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ADDITION FORMULAS FOR MULTIVARIATE HYPERGEOMETRIC POLYNOMIALS

L. N. Djordjević, D. S. Savić
Faculty of Electronic Engineering, University of Niš
Beogradska 14, 18000 Niš, Yugoslavia

Dj. R. Djordjević
Civil Engineering Faculty, University of Niš
Beogradska 14, 18000 Niš, Yugoslavia

Abstract

The proof of addition formulas connecting Appell hypergeometric polynomials in two variables $F_2(a, -m+i, -i;\ p, q;\ x, y)$ and Lauricella polynomials $F_A(a, -m_1, \dots, -m_n;\ p_1, \dots, p_n;\ x_1, \dots, x_n)$ in n variables with Gauss polynomial $F(a, -m;\ s;\ t)$ in one variable, is presented.

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1. Introduction and preliminaries

The aim of this paper is to present addition equalities which connect Appell's hypergeometric polynomials $F_2(a, -m+i, -i; p, q; x, y)$ in two variables and Lauricella's polynomials $F_A(a, -m_1, \dots, -m_n; p_1, \dots, p_n; x_1, \dots, x_n)$ in n variables with Gauss' polynomial $F(a, -m; s; t)$ of one variable.

Some introductory definitions and preliminaries are necessary for giving the proofs. For hypergeometric function (polynomial) the HGF (HGP) notation will be used.

The Gauss HGF $F(\alpha,\beta;\gamma;x)$ [1, p.13] of a real variable x is defined by the series

$$F(\alpha,\beta;\gamma;x) := \sum_{j=0}^{\infty} \frac{(\alpha)_j(\beta)_j}{(\gamma)_j} \frac{x^j}{j!} \quad (|x|<1),$$

where $(\alpha)_0 := 1 \;\; (\alpha \neq 0)$, $\;(\alpha)_n := \alpha(\alpha+1)\dots(\alpha+n-1) \;\; (n \geq 1)$, and the Gauss confluent HGF $F(\beta;\gamma;x)$ of real variable x [1, p.15] using the series

$$F(\beta;\gamma;x) := \sum_{j=0}^{\infty} \frac{(\beta)_j}{(\gamma)_j} \frac{x^j}{j!} \quad (|x|<1).$$

Appell's HGF $F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y)$ in two real variables is defined by the series [1, p. 23]

$$F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y) := \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\alpha)_{j+k}(\beta)_j(\beta')_k}{(\gamma)_j(\gamma')_k} \frac{x^j}{j!}\frac{y^k}{k!} \quad (|x|+|y|<1).$$

The Lauricella HGF in n variables [1, p.41] $F_A(\alpha,\beta_1,\dots,\beta_n;\gamma_1,\dots,\gamma_n;x_1,\dots,x_n)$ is given by the series

$$F_A(\alpha,\beta_1,\dots,\beta_n;\gamma_1,\dots,\gamma_n;x_1,\dots,x_n)$$

$$:= \sum_{j_1,\dots,j_n=0}^{\infty} \frac{(\alpha)_{j_1+\dots+j_n}(\beta_1)_{j_1}\cdots(\beta_n)_{j_n}}{(\gamma_1)_{j_1}\cdots(\gamma_n)_{j_n}} \frac{x_1^{j_1}\cdots x_n^{j_n}}{j_1!\cdots j_n!} \quad (|x_1|+\dots+|x_n|<1).$$

Using the identity $(\alpha)_{j+k} = (\alpha)_k(\alpha+k)_j$ it is easy to show the relation between HGF's F_2 and F [2, p.15]

$$F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y) = \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta')_k}{(\gamma')_k} F(\alpha+k,\beta;\gamma;x)\frac{y^k}{k!}. \tag{1}$$

The corresponding development of HGF F_A in $n+1$ variables, by the variable x_{n+1} is

$$F_A(\alpha,\beta_1,\dots,\beta_n,\beta_{n+1};\gamma_1,\dots,\gamma_n,\gamma_{n+1};x_1,\dots,x_n,x_{n+1})$$

$$= \sum_{j_{n+1}=0}^{\infty} \frac{(\alpha)_{j_{n+1}}(\beta_{n+1})_{j_{n+1}}}{(\gamma_{n+1})_{j_{n+1}}} F_A(\alpha+j_{n+1},\beta_1,\dots,\beta_n;\gamma_1,\dots,\gamma_n; x_1,\dots,x_n)\frac{x_{n+1}^{j_{n+1}}}{j_{n+1}!}.$$

