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ON COLLOCATION METHODS FOR SINGULAR PERTURBATION PROBLEMS

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Abstract

A collocation method using cubic splines for second-order linear sigularly-perturbed two-point boundary value problem is developed. The approximate solution is determined by collocation at the points of a piecewise equidistant mesh of Shishkin type. We improve the method by changing the location of the collocation points from the mesh points to the Gauss-Legendre points. Numerical examples comparing the methods are presented.

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1. Introduction

We consider the numerical solution by collocation methods of the singularly-perturbed second-order linear boundary value problem

$$L_\varepsilon y(x) \equiv -\varepsilon^2 y''(x) + g(x)y(x) = f(x), \tag{1.1}$$

$$y(0) = 0, \quad y(1) = 0 \tag{1.2}$$

where ε is a small positive parameter. Under the basic assumptions

$$g(x), f(x) \in C^k(I), \quad k \in N \tag{1.3}$$

$$g(x) \geq g_0^2 > 0 \quad \text{for all} \quad x \in I,$$

the problem (1.1) has a unique solution $y(x) \in C^{k+2}(I)$.

The reduced problem obtained by setting $\varepsilon = 0$ in (1.1) also has a unique solution $y_r(x) \in C^k(I)$. When $y_r(0) \neq A$ and $y_r(1) \neq B$, $y(x)$ exibits two boundary layers of exponential type at $x = 0$ and $x = 1$, when ε is near to zero.

The problem (1.1) when $\varepsilon = 1$ and the collocation methods with polynomial splines on an regular mesh are considered in [1]. It has been shown that the maximal order of convergence is achieved by collocation at the Gauss-Legandre points. For the cubic spline the forth order of convergence is achieved. The singularly perturbed problem (1.1) and the collocation methods with cubic and tension splines are considered in [2] and the rules for selecting tension parameters and collocation points are developed. The methods converge outside the boundary layer regions.

It is known that most of classical methods fail when ε is small related to the mesh width h used for the discretization of the operator L_ε. In most cases it is impractical to require a partition with subinterval length of order ε. Our aim in this paper is to show that collocation methods with piecewise polynomials can furnish accurate numerical approximations of (1.1), (1.2), (1.3) when h/ε is either small or large, if the mesh used for the discretization is a piecewise equidistant mesh of Shishkin type.

In [3] the collocation with qudratic spline is considered when the collocation points are the midpoints of the piecewise equidistant mesh of Shishkin type. The error for these approximations is $O(n^{-2} \ln^2 n)$, n is the number of the mesh points. The same result is obtained for the collocation with cubic splines for nonlinear problems [4], [5].

In Section 2 we will describe the construction of a piecewise equidistant mesh of Shishkin type. In Section 3, we derive and analyse the method of collocation with cubic splines. We introduce the Gauss-Legendre points for collocation points in order to improve the accuracy of computed approximations. The numerical experiments presented in Section 4 indicate the method which converges uniformly in ε and the error for these approximation is $O(n^{-4}\ln^5 n)$.

2. The mesh

The Shishkin mesh appropriate to the problem (1.1), (1.2) is defined as follows. Given a positive integer n divisible by 4. We divide the interval $[0,1]$ into three subintervalas

$$[0,\delta],\ \ [\delta,1-\delta],\ \ [1-\delta,1].$$

We use equidistant meshes on each subinterval, with $1+n/4$ points in each of $[0,\delta]$ and $[1-\delta,1]$, and $1+n/2$ points in $[\delta,1-\delta]$. Set

$$g = min\{g_0, 1\}.$$

The transition point δ from the fine to the coarse mesh depends on ε and n, and we define

$$\delta = min\{1/4, 4g^{-1}\varepsilon lnn\}.$$

Set $i_0 = n/4$. Then

$$x_{i_0} = \delta \quad \text{and} \qquad x_{n-i_0} = 1-\delta,$$

are the transition points of the mesh:

$$\Delta_S^n : 0 = x_0 < x_1 < \ldots < x_{i_0} < \ldots < x_{n-i_0} < \ldots < x_n = 1. \tag{2.1}$$

