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MATRIX RELATIONS FOR TOTAL COMPLEX DECOMPOSITION OF SWITCHING FUNCTIONS

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Abstract. Some approaches in logical network synthesis use decomposition of a switching function into partial ones, which possess properties convenient for an optimal realization. *The total complex decomposition (TCD)* is one of such methods introduced in previous papers. It appeared to be very useful in the realization employing some logic strictures known as programmable logic arrays.

The method of using TCD in order to find a representation of switching function suitable for realization with chosen architecture has been already proposed. As a continuation of this method, the paper presents matrixes relations for examination of TCD with specific properties.

Classes of matrixes convenient for using fast algorithms of FTF type have been worked out. The proposed procedure examines noted type of decomposition and, in the case of its existence, finds composite expressions needed for direct realization.

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1. Introduction and background

In logic design, complexity of realization of the switching function f might be significantly reduced if f can be represented as a composition of some partial simpler functions. Therefore, the decomposition of switching function has been studied since the early beginning of the switching theory. Researches are oriented to theoretical bases and practical methods for decomposition, which should satisfy different design criteria in order to reduce overall costs of realization. These efforts have been increasing in recent years because of technological demands for networks with very high complexity [1, 2, 3, 4, 5, 6, 7, 16, 18, 19].

Results presented in the paper are a continuation of previous researches on solving the problem of optimal synthesis of a logic network with the specific structure [8, 9, 10, 11, 12, 13, 14].

The total complex decomposition (TCD) is introduced in [9, 12] as a generalization of the complex decomposition and the iterative decomposition, both in disjunctive and in non-disjunctive form, see for example, [3, 4, 6, 7, 15].

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A completely defined switching function $f(\mathbf{X}) : \{0,1\}^n \to \{0,1\}$ is given. Let us split $\mathbf{X} = \{x_1, \ldots, x_n\}$ as follows: $\mathbf{X} = \cup_{i=1}^{r} \mathbf{X}_i$, $1 < r \leq n$, $\mathbf{X}_1 = \{x_1^{(1)}, \ldots, x_{\lambda_1}^{(1)}\}$, $\ldots$, $\mathbf{X}_i = \{x_1^{(i)}, \ldots, x_{\lambda_i}^{(i)}\}$, $\ldots$, $\mathbf{X}_r = \{x_1^{(r)}, \ldots, x_{\lambda_r}^{(r)}\}$, $1 < \lambda_i \leq n$, $x_j^{(i)} \in \mathbf{X}$, $\forall j \in \{1, \ldots, \lambda_i\}$.

Definition 1 $f(\mathbf{X})$ *possesses TCD* if it can be written as

$$f(\mathbf{X}) = g(h_1(\mathbf{X}_1), \ldots, h_i(\mathbf{X}_i), \ldots, h_r(\mathbf{X}_r))$$

where $h_1, \ldots, h_n$ are switching functions.

If it holds $\mathbf{X}_i \cap \mathbf{X}_j \neq \emptyset$, $\forall i, j \in \{1, \ldots, r\}$, $i \neq j$, TCD is *dijunctive* and has the following form

$$f(\mathbf{X}) = g(h_1(x_1^{(1)}, \ldots, x_{\lambda_1}^{(1)}), \ldots, h_r(x_{\lambda_1+\lambda_2+\cdots+\lambda_{r-1}+1}^{(r)}, \ldots, x_n^{(r)}).$$

If it stands $\lambda_i = \lambda$, $\forall i = 1, \ldots, r$, $0 < \lambda \leq n$, TCD is *uniform* and it is reduced to

$$f(\mathbf{X}) = g(h_1(x_1^{(1)}, \ldots, x_{\lambda}^{(1)}), \ldots, h_r(x_1^{(r)}, \ldots, x_{\lambda}^{(r)})).$$ ■

It is shown in [8, 9, 11, 12, 13] that TCD is useful in realization of switching functions by programmable logic devices, in particular PLAs (programmable logic arrays) [2]. In such applications, it is very important to provide procedures for checking TCD which are efficient in terms of space and time.

The mathematical model of PLA with decoders can be expressed as follows [10, 11]

(1) $$f(\mathbf{X}) = g_1(\mathbf{X}) + \cdots + g_j(\mathbf{X}) + \cdots + g_k(\mathbf{X})$$

(2) where $$g_j(\mathbf{X}) = h_1^{(j)}(\mathbf{X}_1) \cdots h_i^{(j)}(\mathbf{X}_i) \cdots h_r^{(j)}(\mathbf{X}_r).$$

(3) The function $$h_i^{(j)}(\mathbf{X}_i) = \prod_{q=0}^{2^\lambda-1} \left(s_q^{(i,j)}(\mathbf{X}_i) + c_q^{(i,j)}(\mathbf{X}_i)\right)$$

is a product of some maxterms in respect to $\mathbf{X}_i$: $s_q^{(i,j)}(\mathbf{X}_i) = \sum_{i=1}^{\lambda} \tilde{x}_{q,l}^{(j,l)}, \tilde{x}_{q,l}^{(j,l)} \in \{\bar{x}_{q,l}^{(j,l)}, x_{q,l}^{(j,l)}\}, x_{q,l}^{(j,l)} \in \mathbf{X}_i$ where the value of $c_q^{(i,j)}(\mathbf{X}_i) \in \{0,1\}$ depends on the appearance of the corresponding maxterm in Eq. (3).

There are some restrictions, caused by technological demands, to realization using a chosen structure: 1. completeness: $\mathbf{X} \setminus \cup_{i=1}^{r} \mathbf{X}_i = \emptyset$, 2. disjunctivity: $\mathbf{X}_i \cap \mathbf{X}_j = \emptyset$, and 3. uniformity: $|\mathbf{X}_i| = \lambda_i = \lambda$. Understanding them, the mathematical model of PLA is expressed in the form of the uniform disjunctive TCD and the uniform non-disjuncrive TCD according to Definition 1. Thus, the practical problem of finding an expression of f suitable for a direct realization using PLA, is transformed to a problem of finding TCD that is described by Eqs. (1) and (2).

The following terms are going to be introduced using a function $f(x_1,\dots,x_6)$ given by a truth vector $\mathbf{F}$ as an example. $\mathbf{X}_1$ is a subset of the first λ elements of $\mathbf{X}$. For $\lambda = 3$, we have $\mathbf{X}_1 = \{x_1, x_2, x_3\}$.

The matrix $\mathbf{F}$ *is partitioned about the variables* $\mathbf{X}_1$ into partitions, for short, $\mathbf{F}$ *by* $\mathbf{X}_1$ as follows

$$\mathbf{F}^{\mathbf{X}_1} = [\ \overbrace{10010101}^{\mathbf{P}_0^{\mathbf{F},\mathbf{X}_1}}\ \overbrace{00000000}^{\mathbf{P}_1^{\mathbf{F},\mathbf{X}_1}}\ \overbrace{10010101}^{\mathbf{P}_2^{\mathbf{F},\mathbf{X}_1}}\ \overbrace{01100100}^{\mathbf{P}_3^{\mathbf{F},\mathbf{X}_1}}$$

$$(4) \qquad \overbrace{01100100}^{\mathbf{P}_4^{\mathbf{F},\mathbf{X}_1}}\ \overbrace{00000000}^{\mathbf{P}_5^{\mathbf{F},\mathbf{X}_1}}\ \overbrace{01100100}^{\mathbf{P}_6^{\mathbf{F},\mathbf{X}_1}}\ \overbrace{11001111}^{\mathbf{P}_7^{\mathbf{F},\mathbf{X}_1}}]^t$$

