NOVI SAD J. MATH.
VOL. 30, NO. 1, 2000, 97-108

SEMIPRIME IDEALS AND IRREDUCIBLE IDEALS OF Γ-SEMIRINGS

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Abstract. We introduce the notions of semiprime ideal, prime radical of an ideal, irreducible ideal in a Γ-semiring and study them via operator semirings of a Γ-semiring.

AMS Mathematics Subject Classification (1991): 16Y60, 16Y99

Key words and phrases: Γ-semiring, operator semiring of a Γ-semiring, semiprime ideal, irreducible ideal, prime radical of an ideal, regular Γ-semiring.

1. Introduction

For symbols and notations we refer to the preliminaries given below. Throughout the paper S will denote a Γ-semiring with zero and unities and $L(R)$ its left(right) operator semiring. In ([2]) we introduced the notions of operator semirings of a Γ-semiring and obtained that lattices of all ideals of S and $L(R)$ are isomorphic and correspondence theorems of the sets of $k-(h-)$ideals of S and $L(R)$ are also obtained. Using these, we studied in ([3]) prime ideals and prime radicals of Γ-semiring via its operator semirings which includes different characterizations of prime ideals and prime radicals.

In this paper we establish an inclusion preserving bijection between the set of all semiprime ideals of S and the set of all semiprime ideals of $L(R)$. We also study prime radical of an ideal of S via operator semirings of S. We then obtain different characterizations of semiprime ideal of S. Lastly we characterize regular Γ-semiring by semiprime ideals and irreducible ideals.

2. Preliminaries

Let S and Γ be two additive commutative semigroups. Then S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \to S$ (images to be denoted by $a\alpha b$ for $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions:
(i) $a\alpha(b + c) = a\alpha b + a\alpha c$
(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$
(iii) $a(\alpha + \beta)c = a\alpha c + a\beta c$
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$ for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

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If A and B are subsets of a Γ-semiring S and $\Delta \subseteq \Gamma$, we denote by $A\Delta B$, the subset of S consisting of all finite sums of the form $\sum a_i\alpha_i b_i$ where $a_i \in A$, $b_i \in B$ and $\alpha_i \in \Gamma$. For the singleton subset $\{x\}$ of S we write $x\Delta B$ instead of $\{x\}\Delta B$.

A right(left) ideal I of a Γ-semiring S is an additive subsemigroup of S such that $I\Gamma S \subseteq I(S\Gamma I \subseteq I)$. If I is both a right and a left ideal of S, then we say that I is a two-sided ideal or simply an ideal of S. For $a \in S$, the principal left (right, two-sided) ideal is denoted by $< a|$ (respectively $|a >, < a >$). A proper ideal P of a Γ-semiring S is said to be prime if for any two ideals H and K of $S, H\Gamma K \subseteq P$ implies that either $H \subseteq P$ or $K \subseteq P$.

Let S be a Γ-semiring and F be the free additive commutative semigroup generated by $S\times\Gamma$ ([5]). Then the relation ρ on F, defined by $\sum_{i=1}^{m}(x_i,\alpha_i)\rho\sum_{j=1}^{n}(y_j,\beta_j)$ if and only if $\sum_{i=1}^{m}x_i\alpha_i a = \sum_{j=1}^{n}y_j\beta_j a$ for all $a \in S$ ($m, n \in Z^+$ = the set of all positive integers), is a congruence on F. The congruence class containing $\sum_{i=1}^{m}(x_i,\alpha_i)$ is denoted by $\sum_{i=1}^{m}[x_i,\alpha_i]$. Then F/ρ is an additive commutative semigroup. Now F/ρ forms a semiring with the multiplication defined by $(\sum_{i=1}^{m}[x_i,\alpha_i])(\sum_{j=1}^{n}[y_j,\beta_j]) = \sum_{i,j}[x_i\alpha_i y_j,\beta_j]$. We denote this semiring by L and call it the left operator semiring of the Γ-semiring S. Dually we define the right operator semiring R of the Γ-semiring S where $R = \{\sum_{i=1}^{m}[\alpha_i,x_i] : \alpha_i \in \Gamma$, $x_i \in S, i = 1,2,\ldots,m;\ m \in Z^+\}$ and the multiplication on R is defined as $(\sum_{i=1}^{m}[\alpha_i,x_i])(\sum_{j=1}^{n}[\beta_j,y_j]) = \sum_{i,j}[\alpha_i,x_i\beta_j y_j]$.