The mesh spacing is given by

$$\begin{aligned} h_j &= 4\delta n^{-1}, && \text{for} \quad j = 0,1,..,i_0-1, n-i_0,...,n-1, \\ h_j &= 2(1-2\delta)n^{-1}, && \text{for} \quad j = i_0,\ldots,n-i_0-1. \end{aligned}$$

We shall assume that $\delta = 4b^{-1}\varepsilon\ln n$ since in the oposite case n^{-1} is very small relative to ε and the method can be analysed using standard techniques. Thus we have that

$$h_j = 16b^{-1}\varepsilon n^{-1}\ln n, \text{ for } j = 0,1,..,i_0-1, n-i_0,...,n-1, \tag{2.2}$$

and

$$n^{-1} \le h_j \le 2n^{-1}, \text{ for } j = i_0,..,n-i_0-1. \tag{2.3}$$

3. Derivation of the scheme

For a given integer n, we denote by Δ^n an arbitraty mesh

$$(3.1) \qquad \Delta^n: \quad 0 = x_0 < x_1 < ... < x_n = 1$$

$$\text{with} \quad h_j = x_{j+1} - x_j \quad \text{for} \quad j = 0, 1, ...n-1.$$

We seek to deterimine an approximation to the solution with a cubic spline $u(x) \in C^1(I)$

$$(3.2) \quad u(x) = u_j + (x - x_j)u'_j + (x - x_j)^2 \frac{u''_j}{2} + (x - x_j)^3 \frac{u'''_j}{6}, \quad x \in [x_j, x_{j+1}].$$

Let $u_j(x)$ be a cubic polynomial which represents the spline $u(x)$ on the interval $[x_j, x_{j+1}]$.

We specify $u(x)$ by collocating at $2n$ points, i.e. by enforcing

$$(3.3) \qquad L_\varepsilon u(z_j) = f(z_j), \quad j = 1, ..., 2n$$

and by requiring $u(x)$ to satisfy the boundary conditions (1.2). We place the collocation points symmetrically disposed on each subinterval:

$$a_j \equiv z_{2j+1} = x_j + t_j h_j, \quad b_j \equiv z_{2j+2} = x_{j+1} - t_j h_j, \quad j = 0, 1, ..., n-1$$

For the appropriate choice of t_j:

$$t_j = (1 - \frac{1}{\sqrt{3}})/2$$

we select the Gauss-Legendre points. Let us introduce the following notations:

$$p_j = a_j - x_j, \; d_j = b_j - x_j, \; g_j = g(a_j), \; \tilde{g}_j = g(b_j),$$

$$f_j = f(a_j), \; \tilde{f}_j = f(b_j), \; j = 0, ..., n-1.$$

Equations (2.3) than can be writen in the form:

$$(3.4) \qquad \begin{array}{lcl} u''_j A_j + u''_{j+1} B_j & = & C_j \\ u''_j D_j + u''_{j+1} G_j & = & N_j, \quad j = 0, 1, ..., n-1 \end{array}$$

where

$$\begin{aligned}
A_j &= \Psi(p_j, g_j) \equiv \varepsilon^2(\tfrac{p_j}{h_j} - 1) + \tfrac{g_j p_j^2}{2}(1 - \tfrac{p_j}{3h_j}) \\
D_j &= \Psi(d_j, \tilde{g}_j) \\
B_j &= \Upsilon(p_j, g_j) \equiv \tfrac{p_j}{h_j}(\tfrac{g_j p_j^2}{6} - \varepsilon^2) \\
G_j &= \Upsilon(d_j, \tilde{g}_j) \\
C_j &= \Omega(p_j, g_j, f_j) \equiv -g_j(u_j + p_j u_j') + f_j \\
N_j &= \Omega(d_j, \tilde{g}_j, \tilde{f}_j).
\end{aligned}$$

Solving the system (3.4) we obtain

$$\begin{aligned}
u''_{j+1} &\equiv O_{1j} = (C_j D_j - N_j A_j)/I_j \\
u''_j &\equiv O_{2j} = (N_j B_j - C_j G_j)/I_j,
\end{aligned} \tag{3.5}$$

$$I_j = B_j D_j - G_j A_j.$$

The continuity condition $u(x) \in C^1(I)$ with (3.5) gives:

$$u'_{j+1} = u'_j + O_{2j}\frac{h_j}{2} + O_{1j}\frac{h_j}{2}, \; j = 0, ..., n-1 \tag{3.6}$$

$$u'_j = \frac{u_{j+1} - u_j}{h_j} - \frac{h_j}{3}O_{2j} - \frac{h_j}{6}O_{1j}, \; j = 0, ..., n-1. \tag{3.7}$$

