Zbornik radova Prirodno-matematičkog fakulteta — Univerzitet u Novom Sadu, knjiga 5 (1975)
Review of Research of Science — University of Novi Sad, Volume 5 (1975)

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GENERALIZED ENTROPY ON INFINITARY QUASIGROUPS

This paper continues the investigation of infinitary quasigroups which begun in [1].

First infinitary quasigroups additive over an abelian group are defined and the existence of such nontrivial quasigroups proved. Some properties of additive infinitary quasigroups are determined. Further, the general solution of the functional equation

$$(1) \qquad A(B_1(x_1^\infty), B_2(y_1^\infty)) = C(D_1(x_1, y_1), D_2(x_2, y_2), \ldots),$$

where A, D_i, $i=1, 2,\ldots$ are binary quasigroups and B_1, B_2, C infinitary quasigroups of the type ω, all defined on the same nonempty set Q, is obtained.

Here the following notation is used: the sequence $x_m, x_{m+1}, \ldots, x_n$ will be denoted by x_m^n or $\{x_i\}_{i=m}^n$. If $m>n$, x_m^n is considered empty and if $m=n$ the x_m^n is the element x_m. The infinite sequence $x_m, x_{m+1}, \ldots, x_n, \ldots$ will be denoted by x_m^∞ or $\{x_i\}_{i=m}^\infty$ (m finite natural number).

The infinite sequence $x, x, \ldots, x, \ldots$ will be denoted by $\overset{\infty}{x}$.

There follow definitions of some basic notions in the theory of infinitary quasigroups.

If Q is a nonempty set and α any ordinal, Q^α denotes the set of all sequences of order type α, of elements from Q. A mapping $A: Q^\alpha \to Q$, $(\alpha \geq \omega)$ will be called an infinitary operation of the type α defined on Q.

A set Q together with an infinitary operation A of type ω will be called an infinitary quasigroup of type ω (briefly ω-quasigroup) if the equation

$$A(a_1^{i-1}, x, a_{i+1}^\infty) = b,$$

has a unique solution x for all a_1^∞, $b \in Q$ and for every positive integer i.

(The definition of a quasigroup of arbitrary type was is given in [1]).

Two quasigroups B and A of the type ω defined on the same set Q are called isotopic if there exists a sequence $T=\alpha_0^\infty$ of permutations of Q such that

$B(x_1^\infty)=\alpha_0^{-1}A(\{\alpha_i x_i\}_{i=1}^\infty)$ for all $x_1^\infty \in Q$. This will be denoted by $B=A^T$. The usual theorems for isotopy of finitary quasigroups are also valid for the infinitary case.

Some other notions and results about infinitary quasigroups can be found in [1].

1°

Definition. *Let Q be a nonempty set, $Q(+)$ an abelian group and $Q\ (R)$ an infinitary quasigroup of the type ω, so that*

$$R(\{x_i+y_i\}_{i=1}^\infty)=R(x_1^\infty)+R(y_1^\infty),$$

for all x_1^∞, $y_1^\infty \in Q$, then ω-quasigroup $Q(R)$ is called additive over abelian group $Q(+)$.

First we shall prove the existence of nontrivial ω-quasigroups additive over an abelian group (an infinitary quasigroup $Q\ (A)$ where set Q contains only one element is called trivial).

Let $Q=\{0, 1,\ldots, m\text{-}1\}$ be the set of residue classes of integers modulo m, and $Q\ (+)$ the additive group of integers modulo m. On set Q^ω of all infinite sequences (of type ω) of elements from Q the following operation is defined:

$$(x_1^\infty)\oplus(y_1^\infty)=(\{x_i+y_i\}_{i=1}^\infty).$$

Set Q^ω obviously forms an abelian group under this operation.

On set Q^ω we introduce the equivalence relation „$\sim$„ in the following way: two sequences from Q^ω are equivalent if and only if they differ only in a finite number of members. It is easy to see that this relation is an equivalence relation.

In set Q^ω, let H be the subset of all sequences in which only a finite number of members are different from 0. It is not difficult to see that H is a subgroup of group $Q^\omega\ (\oplus)$. The elements of the factor group Q^ω/H are in fact equivalence classes from the quotient set $Q^\omega/\sim$. Indeed, two arbitrary sequences from the same coset $(a_1^\infty)\oplus H$ differ only in a finite number of members, that is, they belong to the same equivalence class. If (b_1^∞) and (c_1^∞) are any two sequences from the same equivalence class, then the sequence $(\{b_i-c_i\}_{i=1}^\infty)$ has only a finite number of elements different from 0 and hence belongs to H, that is, (b_1^∞) and (c_1^∞) belong to the same coset in the factor group Q^ω/H.

Now we shall prove that H is a serving*) subgroup of group Q^ω.