For $N \subseteq S$ and $\Delta \subseteq \Gamma$ we denote by $[N,\Delta]$ the set of all finite sums $\sum_{i=1}^{m}[x_i,\alpha_i]$ in L where $x_i \in N$ and $\alpha_i \in \Delta$. Thus in particular $[S,\Gamma] = L$. Similarly, we denote by $[\Delta,N]$ the set of all finite sums $\sum_{j=1}^{n}[\beta_j,y_j]$ in R where $y_j \in N$, $\beta_j \in \Delta$ and in particular $[\Gamma,S] = R$. For simplicity $[\{x\},\Gamma]$ is written as $[x,\Gamma]$ and $[\Gamma,\{x\}]$ is written as $[\Gamma,x]$. We also have $[x,\Gamma] \subseteq P([\Gamma,x] \subseteq P)$ if and only if $[x,\alpha] \in P$ (respectively $[\alpha,x] \in P$) for all $\alpha \in \Gamma$, where P is a subset of L (respectively R) and $x \in S$. For $P \subseteq L(\subseteq R)$ we define $P^+ = \{a \in S : [a,\Gamma] \subseteq P\}$ (respectively $P^* = \{a \in S : [\Gamma,a] \subseteq P\}$). For $Q \subseteq S$ we define $Q^{+'} = \{\sum_{i=1}^{m}[x_i,\alpha_i] \in L : (\sum_{i=1}^{m}[x_i,\alpha_i])S \subseteq Q\}$ where $(\sum_{i=1}^{m}[x_i,\alpha_i])S$ denotes the set of all finite sums $\sum_{i,k} x_i\alpha_i s_k$, $s_k \in S$ and $Q^{*'} = \{\sum_{i=1}^{m}[\alpha_i, x_i] \in R : S(\sum_{i=1}^{m}[\alpha_i, x_i]) \subseteq Q\}$, where $S(\sum_{i=1}^{m}[\alpha_i, x_i])$ is the set of all finite sums $\sum_{k,i} s_k\alpha_i x_i$, $s_k \in S$. For $Q \subseteq S$, $\sum_{i=1}^{m}[x_i, \alpha_i] \in L$ and $\sum_{i=1}^{m}[\alpha_i, x_i] \in R$,

(i) $(\sum_{i=1}^{m}[x_i, \alpha_i])S \subseteq Q$ if and only if $\sum_{i=1}^{m} x_i\alpha_i s \in Q$ for all $s \in S$,

(ii) $S(\sum_{i=1}^{m}[\alpha_i, x_i]) \subseteq Q$ if and only if $\sum_{i=1}^{m} s\alpha_i x_i \in Q$ for all $s \in S$.

If P is an ideal or prime ideal of S, then $P^+(P^*)$ is an ideal or prime ideal of S. If Q is an ideal or prime ideal of S then $Q^{+'}(Q^{*'})$ is an ideal or prime ideal of $L(R)$. For a Γ-semiring S if there exists an element $0 \in S$ such that $0 + x = x$ and $0\alpha x = x\alpha 0 = 0$ for all $x \in S$ and for all $\alpha \in \Gamma$, then 0 is called the zero of the Γ-semiring S and in that case we say that the Γ-semiring S is with zero. In such a case $[0, \alpha]$ is the zero of L and $[\alpha, 0]$ is the zero of R for any $\alpha \in \Gamma$. Again, if there exists an element $\sum_{i=1}^{m}[e_i, \delta_i] \in L(\sum_{j=1}^{n}[\gamma_j, f_j] \in R)$ such that $\sum_{i=1}^{m} e_i\delta_i a = a$ $(\sum_{j=1}^{n} a\gamma_j f_j = a)$ for all $a \in S$ then S is said to have the left unity $\sum_{i=1}^{m}[e_i, \delta_i]$ (respectively the right unity $\sum_{j=1}^{n}[\gamma_j, f_j]$). The left (right) unity of the Γ-semiring S, if it exists, is the identity of the left operator semiring L (respectively the right operator semiring R) of S. If a Γ-semiring S is with zero then for any $a \in S$,

$$< a| = \{ma + \sum_{i=1}^{n} x_i\alpha_i a : m \in Z^+ \cup \{0\}, n \in Z^+, x_i \in S, \alpha_i \in \Gamma\},$$

$$|a >= \{na + \sum_{j=1}^{m} a\beta_j y_j : n \in Z^+ \cup \{0\}, m \in Z^+, y_j \in S, \beta_j \in \Gamma\} \text{ and}$$

$$< a >= \{na + \sum_{k=1}^{p} a\gamma_k z_k + \sum_{t=1}^{s} w_t\delta_t a + \sum_{j=1}^{q} u_j\lambda_j a\mu_j v_j : n \in Z^+ \cup \{0\},\ p, s, q \in Z^+$$
$$z_k, w_t, u_j, v_j \in S \text{ and } \gamma_k, \delta_t, \lambda_j, \mu_j \in \Gamma\},$$ where Z^+ is the set of all positive integers.