It can be shown that the following recurrence relation [3, p. 827] holds for HGF (1):

$$F(\alpha,\beta;\gamma;x) = -\frac{\alpha x}{\gamma}F(\alpha+1,\beta+1;\gamma+1;x) + F(\alpha,\beta+1;\gamma;x). \tag{2}$$

Also, for the Lauricella HGF in n variables the recurrence relation

$$\begin{aligned}
(\beta_1+\cdots+\beta_n)\quad & F_A\quad (\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
= -\frac{\alpha\beta_1}{\gamma_1}x_1\quad & F_A\quad (\alpha+1,\beta_1+1,\beta_2,\ldots,\beta_n;\gamma_1+1,\gamma_2,\ldots,\gamma_n;\\
& \qquad x_1,\ldots,x_n)\\
& \vdots\\
-\frac{\alpha\beta_n}{\gamma_n}x_n\quad & F_A\quad (\alpha+1,\beta_1,\ldots,\beta_{n-1},\beta_n+1;\gamma_1,\ldots,\gamma_{n-1},\gamma_n+1;\\
& \qquad x_1,\ldots,x_n)\\
+\beta_1\quad & F_A\quad (\alpha,\beta_1+1,\beta_2,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)\\
& \vdots\\
+\beta_n\quad & F_A\quad (\alpha,\beta_1,\ldots,\beta_{n-1},\beta_n+1;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n)
\end{aligned} \tag{3}$$

is valid.

The exactness of relations (2) and (3) can be checked by developing their left and right sides in the series and by equalizing the coefficients with monomials of the same degree.

2. Addition formulas for hypergeometric polynomials

For Appell's HGF F_2 and Gauss' HGF F the following assertion was proved in [5, pp. 3-4].

Theorem 1. *Let a,p,q be real numbers $(a,p,q>0)$ and $m \in N_0$. Then it holds that*

$$\begin{aligned}
\sum_{i=0}^{m}\binom{m}{i}\frac{(p)_{m-i}(q)_i}{(a)_m}F_2(a,-m+i,-i;p,q;x,y)\\
= \frac{(p+q)_m}{(a)_m}F(a,-m;p+q;x+y).
\end{aligned} \tag{4}$$

By writting (4) in symmetric form, one gets the equation

$$\sum_{m_1+m_2=m} \frac{m!}{m_1!m_2!}\frac{(p_1)_{m_1}(p_2)_{m_2}}{(a)_m} F_2(a,-m_1,-m_2;p_1,p_2;x_1,x_2)$$

$$= \frac{(p_1+p_2)_m}{(a)_m} F(a,-m;p_1+p_2;x_1+x_2), \tag{5}$$

where $m_1, m_2 \in \{0,1,\dots,m\}, m \in \mathcal{N}_0$.

Let us prove now that for any natural number n $(n \geq 2)$ and any $m \in \mathcal{N}_0$ the general case of equality (5) holds for the Lauricella HGF F_A and the Gauss HGF F. This is given as follows

Theorem 2. *Let a and p_i $(i = 1,\dots,n)$ be real positive numbers and $n \in \mathcal{N}_0$. Then it holds that*

$$\sum_{m_1+\cdots+m_n=m} \frac{m!}{m_1!\cdots m_n!}\frac{(p_1)_{m_1}\cdots(p_n)_{m_n}}{(a)_m} F_A(a,-m_1,\dots,-m_n;$$

$$p_1,\dots,p_n;x_1,\dots,x_n)$$

$$= \frac{(p_1+\cdots+p_n)_m}{(a)_m} F(a,-m;p_1+\cdots+p_n;x_1+\cdots+x_n), \tag{6}$$

where the sum holds for all non-negative integer indices $m_i \in \{0,1,\dots,m\}$, if $m_1+\cdots+m_n = m$.

Proof. By means of (4) the equation (6) is proven for $n = 2$ and $m \in \mathcal{N}_0$. Assuming that (6) holds for $n = s$ and constant m, we suppose that (6) holds for $n = s+1$ too. In order to prove the first part, we shall use the procedure from [4, p. 33] for proving the polynomial formula, and the recurrence relation (3) for the second part.