Substituing (3.5) into (3.7) we get

$$u'_j = \frac{1}{\gamma_j}[\frac{u_{j+1}}{h_j} + u_j\alpha_j + F_j(f_j, h_j, h_j)], \; j = 0, 1, ..., n-1 \tag{3.8}$$

where

$$\begin{aligned}
F_j(f_j, p_j, d_j) &\equiv \tilde{f}_j \tfrac{p_j}{6I_j}(A_j - 2B_j) + f_j \tfrac{d_j}{6I_j}(2G_j - D_j) \\
\gamma_j &= 1 + h_j F_j(g_j, p_j, d_j) \\
\alpha_j &= -\tfrac{1}{h_j} - F(g_j, h_j, h_j)
\end{aligned}$$

Finally, (3.8) and (3.6) give the scheme:

$$r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1} = s_{j-1} f_{j-1} + \tilde{s}_{j-1}\tilde{f}_{j-1} + S_j f_j + \tilde{S}_j \tilde{f}_j, \tag{3.9}$$

$$j = 1, 2, ..., n-1, \quad u_0 = A, \quad u_n = B$$

where

$$\begin{aligned}
\Phi(g_j, p_j, d_j) &\equiv \tfrac{g_j p_j}{2I_j}(-D_j + G_j) + \tfrac{\tilde{g}_j d_j}{2I_j}(A_j - B_j) \\
r_j^+ &= \tfrac{1}{\gamma_j h_j} \\
r_j^c &= \tfrac{\alpha_j}{\gamma_j} - \tfrac{1}{\gamma_{j-1}}\left[\tfrac{1}{h_{j-1}} + \Phi(g_{j-1}, p_{j-1}, d_{j-1})\right] \\
r_j^- &= -\tfrac{\alpha_{j-1}}{\gamma_{j-1}}[1 + h_{j-1}\Phi(g_{j-1}, p_{j-1}, d_{j-1})] - \Phi(g_{j-1}, h_{j-1}, h_{j-1}) \\
s_{j-1} &= \tfrac{h_{j-1}}{6I_{j-1}\gamma_{j-1}}(2G_{j-1} - D_{j-1})(1 + h_{j-1}\Phi(g_{j-1}, p_{j-1}, d_{j-1})) \\
&\quad + \tfrac{h_{j-1}}{2I_{j-1}}(D_{j-1} - G_{j-1}) \\
&\equiv \Gamma(G_{j-1}, D_{j-1}) \\
\tilde{s}_{j-1} &= \Gamma(-B_{j-1}, -A_{j-1}) \\
S_j &= -\tfrac{h_j}{6I_j\gamma_j}(2G_j - D_j) \\
\tilde{S}_j &= -\tfrac{h_j}{6I_j\gamma_j}(-2B_j + A_j)
\end{aligned}$$

From the system (3.9) we obtain u_i and then we can determine u'_j from (3.8), u''_j from (3.5) and $u'''_j = \frac{u''_{j+1} - u''_j}{h_j}$. After that we have the spline $u(x)$ according to (3.2).

For $t_j = 0,\ j = 0, ..., n-1$ the Gauss-Legendre points reduce to the mesh points and the method (3.9) reduces to

$$r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1} = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1}, \tag{3.10}$$

$$j = 1, 2, ..., n-1, \quad u_0 = A, \quad u_n = B$$

where

$$r_j^- = -\frac{\varepsilon^2}{h_{j-1}\tilde{h}_j} + \frac{h_{j-1}g_{j-1}}{6\tilde{h}_j}$$

$$r_j^c = \frac{2\varepsilon^2}{h_j h_{j-1}} + \tfrac{2}{3}g_j$$

$$r_j^+ = -\frac{\varepsilon^2}{h_j\tilde{h}_j} + \frac{h_j g_{j+1}}{6\tilde{h}_j}$$

$$q_j^- = \frac{h_{j-1}}{6\tilde{h}_j}, \quad q_j^c = \frac{4}{6}, \quad q_j^+ = \frac{h_j}{6\tilde{h}_j},$$

where $\tilde{h}_j = \frac{h_j + h_{j-1}}{2}$.