A Γ-semiring S is said to be commutative if $a\alpha b = b\alpha a$ for all $a, b \in S$ and for all $\alpha \in \Gamma$. If S is a commutative Γ-semiring then its operator semirings L and R are commutative.

A non-empty subset H of a Γ-semiring S is said to be an m-system if and only if $c, d \in H$ implies that there exists $p \in S$ and $\alpha, \beta \in \Gamma$ such that $c\alpha p\beta d \in H$.

For preliminaries we refer to ([2]) and ([3]) and for preliminaries of semirings we refer to ([4]).

3. Semiprime ideals of Γ-semirings

Definition 3.1. *Let S be a Γ-semiring. A proper ideal P of S is said to be semiprime if for any ideal A of S, $A\Gamma A \subseteq P$ implies that $A \subseteq P$.*

Obviously a prime ideal of S is also a semiprime ideal.

Example 3.2. Let S be a semiring with the multiplicative identity 1. Then S is a Γ-semiring where $\Gamma = S$ and $a\alpha b$ denotes the product of the elements a, α, b in S. Now any semiprime ideal of the semiring S is a semiprime ideal of the Γ-semiring S.

Lemma 3.3. *Let S be a Γ-semiring and L be its left operator semiring. If P is a semiprime ideal of L then P^+ is a semiprime ideal of S.*

Proof. If P is an ideal of L then P^+ is an ideal of S (Proposition 6.1 [2]). Now suppose $A\Gamma A \subseteq P^+$ where A is an ideal of S. Then $[A\Gamma A, \Gamma] \subseteq P$ i.e. $[A, \Gamma][A, \Gamma] \subseteq P$. Since $[A, \Gamma]$ is an ideal of L and P is a semiprime ideal of L, so $[A, \Gamma] \subseteq P$ whence $A \subseteq P^+$. Hence P^+ is semiprime in S. □

Lemma 3.4. *If Q is a semiprime ideal of a Γ-semiring S then $Q^{+'}$ is a semiprime ideal of L.*

Proof. $Q^{+'}$ is an ideal of L (Proposition 6.2 [2]). Now let $A^2 \subseteq Q^{+'}$ where A is any ideal of L. Then $ALA \subseteq A^2 \subseteq Q^{+'}$. This implies that $A[S, \Gamma]A \subseteq Q^{+'}$. So $(AS)\Gamma(AS) \subseteq Q$. Since AS is an ideal of S and Q is a semiprime ideal of S, $AS \subseteq Q$. Hence $A \subseteq Q^{+'}$ and so $Q^{+'}$ is a semiprime ideal of L. □

Theorem 3.5. *Let S be a Γ-semiring and L be its left operator semiring. Then there exists an inclusion preserving bijection $Q \to Q^{+'}$ between the set of all semiprime ideals of S and the set of all semiprime ideals of L, where Q is a semiprime ideal of S.*

Proof. Let Q be a semiprime ideal of S. Then by Lemma 3.4, $Q^{+'}$ is a semiprime ideal of L. Now $(Q^{+'})^+ = \{s \in S : [s, \Gamma] \subseteq Q^{+'}\} = \{s \in S : [s, \Gamma]S = s\Gamma S \subseteq Q\}$. This implies that $(Q^{+'})^+\Gamma S \subseteq Q$. Since S has the right unity so $(Q^{+'})^+ \subseteq Q$. Again, since $Q\Gamma S \subseteq Q$ so $Q \subseteq (Q^{+'})^+$. Hence $(Q^{+'})^+ = Q$. Now let U be a semiprime ideal of L. Then by Lemma 3.3, U^+ is a semiprime ideal of S. Now $(U^+)^{+'} = \{\sum_i [x_i, \alpha_i] \in L : (\sum_i [x_i, \alpha_i])S \subseteq U^+\} = \{\sum_i [x_i, \alpha_i] \in L :$ $[(\sum_i [x_i, \alpha_i])S, \Gamma] = (\sum_i [x_i, \alpha_i])[S, \Gamma] = (\sum_i [x_i, \alpha_i])L \subseteq U\}$. This implies that