Some partitions are *0-partitons* with only 0s: $\mathbf{P}_{null}^{\mathbf{F},\mathbf{X}_1} = \mathbf{P}_1^{\prime\mathbf{F},\mathbf{X}_1} = \mathbf{P}_5^{\prime\mathbf{F},\mathbf{X}_1}$. Among the *non-0-partitions*, there are *equal partitions.* The *ordered set of different non-0-partitions* is $\Phi'^{\mathbf{X}_1} = \left\{\mathbf{P}_1^{\prime\mathbf{F},\mathbf{X}_1}, \mathbf{P}_2^{\prime\mathbf{F},\mathbf{X}_1}, \mathbf{P}_3^{\prime\mathbf{F},\mathbf{X}_1}\right\}$, where $\mathbf{P}_1^{\prime\mathbf{F},\mathbf{X}_1} = \mathbf{P}_0^{\mathbf{F},\mathbf{X}_1} = \mathbf{P}_2^{\mathbf{F},\mathbf{X}_1}$, $\mathbf{P}_2^{\prime\mathbf{F},\mathbf{X}_1} = \mathbf{P}_3^{\mathbf{F},\mathbf{X}_1} = \mathbf{P}_4^{\mathbf{F},\mathbf{X}_1} = \mathbf{P}_6^{\mathbf{F},\mathbf{X}_1}$, $\mathbf{P}_3^{\prime\mathbf{F},\mathbf{X}_1} = \mathbf{P}_7^{\mathbf{F},\mathbf{X}_1}$. After remaching in Eq. (4)

$$(5)\mathbf{F}^{\mathbf{X}_1} = \left[\ \mathbf{P}_1^{\prime\mathbf{F},\mathbf{X}_1}\ \mathbf{P}_{null}^{\mathbf{F},\mathbf{X}_1}\ \mathbf{P}_1^{\prime\mathbf{F},\mathbf{X}_1}\ \mathbf{P}_2^{\prime\mathbf{F},\mathbf{X}_1}\ \mathbf{P}_2^{\prime\mathbf{F},\mathbf{X}_1}\ \mathbf{P}_{null}^{\mathbf{F},\mathbf{X}_1}\ \mathbf{P}_2^{\prime\mathbf{F},\mathbf{X}_1}\ \mathbf{P}_3^{\prime\mathbf{F},\mathbf{X}_1}\ \right]^t$$

The *diversity coefficient (DC)* is equal to the number of different non-0-partitions, $k_1 = |\Phi'^{\mathbf{X}_1}| = 3$.

The *characteristic vector (CV)* of $\mathbf{F}^{\mathbf{X}_1}$ is

$$\mathbf{F}_{car}^{\mathbf{X}_1} = \left[c_0^{(1)} \cdots c_q^{(1)} \cdots c_{2^\lambda-1}^{(1)}\right]^t,\ c_q^{(1)} = \begin{cases} j & \text{if } \mathbf{P}_q^{\mathbf{F},\mathbf{X}_1} = \mathbf{P}_j^{\prime\mathbf{F},\mathbf{X}_1} \\ 0 & \text{otherwise} \end{cases},\ j \in \{1,\dots,k_1\}$$

The *derived characteristic vector (DCV) by j* of DC is

$$\mathbf{F}_{j\,car}^{\mathbf{X}_1} = \left[c_0^{(1,j)} \cdots c_q^{(1,j)} \cdots c_{2^\lambda-1}^{(1,j)}\right]^t \quad \text{where} \quad c_q^{(1,j)} = \begin{cases} j & \text{if } c_q^{(1)} = j \\ 0 & \text{otherwise} \end{cases}$$

The *normalized form of DCV (NDCV)* is

$$\mathbf{F}_{j\,car}^{\prime\mathbf{X}_1} = \left[c_0^{\prime(1,j)} \cdots c_q^{\prime(1,j)} \cdots c_{2^\lambda-1}^{\prime(1,j)}\right]^t \quad \text{where} \quad c_q^{\prime(1,j)} = \begin{cases} 1 & \text{if } c_q^{(1,j)} \neq 0 \\ 0 & \text{if } c_q^{(1,j)} = 0 \end{cases}$$

The following theorems [9, 13] give a characterization of TCD. The symbols which will appear have the already defined meaning. $\mathbf{F}$ and Φ are related to $f(\mathbf{X})$, $\mathbf{G}$ and Γ to $g(\mathbf{X})$, $\mathbf{H}$ and H to $h(\mathbf{X})$. Recall the meaning: $\mathbf{P}_{null}$ – 0-partiton; Π_{null} – set of 0-partitions; $\Phi', \mathrm{H}', \Gamma'$ – sets of non-0-partitions; $\mathbf{F}_{car}^{\mathbf{X}_i}$, $\mathbf{G}_{car}^{\mathbf{X}_i}$, $\mathbf{H}_{car}^{\mathbf{X}_i}$ – CVs; $\mathbf{F}_{j\,car}^{\mathbf{X}_i}$, $\mathbf{G}_{j\,car}^{\mathbf{X}_i}$, $\mathbf{H}_{j\,car}^{\mathbf{X}_i}$ – DCVs by j; $\mathbf{F}_{j\,car}^{\prime\mathbf{X}_i}$, $\mathbf{G}_{j\,car}^{\prime\mathbf{X}_i}$, $\mathbf{H}_{j\,car}^{\prime\mathbf{X}_i}$ – NDCVs; $\mathbf{F}^{\mathbf{X}_i}$, $\mathbf{G}^{\mathbf{X}_i}$, $\mathbf{H}^{\mathbf{X}_i}$ – truth vectors by $\mathbf{X}_i$.

We are considering functions $f(\mathbf{X})$ and $g_1(\mathbf{X})$ given by $\mathbf{F}^{\mathbf{X}_1} = \left[\mathbf{P}_q^{\mathbf{F},\mathbf{X}_1}\right]_{q=0}^{t\,2^\lambda-1}$ and $\mathbf{G}_j^{\mathbf{X}_1} = \left[\mathbf{P}_q^{\mathbf{G}_j,\mathbf{X}_1}\right]_{q=0}^{t\,2^\lambda-1}$, where DC of $\mathbf{F}^{\mathbf{X}_1}$ is k_1.

Theorem 1. *The function $f(\mathbf{X})$ posseses the uniform non-disjunctive TCD, $f(\mathbf{X}) = \sum_{j=1}^{k_1} g_j(\mathbf{X})$, if it holds: 1. for partitions of $\mathbf{G}_j^{\mathbf{X}_1}$: $\mathbf{P}_q^{\mathbf{G}_j,\mathbf{X}_1} \in \left\{\mathbf{P}_{null}^{\mathbf{F},\mathbf{X}_1}, \mathbf{P}_j^{\prime\mathbf{F},\mathbf{X}_1}\right\}$, where $\mathbf{P}_{null}^{\mathbf{F},\mathbf{X}_1} \in \Pi_{null}^{\mathbf{F},\mathbf{X}_1}$, $\mathbf{P}_j^{\prime\mathbf{F},\mathbf{X}_1} \in \Phi^{\prime\mathbf{X}_1}$*

2. for elements of $\mathbf{G}_{j\,car}^{\mathbf{X}_1}$: $c_q^{(1,j),\mathbf{G}_j} = \begin{cases} j & \text{if } c_q^{(1,j),\mathbf{F}} = 0 \\ 0 & \text{if } c_q^{(1,j),\mathbf{F}} \neq 0 \end{cases}$,

where $c_q^{(1,j),\mathbf{G}_j} \in \mathbf{G}_{j\,car}^{\mathbf{X}_1}$, $c_q^{(1,j),\mathbf{F}} \in \mathbf{F}_{j\,car}^{\mathbf{X}_1}$. ■

An arbitrary function $g(\mathbf{X})$ is given by $\mathbf{G}^{\mathbf{X}_1}$, DC is k_1, CV is $\mathbf{G}_{car}^{\mathbf{X}_1} = \left[c_q^{(1)}\right]_{q=0}^{t\,2^\lambda-1}$. Let the ordered set $\mathbf{X}$ be partitioned as follows: $\mathbf{X} = \mathbf{X}_{\beta_0} = \mathbf{X}_1 \cup \mathbf{X}_{\beta_1}$, $\mathbf{X}_{\beta_1} = \mathbf{X}_2 \cup \mathbf{X}_{\beta_2}, \ldots, \mathbf{X}_{\beta_{i-1}} = \mathbf{X}_i \cup \mathbf{X}_{\beta_i}, \ldots, \mathbf{X}_{\beta_{r-1}} = \mathbf{X}_r \cup \mathbf{X}_{\beta_r}$, $\mathbf{X}_{\beta_r} = \emptyset, |\mathbf{X}_i| = \lambda$, and $\mathbf{X}_{\beta_1} = \mathbf{X}\backslash\mathbf{X}_1$ for $q = 0, \ldots, 2^\lambda - 1$, $i = 1, \ldots, r$, $r = n/\lambda$. $g(\mathbf{X})$ can be decomposed into the uniform disjunctive TCD by employing the following iteration:

