NOVI SAD J. MATH.
VOL. 30, NO. 2, 2000, 15-17

Proceedings of the TARA 2000 Conference
Novi Sad, Yugoslavia, September 6-7, 2000

ON COMPLETNESS CRITERION FOR PARTIAL HYPEROPERATIONS

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Abstract. For a finite set A an analogue of the Slupecki criterion [6] is obtained for the set of partial hyperoperatons This result is of the same type as the one in [3] for the set of partial operations.

AMS Mathematics Subject Classification (1991): 0840

Key words and phrases: clone, completness

1. Preliminaries

Let A be a nonempty set. For a positive integer n, the mapping from A^n into the family $P(A)$ of all subsets of A is called a *partial n-hyperoperation on* A. Denote by $\mathcal{H}_p^{(n)}$ the set of partial n-hyperoperations on A and by $\mathcal{H}_p$ the set of all partial hyperoperations on A i.e. $\mathcal{H}_p = \cup_{n\geq 0}\mathcal{H}_p^{(n)}$. A map $f \in \mathcal{H} = \{f | f : A^n \to P(A) \setminus \{\emptyset\}\}$ is called *hyperoperations* [5]. $(\mathcal{H} \subseteq \mathcal{H}_p)$

If we suppose that there is no difference between an element $a \in A$ and the corresponding one element subset $\{a\}$ of A, then every n-ary operation $f : A^n \to A$ can be considered as a special partial hyperoperation. Partial operations $f : dom(f) \to A$, where $dom(f) \subseteq A^n$, are also special partial hyperoperations. If A is a set, then $|A|$ is the cardinality of A. Namely, $f \in \mathcal{H}_p$ with $|f(x)| \leq 1$ for all $x \in A^n$ de facto is a partial operation on A.

For a positive integer n and for $1 \leq i \leq n$, e_i^n is a *partial n-hyperprojection* if $e_i^n(x_1, \ldots, x_n) = \{x_i\}$ for all $x_1, \ldots, x_n \in A$. (If we do not make difference between a and $\{a\}$, then a partial n-hyperprojection is usual n-ary projection.)

For $n, m \geq 1, f \in \mathcal{H}_p^{(n)}$ and $g_1, \ldots, g_n \in \mathcal{H}_p^{(m)}$, *the composition of f and* $g_1, \ldots, g_n$, denoted by $f(g_1, \ldots, g_n) \in \mathcal{H}_p^{(m)}$, is defined by

$$f(g_1, \ldots, g_n)(x_1, \ldots, x_m) = \bigcup_{\substack{y_i \in g_i(x_1, \ldots, x_m) \\ 1 \leq i \leq n}} f(y_1, \ldots, y_n)$$

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for all $(x_1,\ldots,x_m) \in A^m$.

The set $C \subseteq \mathcal{H}_p$ is a *clone* of partial hyperoperations on A if C is composition closed and C contains all partial n-hyperprojections (for short projections).

For $F \subseteq \mathcal{H}_p$, $\langle F\rangle_{\mathrm{CL}}$ stands for the clone of partial hyperoperations generated by F.

We say that $f: A^n \to A$ depends on its i-th variable if there are $a_1,\ldots,a_{i-1}$, $a_{i+1},\ldots,a_n$ such that $h: A \to A$ defined by the setting
$h(x) := f(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n)$ for all $x \in A$ is non-constant.

Usual n-ary operation $f: A^n \to A$ is *essential* if it results in all values from A and depends on at least two variables. An analogue of the hyperoperation $f: A^n \to P(A)\setminus\{\emptyset\}$ where $|f(x)| = 1$ is called *essential* if $|im(f)| = |A|$ and depends on at least two variables.

A subset F of $\mathcal{H}_p$ is *complete (or primal)* in $\mathcal{H}_p$ if $\langle F\rangle_{\mathrm{CL}} = \mathcal{H}_p$.

2. Slupecki-type criterion

Let $M_1 = \mathcal{H} = \bigcup_{n\in N}\{f \in \mathcal{H}_p^{(n)} : |f(x)| \geq 1 \text{ for all } x \in A^n\}$,
$O_A = \bigcup_{n\in N}\{f \in \mathcal{H}_p^{(n)} : |f(x)| = 1 \text{ for all } x \in A^n\}$ and
$P_A = \bigcup_{n\in N}\{f \in \mathcal{H}_p^{(n)} : |f(x)| \leq 1 \text{ for all } x \in A^n\}$.

It is clear that the previous sets are clones of partial hyperoperations.