$(U^+)^{+'}L \subseteq U$. Since L has the identity (Proposition 5.3 [2]), so this implies that $(U^+)^{+'} \subseteq U$. Again, since $UL \subseteq U$ so $U \subseteq (U^+)^{+'}$. Hence $(U^+)^{+'} = U$. Next, let $P \subseteq Q$, where P and Q are semiprime ideals of S. Let $\sum_i [x_i, \alpha_i] \in P^{+'}$. So $(\sum_i [x_i, \alpha_i]S \subseteq P \subseteq Q$. So $\sum_i [x_i, \alpha_i] \in Q^{+'}$. Hence $P^{+'} \subseteq Q^{+'}$. This completes the proof. □

Theorem 3.6. *Let S be a Γ-semiring. Then for an ideal Q of S the following conditions are equivalent:*

(i) Q is semiprime

(ii) If $a \in S$ such that $a\Gamma S\Gamma a \subseteq Q$ then $a \in Q$.

(iii) $< a > \Gamma < a > \subseteq Q$, $a \in S$ implies that $a \in Q$

(iv) If U is a right ideal of S such that $U\Gamma U \subseteq Q$ then $U \subseteq Q$

(v) If V is a left ideal of S such that $V\Gamma V \subseteq Q$ then $V \subseteq Q$.

Proof. We prove only (i) ⇒ (ii), since all other implications follow easily. Let $a\Gamma S\Gamma a \subseteq Q$, where $a \in S$. $[a\Gamma S\Gamma a, \Gamma] \subseteq [Q, \Gamma]$. Since $Q\Gamma S \subseteq Q$, so $[Q, \Gamma] \subseteq Q^{+'}$. Hence $[a\Gamma S\Gamma a, \Gamma] \subseteq Q^{+'}$. This implies that $[a\alpha s\alpha a, \alpha] \in Q^{+'}$ for all $s \in S$ and for all $\alpha \in \Gamma$. So $[a, \alpha][s, \alpha][a, \alpha] \in Q^{+'}$ for all $s \in S$ and for all $\alpha \in \Gamma$. Since $Q^{+'}$ is a semiprime in L (Lemma 3.4), $[a, \alpha] \in Q^{+'}$ for all $\alpha \in \Gamma$ (Proposition 6.15, [4]) and so $[a, \Gamma] \subseteq Q^{+'}$ whence $a \in (Q^{+'})^+ = Q$ (Theorem 6.3, [2]). □

Definition 3.7. *A non-empty subset N of a Γ-semiring S is said to be an n-system if and only if for any $a \in N$ there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta a \in N$.*

Clearly, every m-system of S is an n-system.

The proofs of the following results are easy and so we omit them.

Theorem 3.8. *A proper ideal Q of a Γ-semiring S is semiprime if and only if Q^c is an n-system of S.*

Proposition 3.9. *If N is an n-system of a Γ-semiring S and $a \in N$, then there exists an m-system M of S contained in N and containing a.*

Proposition 3.10. *A non-empty subset A of a Γ-semiring S is an n-system if and only if it is the union of m-systems of S.*

Combining Theorem 3.8 and Proposition 3.10 we have the following theorem:

Theorem 3.11. *A proper ideal Q of a Γ-semiring S is semiprime if and only if Q^c is the union of m-systems of S.*

Definition 3.12. *Let P be a proper ideal of a Γ-semiring S. Then the prime radical of P is defined as the intersection of all prime ideals of S containing P and is denoted by $r(P)$.*

Proposition 3.13. *Let S be a Γ-semiring and P be a proper ideal of S. If $x \in r(P)$ then $(x\,\gamma)^{n-1}x \in P$ for all $\gamma \in \Gamma$ and for some positive integer $n((x\,\gamma)^0 x = x)$.*

Proof. Let $x \in r(P)$. If possible, suppose for some $\gamma \in \Gamma, (x\,\gamma)^{n-1}x \notin P$ for all positive integers n. Then the m-system $H = \{x, x\gamma x, (x\,\gamma)^2 x, \ldots, (x\,\gamma)^n x, \ldots\}$ is disjoint from P. So, by Proposition 3.16 ([3]), there exists a prime ideal $Q \supseteq P$ such that $Q \cap H = \Phi$. Then $x \notin Q$. So $x \notin r(P)$ - a contradiction. So $(x\,\gamma)^{n-1}x \in P$ for all $\gamma \in \Gamma$ and for some positive integer n. □

As regards the converse we have the following theorem which is a Γ-semiring analogue of characterization of prime radical of an ideal of a commutative semiring.