Theorem 2. *If DC of $\mathbf{H}_{\beta_{i-1}}^{\mathbf{X}_i}$ is equal to 1, $k_1 = 1$, the function $h_{\beta_{i-1}}(\mathbf{X}_{\beta_{i-1}})$ can be written as follows*

$$h_{\beta_{i-1}}(\mathbf{X}_{\beta_{i-1}}) = h_i(\mathbf{X}_i) \cdot h_{\beta_i}(\mathbf{X}_{\beta_i}) \text{ where } \mathbf{X}_{\beta_{i-1}} = \mathbf{X}_i \cup \mathbf{X}_{\beta_i},\ \mathbf{X}_i = \mathbf{X}_i \cap \mathbf{X}_{\beta_i} \neq 0$$

for the initial values $h_{\beta_0}(\mathbf{X}_{\beta_0}) = g(\mathbf{X}) \wedge \mathbf{X}_{\beta_0} = \mathbf{X}$ where $\mathbf{X}_i = \{x_{(i-1)\lambda+1}, \ldots, x_{i\lambda}\}$, $\mathbf{X}_{\beta_i} = \mathbf{X}_{\beta_{i-1}}\backslash\mathbf{X}_i = \{x_{i\lambda+1}, \ldots, x_n\}$. The switching function h_i is defined by the relations

$$h_i(\mathbf{X}_i) = \prod_{q=0}^{2^\lambda-1}\left(\sum_{l=1}^{\lambda}\tilde{x}_l^{(i)} + c_q^{(i)}\right), \qquad \tilde{x}_l^{(i)} \in \{\bar{x}_l^{(i)}, x_l^{(i)}\}, \quad x_l^{(i)} \in \mathbf{X}_i$$

and determined by $\mathbf{H}_{\beta_i}^{\mathbf{X}_i} = \mathbf{P}_i^{\prime\mathbf{H}_{\beta_{i-1}},\mathbf{X}_i}$ where $c_q^{(i)} \in \mathbf{H}_{\beta_{i-1}\,car}^{\prime\mathbf{X}_i}$, $\mathbf{H}_{\beta_{i-1}\,car}^{\prime\mathbf{X}_i}$ is the normalized form of $\mathbf{H}_{\beta_{i-1}\,car}^{\mathbf{X}_i}$, $\mathbf{H}_{\beta_{i-1}\,car}^{\mathbf{X}_i}$ is the CV of $\mathbf{H}_{\beta_{i-1}}^{\mathbf{X}_i}$, $\mathbf{H}_{\beta_{i-1}}^{\mathbf{X}_i}$ is $\mathbf{H}_{\beta_{i-1}}$ by $\mathbf{X}_i$, $\mathbf{H}_{\beta_{i-1}}$ is the truth vector of $h_{\beta_{i-1}}(\mathbf{X}_{\beta_{i-1}})$, $\mathbf{P}_1^{\prime\mathbf{H}_{\beta_{i-1}}} \in \mathrm{H}_{\beta_{i-1}}^{\prime\mathbf{X}_i}$ is a non-0-patrition of $\mathbf{H}_{\beta_{i-1}}^{\mathbf{X}_i}$, $\mathrm{H}_{\beta_{i-1}}^{\prime\mathbf{X}_i}$ is the set of different non-0-partitions of $\mathbf{H}_{\beta_{i-1}}^{\mathbf{X}_i}$. ■

In matrix notation, the TCD which is to be found, is expressed as

$$\mathbf{F} = \sum_{j=1}^{k} \mathbf{G}_j, \quad 1 < k < n \tag{6}$$

where the truth vectors $\mathbf{G}_j$ of the functions g_j satisfy the conditions in Theorems 1 and 2.

These theorems imply an algorithm for checking TCD for a given switching function f. The algorithm consists of the following basic steps

1. Determination of 0-partitions
2. Determination of equal partitions
3. Finding of truth vectors $\mathbf{G}_j$
4. Extraction of partitions

In this paper we present matrix relations for each of these steps of the algorithm for TCD. The ordering of variables considerably influences on the possibility for TCD. Therefore, the reordering of variables, equivalently, the reordering of the elements of the truth vector of the considered function, is frequently required in checking of TCD. For this reason, we also provide matrix relations for reordering of the variables.

2. Reordering of truth vector

Let us analyze $\mathbf{F}$ in order to examine certain properties of f with respect to a subset $\mathbf{X}_1$ of its first λ variables. We are going to define matrixes for a permutation $\mathbf{M}_{i,j}$ of any two variables x_i and x_j. By multiple using these matrixes, $\mathbf{X}_1$ would contain arbitrary λ variables from $\mathbf{X}$.

Let $\mathbf{F}_{i,j}$ denote the truth vector of f where the permutation of its variables x_i and x_j has been done.

$$\mathbf{F}_{i,j} = \mathbf{M}_{i,j} \times \mathbf{F}. \tag{7}$$

To find $\mathbf{M}_{i,j}$ we will exploit the following procedure [17]. One defines a matrix $\mathbf{Q}$ of non-zero elements

$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{11} & q_{11} \end{bmatrix}, \quad q_{i,j} \neq 0, \quad i,j \in \{1,2\} \tag{8}$$

and an auxillary matrix $\mathbf{M}'_{i,j}$

$$\mathbf{M}'_{i,j} = \bigotimes_{s=1}^{n} \mathbf{R}_s \text{ where } \mathbf{R}_s = \begin{cases} \mathbf{Q} & \text{for } s = i \\ \mathbf{Q}^t & \text{for } s = j \\ \mathbf{I} & \text{for } s \in \{0,1,\ldots,n\} \setminus \{i,j\} \end{cases} \tag{9}$$

The matrix $\mathbf{M}_{i,j}$ is obtain from $\mathbf{M}'_{i,j}$ by replacing all the elements which have the form "$q_{ik}q_{jk}$" with 0 or 1, according to the following rule

$$q_{ik}q_{jk} = \begin{cases} 1 & \text{if } q_{ik} = q_{jk} \\ 0 & \text{if } q_{ik} \neq q_{jk} \end{cases} \tag{10}$$

Example. Let a completely defined functon be given: $f(\mathbf{X}) = \overline{x}_1 x_2 + \overline{x}_1\overline{x}_2\overline{x}_3 + x_2\overline{x}_3 x_4 + x_1\overline{x}_2\overline{x}_3$, $\mathbf{X} = \{x_1, x_2, x_3, x_4\}$. The truth vector for f is $\mathbf{F} = [1010\ 1111\ 1000\ 1000]^t$. Let us find $\mathbf{F}_{1,3}$ of f, where $\mathbf{X}_1 = \{x_1, x_3\}$. Therefore, the permutation of x_2 and x_3 should be performed. According to Eqs. (8) and (9) we have

$$\mathbf{M}'_{23} = \mathbf{R}_1 \otimes \mathbf{R}_2 \otimes \mathbf{R}_3 \otimes \mathbf{R}_4 = \mathbf{I} \otimes \mathbf{Q} \otimes \mathbf{Q}^t \otimes \mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \otimes \begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{M}'_{23} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} q_{11}q_{11} & 0 & q_{11}q_{21} & 0 & q_{12}q_{11} & 0 & q_{12}q_{21} & 0 \\ 0 & q_{11}q_{11} & 0 & q_{11}q_{21} & 0 & q_{12}q_{11} & 0 & q_{12}q_{21} \\ q_{11}q_{12} & 0 & q_{11}q_{22} & 0 & q_{12}q_{21} & 0 & q_{12}q_{22} & 0 \\ 0 & q_{11}q_{12} & 0 & q_{11}q_{22} & 0 & q_{12}q_{12} & 0 & q_{12}q_{22} \\ q_{21}q_{11} & 0 & q_{21}q_{21} & 0 & q_{12}q_{12} & 0 & q_{22}q_{21} & 0 \\ 0 & q_{21}q_{11} & 0 & q_{21}q_{21} & 0 & q_{22}q_{11} & 0 & q_{22}q_{21} \\ q_{21}q_{12} & 0 & q_{21}q_{22} & 0 & q_{22}q_{12} & 0 & q_{22}q_{22} & 0 \\ 0 & q_{21}q_{12} & 0 & q_{21}q_{22} & 0 & q_{22}q_{12} & 0 & q_{22}q_{22} \end{bmatrix}$$