Let $(c_1^\infty)\in H$, and n a positive integer and let us suppose that there exists a sequence $(x_1^\infty)\in Q^\omega$ such that

$$n(x_1^\infty)=(c_1^\infty).$$

*A subgroup C of an abelian group G is called a serving (or pure) subgroup if for any element $c\in G$ and any positive integer n for which the equation

$$nx=c,$$

has a solution in G, it has a solution in subgroup C. ([3], p.148).

In the sequence (c_1^∞) only a finite number of members are different from 0. By M we denote the set of all indices of the members which are different from 0. Then $c_i \neq 0$, $i \in M$; $c_k = 0$, $k \in N \setminus M$ (where N is the set of all natural numbers). The sequence $(y_1^\infty) \in H$ defined by $y_i = x_i$, $i \in M$; $y_k = 0$, $k \in N \setminus M$, satisfies the equation

$$n(y_1^\infty) = (c_1^\infty),$$

which means that H is a serving subgroup of the group Q^ω.

Group Q^ω is periodic, orders of all its elements are not greater than m, the same is true of its serving subgroup H and hence*) subgroup H is a direct summand of Q^ω.

This means that there exists a subgroup K of group Q^ω such that Q^ω is the direct sum of the subgroups H and K. Then $H \cap K = \{\overset{\infty}{(0)}\}$ and every sequence from Q^ω can be uniquely represented as the sum of a sequence from H and a sequence from K. Hence, in every coset of the factor group Q^ω/H (i. e. in every equivalence class from the set $Q^\omega/\sim$) there is exactly one sequence from subgroup K.

On set Q an infinitary operation $A: Q^\omega \to Q$ will now be defined. Let A maps all elements from K in 0, i. e. $A\ (x_1^\infty) = 0$, for every $(x_1^\infty) \in K$ If (a_1^∞) is an arbitrary sequence from Q and if (k_1^∞) is the sequence from K which belongs to the same class as (a_1^∞), then $A\ (a_1^\infty)$ is defined by

$$A(a_1^\infty) = \sum_{i=1}^{\infty} (a_i - k_i).$$

$A(a_1^\infty)$ is well defined since $a_i - k_i$ is different from 0 only for a finite number of indices i.

It follows from the results of [1] that $Q(A)$ is an infinitary quasigroup.

We shall prove that $Q(A)$ is an infinitary quasigroup additive over $Q(+)$. Let (x_1^∞) and (y_1^∞) be two arbitrary elements from set Q^ω. If $(s_1^\infty) \in K$ is an element which is equivalent to (x_1^∞), and $(t_1^\infty) \in K$ element equivalent to (y_1^∞), then $(\{x_i + y_i\}_{i=1}^\infty)$ must be equivalent to $(\{s_i + t_i\}_{i=1}^\infty) \in K$ (because cosets of the factor group Q^ω/H conicide with equivalence classes). Then

$$A(x_1^\infty) + A(y_1^\infty) = \sum_{i=1}^{\infty} (x_i - s_i) + \sum_{i=1}^{\infty} (y_i - t_i) =$$

$$= \sum_{i=1}^{\infty} ((x_i + y_i) - (s_i + t_i)) = A(\{x_i + y_i\}_{i=1}^\infty),$$

i. e., A is an infinitary quasigroup additive over $Q(+)$.

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*By the theorem: if a serving subgroup C of an abelian group G is periodic, and the set of orders of its elements is bounded, then C is a direct summand of G ([3], p. 151).

So we have proved the existence of nontrivial infinitary quasigroups additive over an abelian group. We shall now determine some properties of such quasigroups.

Let $Q(R)$ be an infinitary quasigroup additive over an abelian group $Q\ (+)$, that is

$$R(\{x_i+y_i\}_{i=1}^{\infty})=R(x_1^{\infty})+R(y_1^{\infty}), \tag{2}$$

for every (x_1^{∞}), $(y_1^{\infty})\in Q^{\omega}$.

If we put $x_2=x_3=\ldots=y_2=y_3=\ldots=0$ (where 0 is the unit of group $Q(+)$) in (2) we shall have

$$R(x_1+y_1, \overset{\infty}{0})=R(x_1, \overset{\infty}{0})+R(y_1, \overset{\infty}{0}),$$

i. e., the mapping α_1 defined by $\alpha_1 x=R(x, \overset{\infty}{0})$ is an automorphism of group $Q(+)$. If we put $x_2=x_3=\ldots=0$, $y_1=0$, in (2) we obtain

$$R(x_1, y_2^{\infty})=R(x_1, \overset{\infty}{0})+R(0, y_2^{\infty}),$$

$$R(z_1^{\infty})=\alpha_1 z_1+R(0, z_2^{\infty}). \tag{3}$$

Let $R(0, z_2^{\infty})=R_1(z_2^{\infty})$. It is obvious that R_1 is also an infinitary quasigroup additive over $Q(+)$.