Theorem 3.14. *For a commutative Γ-semiring S , $r(Q) = \{s \in S : (s\,\gamma)^{n-1}s \in Q$ for some positive integer n and for all $\gamma \in \Gamma\}$ where Q is a proper ideal of S.*

We also have the following characterization of prime radical of an ideal of a Γ-semiring.

Theorem 3.15. *For a proper ideal Q of a Γ-semiring S, $r(Q) = \{s \in S :$ every m-system in S which contains s has a non-empty intersection with $Q\}$.*

Proof. Let $P' = \{s \in S :$ every m-system in S which contains s has a non-empty intersection with $Q\}$. Now let $s \notin P'$. Then there exists an m-system H in S such that $s \in H$ and $H \cap Q = \Phi$. Then by Proposition 3.16 ([3]), there exists a prime ideal P of S with $P \supseteq Q$ and $P \cap H = \Phi$. So $s \notin P$ and so $s \notin r(Q)$. Next let $s \notin r(Q)$. Then there exists a prime ideal P of S such that $s \notin P$. So $s \in P^c$. Again P^c is an m-system of S (Proposition 3.14 [3]). Since $P \supseteq Q$ so $P^c \cap Q = \Phi$. Thus m-system P^c in S contains s but has an empty intersection with Q. So $s \notin P'$. This completes the proof. □

The following theorem establishes a relation between the prime radicals of a Γ-semiring S and of its left operator semiring.

Theorem 3.16. *Let S be a Γ-semiring and L be its left operator semiring. If P and Q are ideals of L and S respectively then $r(P^+) = r(P)^+$ and $r(Q^{+'}) = r(Q)^{+'}$.*

Proof. First we show that $r(P^+) = r(P)^+$. Let $a \in r(P^+)$ and A be any prime ideal of L containing P. Then $P^+ \subseteq A^+$ (Theorem 6.3[2]). Since A^+ is a prime ideal of S (Lemma 3.2 [3]), so $r(P^+) \subseteq A^+$ whence $a \in A^+$. Hence $[a, \Gamma] \subseteq A$. This implies that $[a, \Gamma] \subseteq r(P)$. So $a \in r(P)^+$. Thus $r(P^+) \subseteq r(P)^+$. Conversely, suppose that $b \in r(P)^+$. Then $[b, \Gamma] \subseteq r(P)$. Let B be any prime ideal of S containing P^+. Then $(P^+)^{+'} \subseteq B^{+'}$ i.e. $P \subseteq B^{+'}$ (Theorem 6.3 [2]). Since $B^{+'}$ is a prime ideal of L (Lemma 3.3 [3]), so $r(P) \subseteq B^{+'}$. Hence $[b, \Gamma] \subseteq B^{+'}$ whence $b \in (B^{+'})^+ = B$. So $b \in r(P^+)$. Thus $r(P)^+ \subseteq r(P^+)$.

So $r(P^+) = r(P)^+$. Now, to prove $r(Q^{+'}) = r(Q)^{+'}$ let $\sum_i [x_i, \alpha_i] \in r(Q)^{+'}$. Then $(\sum_i [x_i, \alpha_i])S \subseteq r(Q)$. Let C be any prime ideal of L containing $Q^{+'}$. Then $(Q^{+'})^+ \subseteq C^+$ i.e. $Q \subseteq C^+$. Since C^+ is prime in S so $r(Q) \subseteq C^+$ and so $(\sum_i [x_i, \alpha_i])S \subseteq C^+$ whence $\sum_i [x_i, \alpha_i] \in (C^+)^{+'} = C$. So $\sum_i [x_i, \alpha_i] \in r(Q^{+'})$. Thus $r(Q)^{+'} \subseteq r(Q^{+'})$. Next let $\sum_j [b_j, \beta_j] \in r(Q^{+'})$ and D be any prime ideal of S containing Q. So $Q^{+'} \subseteq D^{+'}$. Since $D^{+'}$ is a prime ideal of L this implies that $r(Q^{+'}) \subseteq D^{+'}$. So $\sum_j [b_j, \beta_j] \in D^{+'}$. This implies that $(\sum_j [b_j, \beta_j])S \subseteq D$ whence $(\sum_j [b_j, \beta_j])S \subseteq r(Q)$. So $\sum_j [b_j, \beta_j] \in r(Q)^{+'}$. Thus $r(Q^{+'}) \subseteq r(Q)^{+'}$ and so $r(Q^{+'}) = r(Q)^{+'}$. □