Substituting Eq. (10) in the previous one and further in Eq. (7) we get:

$$\mathbf{M}_{23} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix}, \quad \mathbf{F}_{23} = [1011\ 1011\ 1101\ 0000]^t. \qquad \square$$

3. Determination of 0-partitions

Let one should examine if there are 0-partitions, $\mathbf{P}^{\mathbf{F},\mathbf{X}_1}_{null}$, among partitions $\mathbf{P}^{\mathbf{F},\mathbf{X}_1}_{q}$ of $\mathbf{F}^{\mathbf{X}_1}$, $q \in \{0, \ldots, 2^\lambda - 1\}$. We are observing a transformation

$$(11) \qquad \mathbf{Z}^n(\lambda) = \mathbf{T}^n_z(\lambda) \otimes \mathbf{F}$$

The above matrixes have the forms $\mathbf{T}^n_z(\lambda) = \mathbf{I}^{\otimes^\lambda} \otimes \mathbf{B}^{\otimes^{n-\lambda}}$, where $\mathbf{B} = [1\ 1]$, and $\mathbf{Z}^n(\lambda) = [\,z_0\ z_1\ \cdots\ z_{2^\lambda - 1}]^t$ with dimensions $2^n \times 2^n$ and $1 \times 2^\lambda$, respectively. Note that an element z_q is equal to the number of the ones in $\mathbf{P}^{\mathbf{F},\mathbf{X}_1}_q$. Therefore, if $z_q = 0$, $\mathbf{P}^{\mathbf{F},\mathbf{X}_1}_q$ is a 0-partition.

Example (cont.) We are considering $\mathbf{F} = \mathbf{F}_{23}$. The partitions by $\mathbf{X}_1 = \{x_1, x_3\}$ are: $\mathbf{P}^{\mathbf{F},\mathbf{X}_1}_0 = [1011]^t$, $\mathbf{P}^{\mathbf{F},\mathbf{X}_1}_1 = [1011]^t$, $\mathbf{P}^{\mathbf{F},\mathbf{X}_1}_2 = [1101]^t$, $\mathbf{P}^{\mathbf{F},\mathbf{X}_1}_3 = [0000]^t$. Because $z_3 = 0$, $\mathbf{P}^{\mathbf{F},\mathbf{X}_1}_3$ is a 0-partition.

$$\mathbf{Z}_2 = \mathbf{T}^4_z(2) \times \mathbf{F} = \begin{bmatrix} 1111000000000000 \\ 0000111100000000 \\ 0000000011110000 \\ 0000000000001111 \end{bmatrix} \times \begin{bmatrix} \mathbf{P}^{\mathbf{F},\mathbf{X}_1}_0 \\ \mathbf{P}^{\mathbf{F},\mathbf{X}_1}_1 \\ \mathbf{P}^{\mathbf{F},\mathbf{X}_1}_2 \\ \mathbf{P}^{\mathbf{F},\mathbf{X}_1}_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow z_3 = 0 \end{matrix} \qquad \square$$

4. Determination of equal partitions

We should examine whether two arbitrary partitions $\mathbf{P}^{\mathbf{F},\mathbf{X}_1}_i$, $\mathbf{P}^{\mathbf{F},\mathbf{X}_1}_j \in \Phi^{\mathbf{X}_1}$, $i < j$, $\forall i, j \in \{1, \ldots, 2^\lambda - 1\}$ are equal. Let an index be $q = \{1, \ldots, 2^\lambda - 1\}$.

We define auxiliary matrixes $\mathbf{A}^n_{ij}(\lambda)$, $\mathbf{B}^n(\lambda)$, and $\mathbf{S'}^n_{ij}(\lambda)$ with dimensions $2^{n-\lambda}\times 2^n$, $1\times 2^{n-\lambda}$, $2^{n-\lambda}\times 1$, respectively. For $\mathbf{A}_{ij}(\lambda)=\left[a_1\, a_2 \cdots a_{2^\lambda-1}\right]$ there is

$$\mathbf{A}^n_{ij}(\lambda)=\mathbf{A}_{ij}(\lambda)\otimes\mathbf{I}^{\otimes^{n-\lambda}},\quad a_q=\begin{cases}1 & \text{for } q=i\\ -1 & \text{for } q=j\\ 0 & \text{for } q\in\{0,\dots,2^\lambda-1\}\setminus\{i,j\}\end{cases}$$

or $$\mathbf{A}^n_{ij}(\lambda)=\left[\mathbf{A}_1\cdots\mathbf{A}_{2^\lambda-1}\right],\quad \mathbf{A}_q=\begin{cases}\mathbf{I}^{n-\lambda} & \text{for } q=i\\ -\mathbf{I}^{n-\lambda} & \text{for } q=j\\ \mathbf{0}^{n-\lambda} & \text{for } q\in\{0,\dots,2^\lambda-1\}\setminus\{i,j\}\end{cases}$$

$$\mathbf{B}^n(\lambda)=\mathbf{B}^{\otimes^{n-\lambda}},\ \mathbf{B}=\left[1\,1\right],\quad \mathbf{S'}^n_{ij}(\lambda)=\mathbf{A}^n_{ij}(\lambda)\times\mathbf{F}$$

An existence of the noted equality is determined by a matrix

$$\mathbf{S}^n_{ij}(\lambda)=\mathbf{B}^n(\lambda)\times abs\left|\mathbf{S'}^n_{ij}(\lambda)\right| \tag{12}$$

where the elemement $s_{ij}\in\{0,1\}$ has meaning: $s_{ij}=\begin{cases}0 & \Longrightarrow\ \mathbf{P}_i^{\mathbf{F},\mathbf{X}_1}=\mathbf{P}_j^{\mathbf{F},\mathbf{X}_1}\\ 1 & \Longrightarrow\ \mathbf{P}_i^{\mathbf{F},\mathbf{X}_1}\neq\mathbf{P}_j^{\mathbf{F},\mathbf{X}_1}\end{cases}$

For practical applications, there is no sense to use $\lambda>3$. For these cases, we can define matrix relations for a simultaneous exploring of an equality of all the possible partition pairings. For $\lambda=2$ we define: $\mathbf{A}_{ij}=\left[a_1\,a_2\,a_3\,a_4\right]$,
$a_q=\begin{cases}1 & \text{for } q=i\\ -1 & \text{for } q=j\\ 0 & \text{for } q\in\{0,1,2,3\}\setminus\{i,j\}\end{cases}$, $q,i,j\in\{1,2,3,4\}$, $i<j$, $\mathbf{A}^n=\mathbf{A}^2\otimes\mathbf{I}^{\otimes^{n-2}}$, where $\mathbf{A}^2=\left[\mathbf{A}_{01}\,\mathbf{A}_{02}\,\mathbf{A}_{03}\,\mathbf{A}_{10}\,\mathbf{A}_{12}\,\mathbf{A}_{13}\right]^t$, and $\mathbf{B}^n=\mathbf{B}^2\otimes\left[1\,1\right]^{n-2}$, where $\mathbf{B}^2=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\otimes\mathbf{I}$. The matrix

$$\mathbf{S}=\mathbf{B}^4\times abs\left|\mathbf{S'}\right|=\left[s_{01}\,s_{02}\,s_{03}\,s_{10}\,s_{12}\,s_{13}\right]^t,\quad \text{where}\quad \mathbf{S'}=\mathbf{A}^n\times\mathbf{F} \tag{13}$$

has elements $s_{ij}\in\{0,1\}$ with the same meaning as the elements $s_{ij}\in\mathbf{S}^n_{ij}$ in Eq. (12).

Example (cont.)