According to (5), we start from

$$\frac{([p_1+\cdots+p_s]+p_{s+1})_m}{(a)_m} F(a,-m;[p_1+\cdots+p_s]+p_{s+1};$$

$$[x_1+\cdots+x_s]+x_{s+1})$$

$$= \sum_{m_{s+1}=0}^{m} \frac{m!}{(m-m_{s+1})!m_{s+1}!}\frac{(p_1+\cdots+p_s)_{m-m_{s+1}}(p_{s+1})_{m_{s+1}}}{(a)_m} \times$$

$$\times F_2(a,-m_1-\cdots-m_s,-m_{s+1};p_1+\cdots+p_s,p_{s+1};$$

$$x_1+\cdots+x_s,x_{s+1}). \tag{7}$$

Using relation (1) between Appell's HGF F_2 and the Gauss HGF F, the expession on the right hand side of (7) becomes

$$\sum_{m_{s+1}=0}^{m} \frac{m!}{(m-m_{s+1})!m_{s+1}!} \sum_{j_{s+1}=0}^{m_{s+1}} \frac{(a)_{j_{s+1}}(-m_{s+1})_{j_{s+1}}}{(p_{s+1})_{j_{s+1}}} \times$$
$$\times \frac{(p_1+\cdots+p_s)_{m-m_s}(p_{s+1})_{m_{s+1}}}{(a)_m} F(a+j_{s+1}, -m_1-\cdots-m_s;$$
$$p_1+\cdots+p_s; x_1+\cdots+x_s)\frac{x_{s+1}^{j_{s+1}}}{(j_{s+1})!}. \tag{8}$$

Taking into account (6) we obtain from (8)

$$\sum_{m_{s+1}=0}^{m} \frac{m!}{(m-m_{s+1})!m_{s+1}!} \sum_{j_{s+1}=0}^{m_{s+1}} \frac{(a)_{j_{s+1}}(-m_{s+1})_{j_{s+1}}}{(p_{s+1})_{j_{s+1}}} \times$$
$$\times \sum_{m_1+\cdots+m_s=m-m_{s+1}} \frac{(m-m_{s+1})!}{m_1!\cdots m_s!}\frac{(p_1)_{m_1}\cdots(p_s)_{m_s}}{(a)_m} \times \tag{9}$$
$$\times F_A(a+j_{s+1}, -m_1, \cdots, -m_s; p_1, \ldots, p_s; x_1, \ldots, x_s)\frac{x_{s+1}^{j_{s+1}}}{(j_{s+1})!}.$$

From (9), based on the development of HGF F_A in $n+1$ variables, one gets the formula

$$\sum_{m_{s+1}=0}^{m} \frac{m!(p_{s+1})_{m_{s+1}}}{(m-m_{s+1})!m_{s+1}!} \sum_{m_1+\cdots+m_s=m-m_{s+1}} \frac{(m-m_{s+1})!}{m_1!\cdots m_s!}\times$$
$$\times \frac{(p_1)_{m_1}\cdots(p_s)_{m_s}}{(a)_m} F_A(a, -m_1, \cdots, -m_s, -m_{s+1}; \tag{10}$$
$$p_1, \ldots, p_s, p_{s+1}; x_1, \ldots, x_s, x_{s+1}).$$

where $m_1, \ldots, m_s \in \{0, 1, \ldots, m-m_{s+1}\}$ and $m_1+\cdots+m_s = m-m_{s+1}$. This condition can be replaced by the equivalent one $m_1, \ldots, m_s \in \{0, 1, \ldots, m\}$ and $m_1+\cdots+m_s = m-m_{s+1}$. Replacing two sums in (10) with one, the expression (10) becomes

$$\sum_{m_1+\cdots+m_{s+1}=m} \frac{m!}{m_1!\cdots m_s!m_{s+1}!}\frac{(p_1)_{m_1}\cdots(p_s)_{m_s}(p_{s+1})_{m_{s+1}}}{(a)_m} \times$$
$$\times F_A(a, -m_1, \cdots, -m_s, -m_{s+1}; p_1, \ldots, p_s, p_{s+1}; x_1, \ldots, x_s, x_{s+1}).$$

From the fact that (6) holds for $n=2$ and $n=s+1$, assuming it holds for $n=s$, one concludes that (6) holds for any natural number n.

Let us prove now the equality (6) for fixed n, using induction by m. The particular case for $m=1$ can be directly checked as follows.

$$\sum^n \Big[(\frac{p_1}{a}-x_1)+\cdots+(\frac{p_n}{a}-x_n)\Big] = \frac{p_1+\cdots+p_n}{a}-(x_1+\cdots+x_n) \qquad (m=1).$$

Let us suppose that (6) holds for $k=m$. Then, prove that (6) holds for $k=m+1$, i.e.,

$$\begin{aligned}\sum_{m_1+\cdots+m_n=m+1} \frac{(m+1)!}{m_1!\cdots m_n!}\frac{(p_1)_{m_1}\cdots(p_n)_{m_n}}{(a)_{m+1}} F_A(a,-m_1,\dots,-m_n;& \\ p_1,\dots,p_n;x_1,\dots,x_n)& \\ = \frac{(p_1+\cdots+p_n)_{m+1}}{(a)_{m+1}} F(a,-m-1;p_1+\cdots+p_n;x_1+\cdots+x_n).& \end{aligned} \tag{11}$$