The spline $u(x)$ obtained by (3.10) belongs to $C^2(I)$ and the following theorem proofs the uniform convergence of the method in ε:

Theorem 3.1. *Assume that* (1.1),(1.2) *and* (1.3) *hold. Let* $\mathbf{y} = (y(x_0), y(x_1), \ldots, y(x_n))^T$ *be the restriction on the Shishin mesh of the solution* y *to the problem* $L_\varepsilon y(x) = 0$. *Let* $\mathbf{u_S^n} = (u_0, \ldots, u_n)^T$ *be the solution to* (3.10) *on the Shishkin mesh. Then there exists a constant* $C > 0$, *which is independent of* ε *and* n, *such that*

$$\|\mathbf{y} - \mathbf{u_S^n}\|_\infty \leq Mn^{-2} ln^2 n. \tag{3.11}$$

Proof. See [4]□

The analysis of the approximation error of the scheme (3.9) at the transition points is difficult.The analysis in the intervals which contain uniform meshes confirm numerical results which suggest an almost fourth order accuracy, in the maximum norm, over the whole interval $[0, 1]$.

In the next theorem we shall consider the approximation error of the scheme (3.9) on the intervals $[0, x_{i_0-1}]$ and $[x_{n-i_0+1}, 1]$ which contain uniform meshes.

Theorem 3.2. *Let* $y(x) \in C^6[0,1]$ *and let* (1.1), (1.2) *and* (1.3) *hold and let* $u(x)$ *be a solution of* (3.9) *on* Δ_s^n. *Then the error of the approximation* $e(x) = y(x) - u(x)$, *satisfies*

$$|e(x)| \leq Mn^{-4}\ln^5 n, \quad x \in [x_j, x_{j+1}] \quad j = 0, \ldots, i_0 - 2, n - i_0 + 1, \ldots, n-1$$

$$|e(x)| \leq Mn^{-4}, \quad x \in [x_j, x_{j+1}] \quad j = i_0 + 1, \ldots, n - i_0 - 2.$$

Proof. Since the solution of the problem (1.1), (1.2) has the form

$$y(x) = \int_0^1 G(x,t)f(t)dt,$$

according to [1] the difference $e(x) = y(x) - u(x)$ has the form

$$y(x) - u(x) = \int_0^1 G(x,t)\tau(t)dt,$$

$$\tau(t) = L_\varepsilon y(t) - L_\varepsilon u(t) = f(t) - L_\varepsilon u(t),$$

$G(x,t)$ is Green's function associated with operator L_ε and homogeneous boundary conditions on $[0,1]$.

Now we consider one of the intervals $[x_j, x_{j+1}]$. On this interval $\tau(t)$ vanishes at the points a_j and b_j ([1]). Then for any two points σ_1 and σ_2 in $[x_j, x_{j+1}]$ we have

$$G(x,t)\tau(t) = (t-a_j)(t-b_j)(t-\sigma_1)(t-\sigma_2)h_t[\sigma_1,\sigma_2,t] \tag{3.12}$$

where $h_t = G(t,u)\tau[a_j, b_j, u]$.
$f[s_0, s_1, ..., s_k]$ denotes the k-th divided difference of f at the points $s_0, s_1, ..., s_k$.

In [2] the approximation of $G(x,t)$ is given. As a function of t, $G(x,t)$ has boundary layers on bouth sides of $t = x$ and is unbounded as $O(\frac{1}{\varepsilon})$ when $\varepsilon \to 0$.