I. Using Eq. (12).

a) We explore the equality of $\mathbf{P}_0^{\mathbf{F},\mathbf{X}_1}$ and $\mathbf{P}_1^{\mathbf{F},\mathbf{X}_1}$ $\Longrightarrow$ $i=0$, $j=1$.

$$\mathbf{A}^4_{01}(2)=\mathbf{A}_{01}(2)\otimes\mathbf{I}^{\otimes^2}=\left[1\ -1\ 0\ 0\right]\otimes\begin{bmatrix}1&0\\0&1\end{bmatrix}^{\otimes^2}$$

$$=\begin{bmatrix}1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&-1&0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&-1&0&0&0&0&0&0&0&0\end{bmatrix}$$

$$\mathbf{S'}^4_{01}(2)=\mathbf{A}^4_{01}(2)\times\mathbf{F}=\left[0\,0\,0\,0\right]^t,\quad \mathbf{B}^4(2)=\left[1\,1\right]^{\otimes^2}=\left[1\,1\,1\,1\right]$$

$$\mathbf{S}^4_{01}(2)=\mathbf{B}^4(2)\times abs\left|\mathbf{S'}^4_{01}(2)\right|=\left[0\right]\ \Longrightarrow\ \mathbf{P}_0^{\mathbf{F},\mathbf{X}_1},\ \mathbf{P}_1^{\mathbf{F},\mathbf{X}_1}\ \text{– equal partitions.}$$

b) We examine the equality of $\mathbf{P}_0^{\mathbf{F},\mathbf{X}_1}$ and $\mathbf{P}_2^{\mathbf{F},\mathbf{X}_1} \implies i = 0, j = 2$.

$$\mathbf{A}_{02}^4(2) = \mathbf{A}_{02}(2) \otimes \mathbf{I}^{\otimes^2} = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{\otimes^2}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{S'}_{02}^4(2) = \mathbf{A}_{02}^4(2) \times \mathbf{F} = \begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix}^t, \quad \mathbf{B}^4(2) = \begin{bmatrix} 1 1 \end{bmatrix}^{\otimes^2} = \begin{bmatrix} 1\,1\,1\,1 \end{bmatrix}$$

$\mathbf{S}_{02}^4(2) = \mathbf{B}^4(2) \times abs \left| \mathbf{S'}_{02}^4(2) \right| = \begin{bmatrix} 2 \end{bmatrix} \implies \mathbf{P}_0^{\mathbf{F},\mathbf{X}_1}, \mathbf{P}_2^{\mathbf{F},\mathbf{X}_1}$ – not equal partitions.

II. Using Eq. (13): $\mathbf{A}_{01} = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}$, $\mathbf{A}_{02} = \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}$, $\mathbf{A}_{03} = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}$, $\mathbf{A}_{12} = \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}$, $\mathbf{A}_{23} = \begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}$

$$\mathbf{A}^4 = \mathbf{A}^2 \otimes \mathbf{I}^{\otimes^{n-2}} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B}^4 = \mathbf{B}^2 \otimes \begin{bmatrix} 1 1 \end{bmatrix}^{\otimes^2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{S'} = \mathbf{A}^4 \otimes \mathbf{F} = \begin{bmatrix} 0\,0\,0\,0\,0\; -1\,1\,0\,1\,0\,1\,1\,0\; -1\,1\,0\,1\,0\,1\,0\,1\,1\,0\,1 \end{bmatrix}^t$$

$$\mathbf{S} = \mathbf{B}^4 \times abs\,|\mathbf{S'}| = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 2 \\ 2 \\ 3 \end{bmatrix} \begin{matrix} \leftarrow s_{01} = 0 & \implies & \mathbf{P}_0 = \mathbf{P}_1 \\ \leftarrow s_{02} \neq 0 & \implies & \mathbf{P}_0 \neq \mathbf{P}_2 \\ \leftarrow s_{03} \neq 0 & \implies & \mathbf{P}_0 \neq \mathbf{P}_3 \\ \leftarrow s_{12} \neq 0 & \implies & \mathbf{P}_1 \neq \mathbf{P}_2 \\ \leftarrow s_{13} \neq 0 & \implies & \mathbf{P}_1 \neq \mathbf{P}_3 \\ \leftarrow s_{23} \neq 0 & \implies & \mathbf{P}_2 \neq \mathbf{P}_3 \end{matrix}$$

□

5. Finding of truth vector $\mathbf{G}_j$

The truth vector $\mathbf{G}_j$ of $g(\boldsymbol{x})$ is determined by the matrix relation

(14) $$\mathbf{G}_j = \mathbf{C}_j^n(\lambda) \otimes \mathbf{F}$$

The transformation matrix, a square matrix with dimension 2^λ, is given by the following expression

$$\mathbf{C}_j^n(\lambda) = \begin{bmatrix} \left[\mathbf{C}_0^j(\lambda)\right] & \mathbf{0} & \cdots & \\ \mathbf{0} & \left[\mathbf{C}_1^j(\lambda)\right] & & \mathbf{0} \\ \vdots & & \ddots & \\ & \mathbf{0} & & \left[\mathbf{C}_{2^\lambda-1}^j\right] \end{bmatrix}$$

$\mathbf{C}_q^j(\lambda)$ is a square matrix with dimension 2^λ in the following form

$$\mathbf{C}_q^j(\lambda) = \left[c_q^{(j)}\right] \otimes \mathbf{I}^{\otimes^{n-\lambda}} = \begin{cases} \mathbf{I}^{\otimes^{n-\lambda}} & \text{for } c_q^{(j)} = 1 \\ \mathbf{0}^{\otimes^{n-\lambda}} & \text{for } c_q^{(j)} = 0 \end{cases}$$

where $c_q^{(j)}$ is an element of the characteristic vector $\mathbf{G}_{j\,car}^{\mathbf{X}_1} = \mathbf{F}'^{\mathbf{X}_1}_{j\,car}$. $\mathbf{F}'^{\mathbf{X}_1}_{j\,car}$ is a normalized form of $\mathbf{F}_{j\,car}^{\mathbf{X}_1}$ derived from $\mathbf{F}_{car}^{\mathbf{X}_1}$ by j.

Example (cont.)

CV of $\mathbf{F}_1^{\mathbf{X}}$: $\mathbf{F}_{car}^{\mathbf{X}_1} = [1\ 1\ 2\ 0]^t$

DCVs of $\mathbf{F}_{car}^{\mathbf{X}_1}$: $\mathbf{F}_{1\,car}^{\mathbf{X}_1} = \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^t$ $\mathbf{F}_{2\,car}^{\mathbf{X}_1} = \begin{bmatrix}0 & 0 & 2 & 0\end{bmatrix}^t$

NDCVs: $\mathbf{F}'^{\mathbf{X}_1}_{1\,car} = \begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^t$ $\mathbf{F}'^{\mathbf{X}_1}_{2\,car} = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}^t$

Coefficients: $c_0^{(1)}\ c_1^{(1)}\ c_2^{(1)}\ c_3^{(1)}$ $c_0^{(2)}\ c_1^{(2)}\ c_2^{(2)}\ c_3^{(2)}$

$$\mathbf{C}_0^1 = \left[c_0^{(1)}\right] \otimes \mathbf{I}^{\otimes^2} = [1] \otimes = \mathbf{I}^{\otimes^2} = \mathbf{I}^{\otimes^2} \quad \mathbf{C}_0^2 = \left[c_0^{(2)}\right] \otimes \mathbf{I}^{\otimes^2} = [0] \otimes = \mathbf{I}^{\otimes^2} = \mathbf{0}^{\otimes^2}$$

$$\mathbf{C}_1^1 = \left[c_1^{(1)}\right] \otimes \mathbf{I}^{\otimes^2} = [1] \otimes = \mathbf{I}^{\otimes^2} = \mathbf{I}^{\otimes^2} \quad \mathbf{C}_1^2 = \left[c_1^{(2)}\right] \otimes \mathbf{I}^{\otimes^2} = [0] \otimes = \mathbf{I}^{\otimes^2} = \mathbf{0}^{\otimes^2}$$