Lemma 1 *If $f \in \mathcal{H}_p\setminus(P_A\cup M_1)$, then $\langle O_A \cup f\rangle_{\mathrm{CL}} = \mathcal{H}_p$.*

Proof. It is obvious that $\langle O_A \cup f\rangle_{\mathrm{CL}} \subseteq \mathcal{H}_p$. Now we shall prove that $\langle O_A \cup f\rangle_{\mathrm{CL}} \supseteq \mathcal{H}_p$. Let $f \in \mathcal{H}_p\setminus(P_A\cup M_1)$ be an n-ary function. Then, there exist $\mathbf{a} = (a_1,\ldots,a_n) \in A^n$ and $\mathbf{b} = (b_1,\ldots,b_n) \in A^n$ such that $f(\mathbf{a}) = \emptyset$ and $f(\mathbf{b}) = \{c_0,\ldots,c_{p-1}\}$, where $p \geq 2$. Let $h \in \mathcal{H}_p^{(m)}$. We define $f_1,\ldots,f_n \in O_A^{(m)}$ and $g \in O_A^{(m+\ell)}$. If $h(y_1,\ldots,y_m) = \emptyset$ then $(f_1(y_1,\ldots,y_m),\ldots,f_n(y_1,\ldots,y_m)) = (\{a_1\},\ldots,\{a_n\})$ and $g(y_1,\ldots,y_m,x_1,\ldots,x_\ell)$ is arbitrary for all $x_1,\ldots,x_\ell \in A$. If $h(y_1,\ldots,y_m) = \{d_0,d_1,\ldots,d_{q-1}\}, q \geq 1$ then
$(f_1(y_1,\ldots,y_m),\ldots,f_n(y_1,\ldots,y_m)) = (\{b_1\},\ldots,\{b_n\})$ and

$$\begin{aligned}
g(y_1,\ldots,y_m,c_0,\ldots,c_0,c_0) &= \{d_0\}\\
g(y_1,\ldots,y_m,c_0,\ldots,c_0,c_1) &= \{d_1\}\\
&\vdots\\
g(y_1,\ldots,y_m,c_{p-1},\ldots,c_{p-1},c_{p-1}) &= \{d_{q-1}\}
\end{aligned}$$

where $\ell \in \mathbf{N}$ is a number such that $p^{\ell-1} < \max\limits_{(y_1,\ldots,y_m)\in A^m} |h(y_1,\ldots,y_m)| \leq p^\ell$.
More precisely $g(y_1,\ldots,y_m,c_{i_1},\ldots,c_{i_{\ell-1}},c_{i_\ell}) = \{d_i\}$ where

$$i = \begin{cases} i_1p^{\ell-1} + i_2p^{\ell-2} + \ldots + i_\ell p^0 & \text{if } i_1p^{\ell-1} + i_2p^{\ell-2} + \ldots + i_\ell p^0 \leq q-1\\ q-1 & \text{else}\end{cases}$$

Now we can prove that $h = g(e_1^m, \ldots, e_m^m, f(f_1, \ldots, f_n), \ldots, f(f_1, \ldots, f_n))$, which implies $h \in \langle O_A \cup f \rangle_{\mathrm{CL}}$. For $h(y_1, \ldots, y_m) = \emptyset$ the statement is obvious and for $h(y_1, \ldots, y_m) = \{d_0, d_1, \ldots, d_{q-1}\}$ follows:
$g(e_1^m, \ldots, e_m^m, f(f_1, \ldots, f_n), \ldots, f(f_1, \ldots, f_n))(y_1, \ldots, y_m) =$
$g(\{y_1\}, \ldots, \{y_m\}, \{c_0, \ldots, c_{p-1}\}, \ldots, \{c_0, \ldots, c_{p-1}\}) = g(y_1, \ldots, y_m, c_0, \ldots, c_0) \cup$

$\cup (y_1, \ldots, y_m, c_0, \ldots, c_0, c_1) \cup \ldots \ldots \cup g(y_1, \ldots, y_m, c_{p-1}, \ldots, c_{p-1}, c_{p-1}) =$
$= \{d_0, d_1, \ldots, d_{q-1}\} = h(y_1, \ldots, y_m).$

($\subseteq$) It is obvious, since $f \in \mathcal{H}_p$ and $O_A \subseteq \mathcal{H}_p$. □

Theorem 1 *Let A be finite. If $F \subseteq \mathcal{H}_p$ satisfies the following three conditions*
1. F contains an essential operation,
2. F generates all unary operations, and
3. F contains partial hyperoperation $f \in \mathcal{H}_p \setminus (P_A \cup M_1)$,
then F is complete.

Proof. By the Slupecki criterion from 1. and 2. follows $O_A \subseteq \langle F \rangle_{\mathrm{CL}}$, and by the previous lemma and by 3. we obtain $\mathcal{H}_p = \langle O_A \cup f \rangle_{\mathrm{CL}} \subseteq \langle F \rangle_{\mathrm{CL}} \subseteq \mathcal{H}_p$, i.e. $\langle F \rangle_{\mathrm{CL}} = \mathcal{H}_p$. □

In the opposite direction the previous theorem is not valid because it is easy to prove that $\langle P_A \cup M_1 \rangle_{\mathrm{CL}} = \mathcal{H}_p$, or more precisely, we can prove that $\langle O_A \cup \{g\} \cup \{h\} \rangle_{\mathrm{CL}} = \mathcal{H}_p$, where $g \in M_1 \setminus P_A$ and $h \in P_A \setminus M_1$.

Corollary 1 *Let A be finite and $f \in \mathcal{H}_p$. Then, $\{f\}$ is complete if and only if $f \in Poli_A \setminus (P_A \cup M_1)$ and $\langle f \rangle_{\mathrm{CL}}$ contains all unary operations and at least one essential operation.*

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