So, from (3.12) and Theorem 4.1 [1], for each interval $[x_j, x_{j+1}]$ we obtain

$$|\int_{x_j}^{x_{j+1}} G(x,t)\tau(t)dt| \leq \frac{Mh^5}{\varepsilon^5}.$$

At the boundary layer we have

$$\begin{aligned} |e(x)| &= |\textstyle\int_0^{x_{i_0}} G(x,t)\tau(t)dt| = |\sum_{j=0}^{i_0-1} \int_{x_j}^{x_{j+1}} G(x,t)\tau(t)dt| \\ &\leq M\frac{n}{4}\frac{n^{-5}\ln^5 n\varepsilon^5}{\varepsilon^5} \leq Mn^{-4}\ln^5 n, \quad x \in [0, x_{i_0-1}]. \end{aligned}$$

The same estimate holds on the interval $[x_{n-i_0+1}, x_n]$, and on the interval $[x_{i_0+1}, ..., x_{n-i_0-1}]$ we have

$$|e(x)| \leq Mn^{-4}. \Box$$

4. Numerical experiments

Our example is

$$-\varepsilon y'' + y = -\cos^2(\pi x) - 2\varepsilon\pi^2\cos(2\pi x),$$

$$y(0) = 0, \quad y(1) = 0.$$

Its exact solution is

$$y(x) = \frac{e^{\frac{x}{\sqrt{\varepsilon}}} + e^{\frac{x-1}{\sqrt{\varepsilon}}}}{1 + e^{-\frac{1}{\sqrt{\varepsilon}}}} - \cos^2(\pi x).$$

Let $\mathbf{u_G^n} = (u_{G,0}, \ldots, u_{G,n})^T$ be the solution to (2.9) on the Shishkin mesh. Let $\mathbf{u^n} = (u_0, \ldots, u_n)^T$ be the solution to (2.10) on the Shishkin mesh.

We use the double-mesh method to compute the experimental rates of convergence.

For each $n = 2^{-k}$, $k = 5, 6, \ldots, 10$ and $\varepsilon = 2^{-l}$, $l = 1, 2, \ldots, 15$ we shall report

$$E_n = \max_{0 \le j \le n} |y(x_j) - u_j|,$$

and

$$E_{G,n} = \max_{0 \le j \le n} |y(x_j) - u_{G,j}|,$$

in Table 1 and Table 2 below.

Assuming convergence of the order Mn^{-p}, for some p, for fixed ε we compute E_n (and $E_{G,n}$) for two consecutive values of k. Because of

$$\frac{E_n}{E_{2n}} \approx \frac{(n^{-k})^p}{(n^{-2k})^p} = 2^{-p},$$

in Table 3 we estimate the convergence order p for each fixed ε from

$$P_n = \frac{lnE_n - lnE_{2n}}{ln2}, \text{ for } n = 2^k \text{ and } k = 4, 5, \ldots 10,$$

and in Table 4 from

$$P_{G,n} = \frac{lnE_{G,n} - lnE_{G,2n}}{ln2}, \text{ for } n = 2^k \text{ and } k = 4, 5, \ldots 10.$$

Table 1. Errors E_n for scheme (2.10) on Shishkin mesh

l	N					
	32	64	128	256	512	1024
1	2.765(−3)	6.921(−3)	1.731(−4)	4.327(−5)	1.082(−4)	2.705(−6)
2	2.480(−3)	6.208(−3)	1.552(−4)	3.881(−4)	9.704(−6)	2.426(−6)
3	2.134(−3)	5.340(−4)	1.335(−4)	3.339(−5)	8.346(−6)	2.086(−6)
4	1.779(−3)	4.451(−4)	1.113(−4)	2.782(−5)	6.955(−6)	1.739(−6)
5	1.422(−3)	3.556(−4)	8.891(−5)	2.223(−5)	5.557(−6)	1.389(−6)
6	1.017(−3)	2.541(−4)	6.353(−5)	1.589(−5)	3.971(−6)	9.927(−7)
7	1.751(−3)	4.339(−4)	1.085(−4)	2.712(−5)	6.778(−6)	1.695(−6)
8	3.802(−3)	9.336(−4)	2.323(−4)	5.803(−5)	1.450(−5)	3.626(−6)
9	7.580(−3)	1.917(−3)	4.751(−4)	1.187(−4)	2.967(−5)	7.418(−6)
10	1.224(−2)	3.913(−3)	9.612(−4)	2.393(−4)	5.975(−5)	1.493(−5)
11	1.226(−2)	4.241(−3)	1.415(−3)	4.609(−4)	1.120(−4)	2.992(−5)
12	1.226(−2)	4.243(−3)	1.415(−3)	4.611(−4)	1.457(−4)	4.494(−5)
13	1.226(−2)	4.243(−3)	1.415(−3)	4.612(−4)	1.457(−4)	4.494(−5)
14	1.226(−2)	4.243(−3)	1.416(−3)	4.612(−4)	1.457(−4)	4.495(−5)
15	1.227(−2)	4.243(−3)	1.416(−3)	4.612(−4)	1.457(−4)	4.495(−5)

