}$ and $\{Tx_n\}$ are both Cauchy sequences.

For integer $n \geq 2$ define

$$B_n = \{x_n\}_{i\geq n} \cup \{Tx_i\}_{i\geq n},$$

$$\beta_n = diam B_n.$$

As above, one can show that

$$\beta_n = \sup\{d(x_n, Tx_j);\ \ j \geq n\}$$

and that $\{\beta_n\}$ is nonincreasing and bounded, and therefore there exists a limit and it must be zero. See [1]. So $\{x_n\}$ and $\{Tx_n\}$ are Cauchy sequences.

Since M is complete and C is closed, there exists $z \in C$ such that $z = \lim_{n\to\infty} x_n$.

Further, from $d(x_n, Tx_n) \leq \beta_n$, $\beta_n \to 0$, $n \to \infty$ it follows that $\lim_{n\to\infty} Tx_n = z$.

Assume that $z \neq Tz$. Then

$$d(Tx_n, Tz) \leq q \cdot \max\{d(x_n, z),$$

$$d(x_n, Tx_n), d(z, Tz), d(z, Tx_n), d(x_n, Tz)\}.$$

For $n \to \infty$ we have

$$d(z, \dot{T}z) = \lim_{n\to\infty} d(Tx_n, Tz) \leq q \cdot d(z, Tz) < d(z, Tz)$$

which is a contradiction. Therefore $z = Tz$. The contractive assumption implies uniqueness of the fixed point.

Remark. If we suppose T to be continuous and C compact then we may replace the condition $(*)$ by the following: for all $x, y \in C$

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty),$$

$$d(x, Ty), d(y, Tx)\}.$$

The proof is as in [1].

References

[1] Ćirić, Lj. B., Quasi contraction nonself mappings on Banach spaces, Srpska Akademija nauka i umetnosti (in print).

[2] Takahashi, W., A convexity in metric space and nonexpansive mappings, I, Kodai Math. Sem. Pep., 22 (1970), 142-149.