Corollary 3.17. *For any proper ideal Q of a Γ-semiring S, $r(Q)$ is the smallest semiprime ideal containing Q.*

Proof. It can easily be proved that $r(Q)$ is a semiprime ideal of S. Let $Q \subseteq A$ where A is a semiprime ideal of S. Then $Q^{+'} \subseteq A^{+'}$. Since $r(Q^{+'})$ is the smallest semiprime ideal of L containing $Q^{+'}$ ([4]) and $A^{+'}$ is a semiprime ideal of L so $r(Q^{+'}) \subseteq A^{+'}$ whence by Theorem 3.16, $r(Q)^{+'} \subseteq A^{+'}$ and so $r(Q)^{+'+} \subseteq A^{+'+}$ i.e. $r(Q) \subseteq A$. Hence the corollary. □

Now we obtain the following characterization of semiprime ideal:

Theorem 3.18. *A proper ideal Q of a Γ-semiring S is semiprime if and only if $r(Q) = Q$.*

Proof. Let Q be semiprime in S. Then by Lemma 3.4, $Q^{+'}$ is semiprime in L. So $r(Q^{+'}) = Q^{+'}$ (Proposition 6.19 [4]). So by Theorem 3.16, $r(Q)^{+'} = r(Q^{+'}) = Q^{+'}$ whence $(r(Q)^{+'})^+ = (Q^{+'})^+$ i.e. $r(Q) = Q$. □

Converse is obvious.

Proposition 3.19. *If P and Q are proper ideals of a Γ-semiring S then*

(i) $P \subseteq Q$ *implies that* $r(P) \subseteq r(Q)$

(ii) $r(r(P)) = r(P)$

(iii) $r(P+Q) = r(r(P) + r(Q))$

Proof. We omit the proof as it is a matter of routine verification. □

Proposition 3.20. *If P and Q are proper ideals of a commutative Γ-semiring S then $r(P\Gamma Q) = r(P \cap Q) = r(P) \cap r(Q)$.*

Proof. Clearly, $r(P\Gamma Q) \subseteq r(P \cap Q) \subseteq r(P) \cap r(Q)$. Now let $a \in r(P) \cap r(Q)$. Then by Theorem 3.13, $(a\ \gamma)^{n-1}a \in P$ and $(a\ \gamma)^{m-1}a \in Q$ for all $\gamma \in \Gamma$ and for some positive integers n, m. Then $(a\ \gamma)^{t-1}a \in P\Gamma Q$ for all $\gamma \in \Gamma$ and for some positive integer t. Since S is commutative, this implies that $a \in r(P\Gamma Q)$ (Theorem 3.14). Hence $r(P) \cap r(Q) \subseteq r(P\Gamma Q)$. This completes the proof. □

Definition 3.21. *A proper ideal I of a Γ-semiring is said to be irreducible if for the ideals H and K of S, $I = H \cap K$ implies that $I = H$ or $I = K$.*

Definition 3.22. *A proper ideal I of a Γ-semiring S is said to be strongly irreducible if for ideals H and K of S, $H \cap K \subseteq I$ implies that $H \subseteq I$ or $K \subseteq I$.*

Clearly, a strongly irreducible ideal is irreducible and a prime ideal is strongly irreducible.

Definition 3.23. *A non-empty subset A of a Γ-semiring S is said to be an I-system if $a, b \in A$ implies that $< a > \cap < b > \cap A \neq \Phi$.*

The proofs of the following results are matter of routine verification and so omitted.

Theorem 3.24. *The following conditions on an ideal I of a Γ-semiring S are equivalent:*

(i) I is strongly irreducible

(ii) For $a, b \in S$, $< a > \cap < b > \subseteq I$ implies that $a \in I$ or $b \in I$.

(iii) I^c is an I-system.