$$\mathbf{C}_2^1 = \left[c_2^{(1)}\right] \otimes \mathbf{I}^{\otimes^2} = [0] \otimes = \mathbf{I}^{\otimes^2} = \mathbf{0}^{\otimes^2} \quad \mathbf{C}_2^2 = \left[c_2^{(2)}\right] \otimes \mathbf{I}^{\otimes^2} = [1] \otimes = \mathbf{I}^{\otimes^2} = \mathbf{0}^{\otimes^2}$$

$$\mathbf{C}_3^1 = \left[c_3^{(1)}\right] \otimes \mathbf{I}^{\otimes^2} = [0] \otimes = \mathbf{I}^{\otimes^2} = \mathbf{0}^{\otimes^2} \quad \mathbf{C}_3^2 = \left[c_3^{(2)}\right] \otimes \mathbf{I}^{\otimes^2} = [0] \otimes = \mathbf{I}^{\otimes^2} = \mathbf{0}^{\otimes^2}$$

$$\mathbf{G}_1 = \mathbf{C}_1^4(2) \times \mathbf{F} = \begin{bmatrix} \mathbf{I}^{\otimes^2} & & & \\ & \mathbf{I}^{\otimes^2} & \mathbf{0} & \\ & \mathbf{0} & \mathbf{0}^{\otimes^2} & \\ & & & \mathbf{0}^{\otimes^2} \end{bmatrix} \times \begin{bmatrix}1\\0\\1\\1\\1\\0\\1\\1\\1\\1\\0\\1\\0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\\1\\1\\0\\1\\1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}$$

$$\mathbf{G}_2 = \mathbf{C}_2^4(2) \times \mathbf{F} = \begin{bmatrix} \mathbf{0}^{\otimes^2} & & & \\ & \mathbf{0}^{\otimes^2} & \mathbf{0} & \\ & \mathbf{0} & \mathbf{I}^{\otimes^2} & \\ & & & \mathbf{0}^{\otimes^2} \end{bmatrix} \times \begin{bmatrix}1\\0\\1\\1\\1\\0\\1\\1\\1\\1\\0\\1\\0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\\1\\1\\0\\1\\0\\0\\0\\0\end{bmatrix}$$

□

6. Extraction of the partition

A partition $\mathbf{P}_q^{\mathbf{F},\mathbf{X}_i}$ can be extracted from $\mathbf{F}$ using the following matrix transformation

$$\mathbf{P}_q^{\mathbf{F},\mathbf{X}_i} = \mathbf{D}_i^n(\lambda) \times \mathbf{F} \tag{15}$$

where $\quad \mathbf{D}_i^n(\lambda) = \left[\, d_0\, d_1 \cdots d_{2^\lambda - 1} \,\right] \otimes \mathbf{I}^{\otimes^{n-\lambda}}, \quad d_i = \begin{cases} 1 \text{ for } i = q \\ 0 \text{ for } i \neq q \end{cases},$

$\forall i, q \in \{0, 1, \ldots, 2^\lambda - 1\}$. This matrix has the following form

$$\mathbf{D}_i^n(\lambda) = \left[\, \mathbf{D}_0(\lambda)\, \mathbf{D}_1(\lambda) \cdots \mathbf{D}_{2^\lambda - 1}(\lambda) \,\right], \quad \mathbf{D}_i(\lambda) = \begin{cases} \mathbf{I}^{\otimes^{n-\lambda}} \text{ for } i = q \\ \mathbf{0}^{\otimes^{n-\lambda}} \text{ for } i \neq q \end{cases}.$$

Example (cont.)

We are extracting the partition $\mathbf{P}_0^{\mathbf{F},\mathbf{X}_i}$

$$\mathbf{D}_0^4(2) = \left[\, 1\,0\,0\,0 \,\right] \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{\otimes^2} = \begin{bmatrix} 1000000000000000 \\ 0100000000000000 \\ 0010000000000000 \\ 0001000000000000 \end{bmatrix}$$

$$\mathbf{P}_q^{\mathbf{F},\mathbf{X}_i} = \mathbf{D}_0^4(2) \times \mathbf{F} = \left[\, \mathbf{I}^{\otimes^2}\ \mathbf{0}^{\otimes^2}\ \mathbf{0}^{\otimes^2}\ \mathbf{0}^{\otimes^2} \,\right] \times \begin{bmatrix} \mathbf{P}_0^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_1^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_2^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_3^{\mathbf{F},\mathbf{X}_1} \end{bmatrix} = \left[\mathbf{P}_0^{\mathbf{F},\mathbf{X}_1} \right] = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \square$$

7. Illustrative example

The following example illustrates the procedure of checking TCD by using the matrix relations introduced above. A completely defined switching function $f(\mathbf{X})$, $\mathbf{X} = \{x_1, \ldots, x_6\}$, is given by truth vector:

$\mathbf{F} = [\,1001100100100010\ 0001100000000010\ 1101110100110011\ 0001110000000011\,]^t$

It can be shown that TCD can not be found for that ordering of variables.

Let us consider rearranged $\mathbf{F}$ for a new ordering $\mathbf{X} = \{x_3, x_5, x_2, x_4, x_2, x_6\}$ which can be obtained using Eqs. (7)–(10) for two successive replacements: 1. x_1 and x_3, 2. x_2 and x_5.

$$\mathbf{F}_{1,3} = \mathbf{M}_{1,3} \times \mathbf{F} =$$
$$= [\,1001100100100010\ \ 0001100000000010\ \ 1101110100110011\ \ 0001110000000011\,]^t$$

using $\quad \mathbf{M}'_{13} = \mathbf{Q} \otimes \mathbf{I} \otimes \mathbf{Q}^t \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \otimes \mathbf{I} \otimes \begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{I}$

and the following transformation applied onto the $\mathbf{F}_{1,3} = \mathbf{F}$

$$\mathbf{F}_{2,5} = \mathbf{M}_{2,5} \times \mathbf{F} = \mathbf{F}^{\mathbf{X}_1} =$$

$$[\,\overbrace{1010001011110011}^{\mathbf{P}_0^{\mathbf{F},\mathbf{X}_1}}\ \overbrace{0101010001010100}^{\mathbf{P}_1^{\mathbf{F},\mathbf{X}_1}}\ \overbrace{0000000000000000}^{\mathbf{P}_2^{\mathbf{F},\mathbf{X}_1}}\ \overbrace{1010001011110011}^{\mathbf{P}_3^{\mathbf{F},\mathbf{X}_1}}\,]^t$$

using $\mathbf{M}'_{25} = \mathbf{I} \otimes \mathbf{Q} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{Q}^t \otimes \mathbf{I} = \mathbf{I} \otimes \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \otimes \mathbf{I} \otimes \mathbf{I} \otimes \begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix} \otimes \mathbf{I}$

Eq. (11) *detects the 0-partition*

$$\mathbf{Z}_6(2) = \mathbf{T}_z^6(2) \times \mathbf{F}$$

$$= \left(\begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix}^{\otimes^2} \otimes [1\,1]^{\otimes^4} \right) \times \begin{bmatrix} \mathbf{P}_0^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_1^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_2^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_3^{\mathbf{F},\mathbf{X}_1} \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 0 \\ 9 \end{bmatrix} \leftarrow z_2 = 0 \Longrightarrow \mathbf{P}_2^{\mathbf{F},\mathbf{X}_1} = \mathbf{P}_{null}^{\mathbf{F},\mathbf{X}_1}$$