Starting from the recurrence relation (2), then, after introducing the replacements $\alpha=a$, $\beta=-m-1$, $\gamma=p_1+\cdots+p_n$, $x=x_1+\cdots+x_n$ and multiplying by $\frac{(p_1+\cdots+p_n)_{m+1}}{(a)_{m+1}}$, we obtain

$$\begin{aligned}\frac{(p_1+\cdots+p_n)_{m+1}}{(a)_{m+1}} F(a,-m-1,p_1+\cdots+p_n;x_1+\cdots+x_n)& \\ -(x_1+\cdots+x_n)\frac{p_1+\cdots+p_n+1)_m}{(a+1)_m}\times& \\ \times F(a+1,-m;p_1+\cdots+p_n+1;x_1+\cdots+x_n)& \\ +\frac{(p_1+\cdots+p_n)_{m+1}}{(a)_{m+1}} F(a,-m;p_1+\cdots+p_n;x_1+\cdots+x_n).& \end{aligned} \tag{12}$$

The according to (6) right-hand side of (12) can be written as

$$\begin{aligned} &- x_1 \sum_{m_1+\cdots+m_n=m} \frac{m!}{m_1!\cdots m_n!}\frac{(p_1+1)_{m_1}\cdots(p_n)_{m_n}}{(a+1)_m}\times \\ &\qquad \times F_A(a+1,-m_1,\dots,-m_n;p_1+1,p_2,\dots,p_n;x_1,\dots,x_n) \\ &\vdots \\ &- x_n \sum_{m_1+\cdots+m_n=m} \frac{m!}{m_1!\cdots m_n!}\frac{(p_1)_{m_1}\cdots(p_n+1)_{m_n}}{(a+1)_m}\times \\ &\qquad \times F_A(a+1,-m_1,\dots,-m_n;p_1,\dots,p_n+1;x_1,\dots,x_n) \\ &+ \frac{p_1+\cdots+p_n+m}{a+m} \sum_{m_1+\cdots+m_n=m} \frac{m!}{m_1!\cdots m_n!}\frac{(p_1)_{m_1}\cdots(p_n)_{m_n}}{(a)_m}\times \\ &\qquad \times F_A(a,-m_1,\dots,-m_n;p_1,\dots,p_n;x_1,\dots,x_n). \end{aligned}$$

Incrementing the summation index by 1, (12) becomes

$$
\begin{aligned}
- & \quad x_1 \sum_{m_1+\cdots+m_n=m+1} \frac{m!}{(m_1-1)!\cdots m_n!} \frac{(p_1+1)_{m_1-1}\cdots(p_n)_{m_n}}{(a+1)_m} \times \\
& \qquad \times F_A(a+1,-m_1+1,\ldots,-m_n;p_1+1,p_2,\ldots,p_n;x_1,\ldots,x_n) \\
\vdots & \\
- & \quad x_n \sum_{m_1+\cdots+m_n=m+1} \frac{m!}{m_1!\cdots(m_n-1)!} \frac{(p_1)_{m_1}\cdots(p_n+1)_{m_n-1}}{(a+1)_m} \times \\
& \qquad \times F_A(a+1,-m_1,\ldots,-m_n+1;p_1,\ldots,p_n+1;x_1,\ldots,x_n) \\
+ & \quad \sum_{m_1+\cdots+m_n=m+1} \frac{p_1+m_1-1}{a+m} \frac{m!}{(m_1-1)!\cdots m_n!} \frac{(p_1)_{m_1-1}\cdots(p_n)_{m_n}}{(a)_m} \times \\
& \qquad \times F_A(a,-m_1+1,\ldots,-m_n;p_1,\ldots,p_n;x_1,\ldots,x_n) \\
\vdots & \\
+ & \quad \sum_{m_1+\cdots+m_n=m+1} \frac{p_n+m_n-1}{a+m} \frac{m!}{m_1!\cdots(m_n-1)!} \frac{(p_1)_{m_1}\cdots(p_n)_{m_n-1}}{(a)_m} \times \\
& \qquad \times F_A(a,-m_1,\ldots,-m_n+1;p_1,\ldots,p_n;x_1,\ldots,x_n).
\end{aligned}
$$