By an analogous procedure, we obtain

$$R_1(z_2^{\infty})=\alpha_2 z_2+R_1(0, z_3^{\infty})$$

or

$$R_1(z_2^{\infty})=\alpha_2 z_2+R_2(z_3^{\infty}), \tag{4}$$

where α_2 is an automorphism of the group $Q(+)$, and R_2 is an infinitary quasigroup additive over $Q(+)$.

From (3) and (4) it follows

$$R(z_1^{\infty})=\alpha_1 z_1+\alpha_2 z_2+R_2(z_3^{\infty}).$$

If we continue this procedure after n steps we shall have

$$R(z_1^{\infty})=\alpha_1 z_1+\alpha_2 z_2+\ldots+\alpha_n z_n+R_n(z_{n+1}^{\infty}), \tag{5}$$

and so we have found that every infinitary quasigroup $Q\ (R)$ additive over an abelian group Q $(+)$ can be represented in the form of (5), where n is an arbitrary natural number, α_i, $i=1, 2,\ldots,n$ automorphisms of the group $Q(+)$ and $Q(R_n)$ an infinitary quasigroup additive over $Q(+)$.

2°

Now we shall consider the functional equation

$$(1) \qquad A(B_1(x_1^\infty), B_2(y_1^\infty)) = C(D_1(x_1, y_1), D_2(x_2, y_2), \ldots),$$

where A, $D_i = 1, 2, \ldots$ are binary quasigroups and B_1, B_2, C infinitary quasigroups of type ω, all defined on the same nonempty set Q.

Let i, j be arbitrary natural numbers (here we suppose $i<j$, the procedure is analogous if $i>j$). If in (1) we replace variables x_1^{i-1}, x_{i+1}^{j-1}, x_{j+1}^∞, y_1^{i-1}, y_{i+1}^{j-1}, y_{j+1}^∞ by fixed elements a_1^{i-1}, a_{i+1}^{j-1}, a_{j+1}^∞, b_1^{i-1}, b_{i+1}^{j-1}, b_{j+1}^∞ from Q respectively, we obtain

$$(6) \qquad A(B'_1(x_i, x_j), B'_2(y_i, y_j)) = C'(D_i(x_i, y_i), D_j(x_j, y_j)),$$

where $B'_1(x_i, x_j) = B_1(a_1^{i-1}, x_i, a_{i+1}^{j-1}, x_j, a_{j+1}^\infty)$, $\quad B'_2(y_i, y_j) =$

$$= B_2(b_1^{i-1}, y_i, b_{i+1}^{j-1}, y_j, b_{j+1}^\infty),$$

$$C'(x, y) = C(D_1(a_1, b_1), \ldots, D_{i-1}(a_{i-1}, b_{i-1}), x, D_{i+1}(a_{i+1}, b_{i+1}), \ldots$$

$$\ldots, D_{j-1}(a_{j-1}, b_{j-1}), y, D_{j+1}(a_{j+1}, b_{j+1}), \ldots).$$ The equation

(6) is the functional equation of generalized entropy on binary quasigroups and from [2] it follows that

$$(7) \qquad A(x, y) = \alpha x + \beta y,$$

$$(8) \qquad D_i(x, y) = \varphi_i^{-1}(\gamma_i x + \theta_i y),$$

$$(9) \qquad D_j(x, y) = \varphi_j^{-1}(\gamma_j x + \theta_j y),$$

where $Q(+)$ is an abelian group and α, β, φ_i, φ_j, γ_i, γ_j, θ_i, θ_j permutations of set Q. We shall show that this abelian group does not depend upon the choice of i and j. If we choose some other natural numbers k and m instead of i and j we find analogously that there exists an abelian group $Q(\oplus)$ such that

$$(7') \qquad A(x, y) = \alpha' x \oplus \beta' y,$$

where α' and β' are permutations of set Q. From (7) and (7') we have

$$\alpha x + \beta y = \alpha' x \oplus \beta' y,$$

and so $+$ and $\oplus$ are isotopic groups, and by Albert's familiar theorem they are isomorphic. Hence, in the equations (7), (8), (9) the abelian group $Q(+)$ is fixed for every i, j.