Lemma 3.25. *Let L be the left operator semiring of a Γ-semiring S. Then for any two ideals A and B of L, $(A \cap B)^+ = A^+ \cap B^+$.*

Lemma 3.26. *For any two ideals A and B of a Γ-semiring S, $(A \cap B)^{+'} = A^{+'} \cap B^{+'}$.*

Proposition 3.27. *Let L be the left operator semiring of a Γ-semiring S. If P is a strongly irreducible ideal of S then $P^{+'}$ is strongly irreducible in L.*

Proof. Let $A \cap B \subseteq P^{+'}$ where A and B are ideals of L. Then $(A \cap B)^+ \subseteq (P^{+'})^+$ (Theorem 6.3 [2]). So $A^+ \cap B^+ \subseteq P$ (Lemma 3.25). Since A^+ and B^+ are ideals of S (Proposition 6.1 [2]) and P is strongly irreducible in S this implies that $A^+ \subseteq P$ or $B^+ \subseteq P$. So $A = (A^+)^{+'} \subseteq P^{+'}$ or $B = (B^+)^{+'} \subseteq P^{+'}$ (Theorem 6.3 [2]). Hence $P^{+'}$ is strongly irreducible in L. □

Proposition 3.28. *If P is a strongly irreducible ideal of L then P^+ is strongly irreducible in S.*

Proof. Let $A \cap B \subseteq P^+$, where A and B are ideals of S. Then $(A \cap B)^{+'} \subseteq (P^+)^{+'}$ (Theorem 6.3 [2]). This implies that $A^{+'} \cap B^{+'} \subseteq P$ (Lemma 3.26). Since $A^{+'}$ and $B^{+'}$ are ideals of L (Proposition 6.2 [2]) and P is strongly irreducible in L so this implies that $A^{+'} \subseteq P$ or $B^{+'} \subseteq P$ ([4]) whence $A = (A^{+'})^+ \subseteq P^+$ or $B = (B^{+'})^+ \subseteq P^+$ (Theorem 6.3 [2]). Hence P^+ is strongly irreducible in S. □

Similar are the proofs of the following two propositions:

Proposition 3.29. *If P is an irreducible ideal of a Γ-semiring S then is irreducible in L.*

Proposition 3.30. *If P is an irreducible ideal of L then P^+ is irreducible in the Γ-semiring S.*

The following theorem gives a sufficient condition for a semiprime ideal to be a prime ideal in a Γ- semiring.

Theorem 3.31. *A proper ideal I of Γ-semiring S is prime if and only if it is semiprime and strongly irreducible.*

Proof. Suppose that I be semiprime and strongly irreducible in S. Then $I^{+'}$ is semiprime (Lemma 3.4) and strongly irreducible (Proposition 3.27) in L. So $I^{+'}$ is prime (Proposition 6.26 [4]) in L. So $I = (I^{+'})^+$ is prime (Lemma 3.2 and Theorem 3.17 [3]) in S. The converse follows easily. □

Proposition 3.32. *Let a be a non-zero element of a Γ-semiring S and let I be a proper ideal of S not containing a. Then there exists an irreducible ideal H of S containing I and not containing a.*

Proof. By Zorn's Lemma, the set of all ideals of S containing I and not containing a has a maximal element H. We prove that H is irreducible. If possible, suppose $H = H' \cap H''$, where H' and H'' are both ideals of S properly containing H. Then H' and H'' both contain I and so by the maximality of H, $a \in H'$ and $a \in H''$. Thus $a \in H' \cap H'' = H$ - a contradiction. Hence $H = H'$ or $H = H''$ whence H is irreducible. □

Theorem 3.33. *Any proper ideal I of a Γ-semiring S is the intersection of all irreducible ideals containing it.*

Proof. Since I is a proper ideal of S so there exists $a(\neq 0) \in S$ such that $a \notin I$. So by Proposition 3.32, there exists an irreducible ideal H of S containing I and not containing a. Let I' be the intersection of all irreducible ideals of S containing I. Then $I \subseteq I'$. If this inclusion is proper then there exists an element $b \in I' \setminus I$. But by Proposition 3.32, there exists an irreducible ideal K of S containing I and not containing b. Now $I' \subseteq K$ such that $b \notin K$. So $b \notin I'$ - a contradiction. Hence $I = I'$. □

Remark 3.34. Results similar to Lemmas 3.3, 3.4, 3.25, 3.26, Theorems 3.5, 3.16 and Propositions 3.27, 3.28, 3.29,3.30 can be obtained for the right operator semiring R of the Γ-semiring S.