By further transformations, the previous expression assumes the form

$$
\begin{aligned}
- & \quad \frac{x_1}{m+1} \sum_{m_1+\cdots+m_n=m+1} \frac{m_1(m+1)!}{m_1!\cdots m_n!} \frac{a}{p_1} \frac{(p_1)_{m_1}\cdots(p_n)_{m_n}}{(a)_{m+1}} \times \\
& \qquad \times F_A(a+1,-m_1+1,\ldots,-m_n;p_1+1,p_2,\ldots,p_n;x_1,\ldots,x_n) \\
\vdots & \\
- & \quad \frac{x_n}{m+1} \sum_{m_1+\cdots+m_n=m+1} \frac{m_n(m+1)!}{m_1!\cdots(m_n)!} \frac{a}{p_n} \frac{(p_1)_{m_1}\cdots(p_n)_{m_n}}{(a)_{m+1}} \times \\
& \qquad \times F_A(a+1,-m_1,\ldots,-m_n+1;p_1,\ldots,p_n+1;x_1,\ldots,x_n) \\
+ & \quad \frac{1}{m+1} \sum_{m_1+\cdots+m_n=m+1} \frac{m_1(m+1)!}{m_1!\cdots m_n!} \frac{(p_1)_{m_1}\cdots(p_n)_{m_n}}{(a)_{m+1}} \times \\
& \qquad \times F_A(a,-m_1+1,\ldots,-m_n;p_1,\ldots,p_n;x_1,\ldots,x_n) \qquad (13) \\
\vdots & \\
+ & \quad \frac{1}{m+1} \sum_{m_1+\cdots+m_n=m+1} \frac{m_n(m+1)!}{m_1!\cdots m_n!} \frac{(p_1)_{m_1}\cdots(p_n)_{m_n}}{(a)_{m+1}} \times \\
& \qquad \times F_A(a,-m_1,\ldots,-m_n+1;p_1,\ldots,p_n;x_1,\ldots,x_n).
\end{aligned}
$$

By factorizing, we get from (13)

$$
\begin{aligned}
&\sum_{m_1+\cdots+m_n=m+1} \frac{(m+1)!}{m_1!\cdots m_n!}\frac{(p_1)_{m_1}\cdots(p_n)_{m_n}}{(a)_{m+1}} \times \\
&\times\Big[- x_1\frac{am_1}{p_1}F_A(a+1,-m_1+1,\ldots,-m_n;p_1+1,p_2,\ldots,p_n; \\
&\qquad x_1,\ldots,x_n) \\
&\qquad \vdots \\
&\quad - x_n\frac{am_n}{p_n}F_A(a+1,-m_1,\ldots,-m_n+1;p_1,\ldots,p_n+1; \\
(14)\quad &\qquad x_1,\ldots,x_n) \\
&\quad + m_1F_A(a,-m_1+1,\ldots,-m_n;p_1,\ldots,p_n;x_1,\ldots,x_n) \\
&\qquad \vdots \\
&\quad + m_nF_A(a,-m_1,\ldots,-m_n+1;p_1,\ldots,p_n;x_1,\ldots,x_n)\Big] \\
&\qquad /(m+1).
\end{aligned}
$$

Involving (3), (14) becomes the left-hand side of (11), what was to be proven.

Since the induction for fixed m has been proved for n, and for fixed n for m, we conclude that formula (6) holds for every $n, m \in \mathcal{N}_0$, which completes the proof.

Theorem 3. *Let* p_i $(i = 1,\ldots,n)$ *be real positive numbers, and* $m \in \mathcal{N}_0$. *Then it holds that*

$$
\begin{aligned}
&\sum_{m_1+\cdots+m_n=m} \frac{m!}{m_1!\cdots m_n!}(p_1)_{m_1}\cdots(p_n)_{m_n}F_A(-m_1,\ldots,-m_n;p_1,\ldots,p_n; \\
&\qquad x_1,\ldots,x_n) \\
&= (p_1+\cdots+p_n)_mF(-m;p_1+\cdots+p_n;x_1+\cdots+x_n),
\end{aligned}
$$

where

$$
\begin{aligned}
&F_A(-m_1,\ldots,-m_n;p_1,\ldots,p_n;x_1,\ldots,x_n) \\
&\qquad := F(-m_1;p_1;x_1)\cdots F(-m_n;p_n;x_n).
\end{aligned}
$$

Proof. The proof is the same as the proof of Theorem 2 and will be omitted.

The addition formulas for multivariate orthogonal polynomials, for $n-$dimensional simplex, 2^nth part of $n-$dimensional space, $n-$dimensional sphere, and entire $n-$dimensional space, are presented in [5, pp. 6-7].

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