Table 2. Errors $E_{G,n}$ for scheme (2.9) on Shishkin mesh

l	N					
	32	64	128	256	512	1024
1	9.573(−7)	5.975(−8)	3.733(−9)	2.335(−10)	1.523(−11)	2.877(−12)
2	9.060(−7)	5.655(−8)	3.533(−9)	2.207(−10)	1.323(−11)	6.460(−12)
3	8.470(−7)	5.288(−8)	3.304(−9)	2.066(−10)	1.300(−11)	1.524(−12)
4	8.088(−7)	5.050(−8)	3.156(−9)	1.972(−10)	1.227(−11)	5.638(−13)
5	8.247(−7)	5.150(−8)	3.218(−9)	2.011(−10)	1.254(−11)	7.598(−13)
6	8.890(−7)	5.551(−8)	3.469(−9)	2.168(−10)	1.361(−11)	9.140(−13)
7	9.644(−7)	5.999(−8)	3.743(−9)	2.340(−10)	1.469(−11)	7.919(−13)
8	4.975(−6)	3.060(−7)	1.905(−8)	1.189(−9)	7.441(−11)	5.094(−12)
9	2.056(−5)	1.309(−6)	8.123(−8)	5.079(−9)	3.173(−10)	2.002(−11)
10	5.006(−5)	5.395(−6)	3.323(−7)	2.069(−8)	1.292(−9)	8.081(−11)
11	5.027(−5)	6.338(−6)	7.212(−7)	7.679(−8)	5.191(−9)	3.244(−10)
12	5.036(−5)	6.346(−6)	7.220(−7)	7.688(−8)	7.688(−9)	7.319(−10)
13	6.995(−5)	6.349(−6)	7.222(−7)	7.690(−8)	7.690(−9)	7.321(−10)
14	1.530(−4)	8.061(−6)	7.223(−7)	7.691(−8)	7.690(−9)	7.322(−10)
15	2.667(−4)	2.069(−5)	8.033(−7)	7.691(−8)	7.690(−9)	7.322(−10)

Table 3. Estimated convergence orders P_n for scheme (2.10)

l	N				
	32	64	128	256	512
1	2.00	2.00	2.00	2.00	2.00
2	2.00	2.00	2.00	2.00	2.00
3	2.00	2.00	2.00	2.00	2.00
4	2.00	2.00	2.00	2.00	2.00
5	2.00	2.00	2.00	2.00	2.00
6	2.00	2.00	2.00	2.00	2.00
7	2.01	2.00	2.00	2.00	2.00
8	2.02	2.01	2.00	2.00	2.00
9	1.98	2.01	2.00	2.00	2.00
10	1.64	2.02	2.00	2.00	2.00
11	1.53	1.58	1.62	1.94	2.00
12	1.53	1.58	1.62	1.66	1.70
13	1.53	1.58	1.62	1.66	1.70
14	1.53	1.58	1.62	1.66	1.70
15	1.53	1.58	1.62	1.66	1.70

Table 4. Estimated convergence orders $P_{G,n}$ for scheme (2.9)

l	N				
	32	64	128	256	512
1	4.00	4.00	4.00	3.94	2.40
2	4.00	4.00	4.00	4.06	1.03
3	4.00	4.00	4.00	4.00	3.09
4	4.00	4.00	4.00	4.00	4.44
5	4.00	4.00	4.00	4.00	4.04
6	4.00	4.00	4.00	3.99	3.90
7	4.00	4.00	4.00	3.99	4.21
8	4.02	4.01	4.00	4.00	3.87
9	3.97	4.01	4.00	4.00	4.00
10	3.21	4.02	4.00	4.00	4.00
11	2.99	3.13	3.23	3.89	4.00
12	3.00	3.14	3.23	3.32	3.39
13	3.46	3.13	3.23	3.32	3.39
14	4.25	3.48	3.23	3.32	3.39
15	3.69	4.69	3.38	3.38	3.30

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