We are going to use Eq. (13) to *determine of equal partitions*:

$$\mathbf{S}' = \mathbf{A}^6 \otimes \mathbf{F} = \mathbf{A}^2 \otimes \mathbf{I}^{\otimes^4} \otimes \mathbf{F} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix}^{\otimes^4} \otimes \mathbf{F}$$

$$\mathbf{B}^6 = \mathbf{B}^2 \otimes [1\,1]^{\otimes^4} = \begin{bmatrix} \mathbf{I}\,0\,0 \\ 0\,\mathbf{I}\,0 \\ 0\,0\,\mathbf{I} \end{bmatrix} \otimes [1\,1]^{\otimes^4}$$

$$\mathbf{S} = \mathbf{B}^6 \times abs\,|\mathbf{S}'| = \begin{bmatrix} 11 \\ 9 \\ 0 \\ 6 \\ 11 \\ 9 \end{bmatrix} \begin{matrix} \leftarrow s_{01} \neq 0 & \Longrightarrow & \mathbf{P}_0^{\mathbf{F},\mathbf{X}_1} \neq \mathbf{P}_1^{\mathbf{F},\mathbf{X}_1} \\ \leftarrow s_{02} \neq 0 & \Longrightarrow & \mathbf{P}_0^{\mathbf{F},\mathbf{X}_1} \neq \mathbf{P}_2^{\mathbf{F},\mathbf{X}_1} \\ \leftarrow s_{03} = 0 & \Longrightarrow & \mathbf{P}_0^{\mathbf{F},\mathbf{X}_1} = \mathbf{P}_3^{\mathbf{F},\mathbf{X}_1} \\ \leftarrow s_{12} \neq 0 & \Longrightarrow & \mathbf{P}_1^{\mathbf{F},\mathbf{X}_1} \neq \mathbf{P}_2^{\mathbf{F},\mathbf{X}_1} \\ \leftarrow s_{13} \neq 0 & \Longrightarrow & \mathbf{P}_1^{\mathbf{F},\mathbf{X}_1} \neq \mathbf{P}_3^{\mathbf{F},\mathbf{X}_1} \\ \leftarrow s_{23} \neq 0 & \Longrightarrow & \mathbf{P}_2^{\mathbf{F},\mathbf{X}_1} \neq \mathbf{P}_3^{\mathbf{F},\mathbf{X}_1} \end{matrix}$$

The relation between partitions of f and g_js as well as its characteristic vectors is as follows

$$\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} \quad \underset{\mathbf{F}_{car}^{\mathbf{X}_1}}{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}} = \underset{\mathbf{F}_{1\,car}^{\mathbf{X}_1}}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}} + \underset{\mathbf{F}_{2\,car}^{\mathbf{X}_1}}{\begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}} \quad \underset{\mathbf{F}^{\mathbf{X}_1}}{\begin{bmatrix} \mathbf{P}_1'^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_2'^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_{null}'^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_1'^{\mathbf{F},\mathbf{X}_1} \end{bmatrix}} = \underset{\mathbf{G}_1^{\mathbf{X}_1}}{\begin{bmatrix} \mathbf{P}_1'^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_{null}'^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_{null}'^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_1'^{\mathbf{F},\mathbf{X}_1} \end{bmatrix}} + \underset{\mathbf{G}_2^{\mathbf{X}_1}}{\begin{bmatrix} \mathbf{P}_{null}'^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_2'^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_{null}'^{\mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_{null}'^{\mathbf{F},\mathbf{X}_1} \end{bmatrix}} \quad \underset{\mathbf{G}_{1\,car}^{\mathbf{X}_1}}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \; \underset{\mathbf{G}_{2\,car}^{\mathbf{X}_1}}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$$

q

For *finding truth vectors* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ we will use Eq. (14):

$$\mathbf{C}_0^1 = [1] \otimes \begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix}^{\otimes^4} \quad \mathbf{C}_1^1 = [0] \otimes \begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix}^{\otimes^4} \quad \mathbf{C}_2^1 = [0] \otimes \begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix}^{\otimes^4} \quad \mathbf{C}_3^1 = [1] \otimes \begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix}^{\otimes^4}$$

$$\mathbf{C}_0^2 = [0] \otimes \begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix}^{\otimes^4} \quad \mathbf{C}_1^2 = [0] \otimes \begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix}^{\otimes^4} \quad \mathbf{C}_2^2 = [1] \otimes \begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix}^{\otimes^2} \quad \mathbf{C}_3^2 = [0] \otimes \begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix}^{\otimes^4}$$

$$\mathbf{G}_1 = \mathbf{C}_1^6(2) \times \mathbf{F} = \begin{bmatrix} \mathbf{I}^{\otimes 2} & & & \\ & \mathbf{0}^{\otimes 2} & 0 & \\ & & \mathbf{0}^{\otimes 2} & \\ 0 & & & \mathbf{I}^{\otimes 2} \end{bmatrix} \times \begin{bmatrix} \mathbf{P}_1^{\prime \mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_2^{\prime \mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_{null}^{\prime \mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_1^{\prime \mathbf{F},\mathbf{X}_1} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_1^{\prime \mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_{null}^{\prime \mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_{null}^{\prime \mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_1^{\prime \mathbf{F},\mathbf{X}_1} \end{bmatrix}$$

$\mathbf{G}_1 = [\,1010001011110011\;0000000000000000\;0000000000000000\;1010001011110011\,]^t$

$$\mathbf{G}_2 = \mathbf{C}_2^6(2) \times \mathbf{F} = \begin{bmatrix} \mathbf{0}^{\otimes 2} & & & \\ & \mathbf{0}^{\otimes 2} & 0 & \\ & & \mathbf{I}^{\otimes 2} & \\ 0 & & & \mathbf{I}^{\otimes 2} \end{bmatrix} \times \begin{bmatrix} \mathbf{P}_1^{\prime \mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_2^{\prime \mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_{null}^{\prime \mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_1^{\prime \mathbf{X}_1} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{null}^{\prime \mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_{null}^{\prime \mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_2^{\prime \mathbf{F},\mathbf{X}_1} \\ \mathbf{P}_{null}^{\mathbf{F},\mathbf{X}_1} \end{bmatrix}$$

$\mathbf{G}_2 = [\,0000000000000000\;0001100000000010\;0000000000000000\;0000000000000000\,]^t$

A further decomposition of the functions $\mathbf{G}_j$ is based on the iterative procedure given in Theorem 2. We consider the functions $\mathbf{H}_{\beta_0}^{(1)}(\mathbf{X}_{\beta_0}) = \mathbf{G}_1$ and $\mathbf{H}_{\beta_0}^{(2)}(\mathbf{X}_{\beta_0}) = \mathbf{G}_2$ where $\mathbf{X}_{\beta_0} = \mathbf{X}$.

The Eq. (15) gives a way to *extract the non-0-partitions* from $\mathbf{G}_1$ and $\mathbf{G}_2$, that is $\mathbf{H}_{\beta_1}^{(1)}$ and $\mathbf{H}_{\beta_1}^{(2)}$ in order to get truth vectors of the functions $h_{\beta_1}^{(1)}(\mathbf{X}_{\beta_1})$ and $h_{\beta_1}^{(2)}(\mathbf{X}_{\beta_1})$, where $\mathbf{X}_{\beta_1} = \mathbf{X}_{\beta_0} \backslash \mathbf{X}_1 = \{x_1, x_2, x_4, x_6\}$:

$$\mathbf{H}_{\beta_1}^{(1)} = \mathbf{D}_0^6(2) \times \mathbf{H}_{\beta_0}^{(1),\mathbf{X}_1} = \left[\mathbf{I}^{\otimes 4}\;\mathbf{0}^{\otimes 4}\;\mathbf{0}^{\otimes 4}\;\mathbf{0}^{\otimes 4}\right] \otimes \mathbf{H}_{\beta_0}^{(1)} = \left[\mathbf{P}_0^{\mathbf{H}_{\beta_0}^{(1)},\mathbf{X}_1}\right]$$

$$\mathbf{H}_{\beta_1}^{(1)} = [1010001011110011]^t$$

$$\mathbf{H}_{\beta_1}^{(2)} = \mathbf{D}_0^6(2) \times \mathbf{H}_{\beta_0}^{(2),\mathbf{X}_1} = \left[\mathbf{0}^{\otimes 4}\;\mathbf{0}^{\otimes 4}\;\mathbf{I}^{\otimes 4}\;\mathbf{0}^{\otimes 4}\right] \otimes \mathbf{H}_{\beta_0}^{(2)} = \left[\mathbf{P}_0^{\mathbf{H}_{\beta_0}^{(2)},\mathbf{X}_1}\right]$$