From (6), (7) and (8) there follows

$$\alpha B_1(x_1^\infty)+\beta B_2(y_1^\infty)=C(\{\varphi_i^{-1}(\gamma_i x_i+\theta_i y_i)\}_{i=1}^\infty). \tag{10}$$

Putting $\gamma_i^{-1}x_i$ instead of x_i, $\theta_i^{-1}y_i$ instead of y_i and introducing isotopes

$$B_1^{(\alpha^{-1},\ \{\gamma_i^{-1}\}_{i=1}^\infty)}=\bar{B}_1,\ B_2^{(\beta^{-1},\{\theta_i^{-1}\}_{i=1}^\infty)}=\bar{B}_2, \tag{11}$$

we obtain

$$\bar{B}_1(x_1^\infty)+\bar{B}_2(y_1^\infty)=C(\{\varphi_i^{-1}(x_i+y_i)\}_{i=1}^\infty). \tag{12}$$

If we introduce the isotope $C^{(\varepsilon,\ \{\varphi_i^{-1}\}_{i=1}^\infty)}=\bar{C}$, (where ε denotes identity permutation) we shall have

$$\bar{B}_1(x_1^\infty)+\bar{B}_2(y_1^\infty)=\bar{C}(\{x_i+y_i\}_{i=1}^\infty). \tag{13}$$

Fixing in (13) first $y_i=0$, $i=1,2,\ldots$ and then $x_i=0$, $i=1.2,\ldots$ (by 0 we denote the unit of the group $Q(+)$) it follows

$$\bar{B}_1(x_1^\infty)+a=\bar{C}(x_1^\infty), \tag{14}$$

$$\bar{B}_2(y_1^\infty)+b=\bar{C}(y_1^\infty), \tag{15}$$

where $\bar{B}_2(\overset{\infty}{0})=a$, $\bar{B}_1(\overset{\infty}{0})=b$, and by (13)

$$\bar{C}(\{x_i+y_i\}_{i=1}^\infty)=\bar{C}(x_1^\infty)+\bar{C}(y_1^\infty)-c, \tag{16}$$

where $c=a+b$.
Putting

$$\bar{C}(x_1^\infty)-c=R(x_1^\infty), \tag{17}$$

from (16) it follows that R is an infinitary quasigroup additive over $(+)$, i. e.

$$R(\{x_i+y_i\}_{i=1}^\infty)=R(x_1^\infty)+R(y_1^\infty). \tag{18}$$

We shall express B_1, B_2 and C by R. From (11), (14) and (17) we obtain

$$B_1(x_1^\infty)=\alpha^{-1}\bar{B}_1(\{\gamma_i x_i\}_{i=1}^\infty),$$

$$\bar{B}_1(x_1^\infty)=\bar{C}(x_1^\infty)-a=R(x_1^\infty)+c-a=R(x_1^\infty)+b.$$

Hence,

(19) $$B_1(x_1^\infty)=\alpha^{-1}(R(\{\gamma_i x_i\}_{i=1}^\infty)+b).$$

Similarly,

(20) $$B_2(x_1^\infty)=\beta^{-1}(R(\{\theta_i x_i\}_{i=1}^\infty)+a).$$

Since $C=\bar{C}^{(\varepsilon,\ \varphi_1^\infty)}$ it follows

(21) $$C(x_1^\infty)=\bar{C}(\{\varphi_i x_i\}_{i=1}^\infty)=R(\{\varphi_i x_i\}_{i=1}^\infty)+c.$$

On the basis of previous results, we can formulate the following theorem:

Theorem. *All solutions of the functional equation (1) are given by equations (7), (8), (19), (20), (21) where $Q\ (+)$ is an arbitrary abelian group, $Q\ (R)$ an arbitrary infinitary quasigroup of type ω additive over $Q\ (+)$, α, β, $\gamma_i\ \theta_i$, φ_i, $i=1, 2, \ldots$ arbitrary permutations of the set Q, a, b, c arbitrary elements from Q such that $a+b=c$.*

It has been shown that the conditions of the theorem are necessary. It is simple to check that they are sufficient:

$$A(B_1(x_1^\infty),\ B_2(y_1^\infty))=\alpha B_1(x_1^\infty)+\beta B_2(y_1^\infty)=$$

$$=R(\{\gamma_i x_i\}_{i=1}^\infty)+b+R(\{\theta_i y_i\}_{i=1}^\infty)+a=$$

$$=R(\{\gamma_i x_i+\theta_i x_i\}_{i=1}^\infty)+c.$$

$$C(D_1(x_1,\ y_1),\ D_2(x_2,\ y_2),\ldots)=$$

$$=R(\{\varphi_i D_i(x_i,\ y_i)\}_{i=1}^\infty)+c=R(\{\gamma_i x_i+\theta_i y_i\}_{i=1}^\infty)+c.$$

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UOPŠTENA ENTROPIJA NA INFINITARNIM KVAZIGRUPAMA

Izvod

U ovom radu nastavljeno je ispitivanje infinitarnih kvazigrupa početo u [1].

Definisane su infinitarne kvazigrupe aditivne nad Abelovom grupom i dokazana egzistencija takvih netrivijalnih infinitarnih kvazigrupa. Ispitane su neke osobine infinitarnih kvazigrupa aditivnih nad Abelovom grupom. Određena su sva rešenja funkcionalne jednačine na infinitarnim kvazigrupama koja predstavlja uopštenje funkcionalne jednačine opšte entropije.