$$\mathbf{H}_{\beta_1}^{(2)} = [0001100000000010]^t$$

It can be shown that TCD of $h_{\beta_1}^{(1)}(\mathbf{X}_{\beta_1})$ and $h_{\beta_1}^{(2)}(\mathbf{X}_{\beta_1})$ can not be found for the present ordering of the variables. Let us reorder them into $\mathbf{X}_{\beta_1} = \{x_1, x_6, x_2, x_4\}$, which causes a new ordering in the vectors $\mathbf{H}_{\beta_1}^{(1),\mathbf{X}_2}$ and $\mathbf{H}_{\beta_1}^{(2),\mathbf{X}_2}$ for $\mathbf{X}_2 = \mathbf{X}_{\beta_1} \backslash \mathbf{X}_1 = \{x_1, x_6\}$. Using Eqs. (7)–(10), one gets

$$\mathbf{H}_{\beta_1 2,4}^{(1),\mathbf{X}_2} = \mathbf{M}_{2,4} \times \mathbf{H}_{\beta_1}^{(1)} = [1010001011110011]^t$$

using $\quad \mathbf{M}'_{24} = \mathbf{I} \otimes \mathbf{Q} \otimes \mathbf{I} \otimes \mathbf{Q}^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix}$

and the following transformation applied to $\mathbf{H}^{(1),\mathbf{X}_2}_{\beta_1 3,4} = \mathbf{H}^{(1)}_{\beta_1}$

$$\mathbf{H}^{(1),\mathbf{X}_2}_{\beta_1 3,4} = \mathbf{M}_{3,4} \times \mathbf{H}^{(1)}_{\beta_1} = \mathbf{H}^{(1),\mathbf{X}_2}_{\beta_1} = [\overbrace{1101}^{\mathbf{P}_0^{\mathbf{H}^{(1)}_{\beta_1}}}\ \overbrace{0000}^{\mathbf{P}_1^{\mathbf{H}^{(1)}_{\beta_1}}}\ \overbrace{1101}^{\mathbf{P}_2^{\mathbf{H}^{(1)}_{\beta_1}}}\ \overbrace{1101}^{\mathbf{P}_3^{\mathbf{H}^{(1)}_{\beta_1}}}]^t$$

using $\mathbf{M}'_{34} = \mathbf{I} \otimes \mathbf{I} \otimes \mathbf{Q} \otimes \mathbf{Q}^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \otimes \begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix}.$

By continuing this procedure we will get the characteristic vectors $\mathbf{H}^{(1),\mathbf{X}_i}_{\beta_i - 1\, car}$ and $\mathbf{H}^{(2),\mathbf{X}_i}_{\beta_i - 1\, car}$. They give us the all information for direct writing of decompositive expressions as shown in [13]. Here, only an example is shown.

$$\mathbf{H}^{(1),\mathbf{X}_1}_{1\, car} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \leftarrow 00 \leftarrow 1 \\ \leftarrow 01 \leftarrow x_3 + \bar{x}_5 \\ \leftarrow 10 \leftarrow \bar{x}_3 + x_5 \\ \leftarrow 01 \leftarrow 1 \end{matrix} \qquad \mathbf{H}^{(1),\mathbf{X}_2}_{2\, car} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \begin{matrix} \leftarrow 00 \leftarrow 1 \\ \leftarrow 01 \leftarrow x_1 + \bar{x}_6 \\ \leftarrow 10 \leftarrow 1 \\ \leftarrow 01 \leftarrow 1 \end{matrix}$$

TCD of the given function in the requested form is as follows

$$g_1(\mathbf{X}) = \underbrace{(x_3 + \bar{x}_5)(\bar{x}_3 + x_5)}_{h_1^{(1)}(\mathbf{X}_1)}\underbrace{(x_1 + \bar{x}_6)}_{h_2^{(1)}(\mathbf{X}_2)}\underbrace{(\bar{x}_2 + x_4)}_{h_3^{(1)}(\mathbf{X}_3)}$$

$$g_2(\mathbf{X}) = \underbrace{(x_3 + x_5)(\bar{x}_3 + x_5)(x_3 + \bar{x}_5)}_{h_1^{(2)}(\mathbf{X}_1)}\underbrace{(x_1 + \bar{x}_6)}_{h_2^{(2)}(\mathbf{X}_1)}\underbrace{(x_2 + \bar{x}_4)}_{h_3^{(2)}(\mathbf{X}_3)}$$

$$f(\mathbf{X}) = \underbrace{(x_3 + \bar{x}_5)(\bar{x}_3 + x_5)(x_1 + \bar{x}_6)(\bar{x}_2 + x_4)}_{g_1(\mathbf{X})} +$$

$$+ \underbrace{(x_3 + x_5)(\bar{x}_3 + x_5)(x_3 + \bar{x}_5)(x_1 + \bar{x}_6)(x_2 + \bar{x}_4)}_{g_2(\mathbf{X})}$$

8. Conclusion

In the paper, matrix relations for finding the total complex decomposition of a switching function are introduced. They deal with groups of elements in a truth vector, which is called partitions.

The proposed relations are a base for procedure deriving, which can be easily translated into a program realization.

References

[1] Asari, V.K., Eswarn, C., An optimization technique for the design of multiple valued PLA's, IEEE Transaction on Computers, vol. 43, no. 1, Jan. 1994, 118-122.

[2] Bolton, M., Digital System Design with Programmable Logic, Addison - Wesley, England, 1990.

[3] Curtis, H.A., A New Approach to the Design of Switching Functions, D. van Nostrand, New York, 1962.

[4] Davio, M., Deschamps, J.-P., Thayse, A., Digital Systems with Algorithms Implementation, Wiley, New York, 1983.

[5] Davio, M., Deschamps, J.-P., Thayse, A., Discrete and Switching Functions, McHill, New York, 1978.

[6] Hurst, S.L., Miller, D.M., Muzio, J.C., Spectral Technics in Digital Logic, Academic Press, New York, 1985.

[7] Pibilo, S.L., Enin, S.V., Synthesis of Combination Networks Using Methods of Functional Decomposition, Nauka i Tehnika, Minsk, 1987. (in Russian)

[8] Rakovac, S., One method of optimal realization of switching functions using three-level PLA, in Proc. 38th Conf. of ETRAN, vol. 1, pp. 39-40, Belgrade, 1994. (in Serbian)

[9] Rakovac, S., Total disjunctive decomposition of switching functions, in Abs. 38th Int. Conf. on Algebra, Logic and Discrete Mathematics, pp. 98, Niš, Yugoslavia, 1995.

[10] Rakovac, S., Optimization of three-level PLA's using symmetry of switching functions, in Proc. 39th Conf. of ETRAN, vol. 3, pp. 138-141, Belgrade, 1995. (in Serbian)

[11] Rakovac, S., Optimization of Programmable Logic Arrays Using Proprieties of Symmetry of Switching Functions, M. Sc. thesis, University of Niš, Yugoslavia, June 1995. (in Serbian)

[12] Rakovac, S., Application of principle of decomposition of switching functions in digital systems, in Proc. 3rd Int. Conf. in Telecommunications in Modern Satellite, Cable and Broadcasting Services, TELSIKS'97, pp. 744-747, Niš, Yugoslavia, 1997.

[13] Rakovac, S., Procedure for total complex decomposition of switching functions, in Proc. 5th Int. Conf. in Pattern Recognition and Information Processing PRIP'99, pp. 187-195, Minsk, Belarus, 1999.

[14] Rakovac, S., Fast algorithms for total complex decomposition of switching functions, (prepared for submission)

[15] Sasao, T., Switching Theory for Logic Synthesis, Kluwer Academic Publishers, 1999.

[16] Shestakov, E., Doudkin, A., Method of coding intermediate variables for decomposition of a system of incompletely defined Boolean functions, in Proc. 5th Int. Conf. in Pattern Recognition and Information Processing PRIP'99, pp. 207-214, Minsk, Belarus, 1999.

[17] Stanković, R.S., Denić, N., Program for the detection of symmetry and co-symmetry properties of switching functions, Research reports in applied mathematics, Multiple-valued logic, Rep. No. 5, Faculty of Electronic Engineering, Niš, Yugoslavia, 1991.

[18] Rose, J., Gamal, A., Vincentilli, A.S., Architecture of Field-Programmable Gate Arrays, Proceeding of the IEEE, vol. 81, no. 7, pp. 1013-1029, July 1993.

[19] Yanushkevich, S., Logic Differential Calculus in Multi-Valued Design, Technical University of Szczecin, Szczecin, Poland, 1998.