4. Regular Γ-semiring

Definition 4.1. *A Γ-semiring S is said to be regular if for all $a \in S$, $a \in a\Gamma S\Gamma a$ where $a\Gamma S\Gamma a$ denotes the finite sum of elements of the form $a\alpha s\alpha a$ for all $\alpha, \beta \in \Gamma, s \in S$ ([1]).*

Remark 4.2. Since any semiring S can be considered as a Γ-semiring where $\Gamma = S$ so in that case the above definition of regularity of a Γ-semiring S reduces to the definition of multiplicative regularity of a semiring S namely " A semiring S is multiplicatively regular if for any $a \in S, a = axa$ for some $x \in S$".

Definition 4.3. *An ideal P of a Γ-semiring S is idempotent if $P\Gamma P = P$.*

Proposition 4.4. *A Γ-semiring S is regular if and only if $A\Gamma B = A \cap B$ for all right ideals A and for all left ideals B of S.*

Proof. Let the Γ-semiring S be regular and A and B be respectively a right and a left ideal of S. Then $A\Gamma B \subseteq A \cap B$. Now suppose $x \in A \cap B$. Since S is regular so $x \in x\Gamma S\Gamma x \subseteq A\Gamma x$ (since $x\Gamma S \subseteq A\Gamma S \subseteq A$) $\subseteq A\Gamma B$. Thus $A \cap B \subseteq A\Gamma B$. Hence $A\Gamma B = A \cap B$. Conversely, suppose $A\Gamma B = A \cap B$ where A and B are respectively right and left ideals of S. Let $x \in S$. Then $x\Gamma S$ is a right ideal and $S\Gamma x$ is a left ideal of S. So $(x\Gamma S)\Gamma(S\Gamma x) = (x\Gamma S) \cap (S\Gamma x)$. Since S has the right unity and left unity , $x \in x\Gamma S$ and $x \in S\Gamma x$ and so $x \in (x\Gamma S) \cap (S\Gamma x) = (x\Gamma S)\Gamma(S\Gamma x) \subseteq x\Gamma S\Gamma x$. Hence S is regular. □

Theorem 4.5. *A commutative Γ-semiring S is regular if and only if every ideal of S is idempotent.*

Proof. Let the Γ-semiring S be regular and I be any ideal of S. Then $I\Gamma I \subseteq I\Gamma S \subseteq I$. Let $a \in I$. Now $a \in a\Gamma S\Gamma a \subseteq I\Gamma S\Gamma I \subseteq I\Gamma I$. So $I \subseteq I\Gamma I$. Hence $I = I\Gamma I$ i.e. I is idempotent. Conversely, suppose that every ideal of S is idempotent. Let A and B two ideals of S. Then $A\Gamma B \subseteq A \cap B$. Also $(A\cap B)\Gamma(A\cap B) \subseteq A\Gamma B$. Since $A \cap B$ is an ideal of S, $(A\cap B)\Gamma(A\cap B) = A \cap B$. So $A \cap B \subseteq A\Gamma B$. Hence $A\Gamma B = A \cap B$ and so by proposition 4.4, S is a regular Γ-semiring. Hence the theorem. □

Theorem 4.6. *A commutative Γ-semiring S is regular if and only if every proper ideal of S is semiprime.*

Proof. Let the Γ-semiring S be regular and P be any proper ideal of S with $A\Gamma A \subseteq P$ where A is any ideal of S. Since $A\Gamma A = A$ (Theorem 4.5) so $A \subseteq P$. Hence P is semiprime in S. Conversely, suppose that every proper ideal of S is semiprime. Then let $x \in S$. Since S is commutative, $x\Gamma S\Gamma x$ is an ideal of S. If $x\Gamma S\Gamma x$ is an improper ideal then we are done. So let $x\Gamma S\Gamma x$ be proper. Then it is semiprime. Again since $< x > \Gamma < x > \subseteq x\Gamma S\Gamma x$ so $< x > \subseteq x\Gamma S\Gamma x$ whence $x \in x\Gamma S\Gamma x$. Hence S is a regular Γ-semiring. □

Remark: Commutativity of S is not required in the necessity part of the above two theorems.

Combining Theorem 3.17 and Theorem 4.6 we have the following theorem:

Theorem 4.7. *A commutative* Γ*-semiring* S *is regular if and only if for every proper ideal* Q *of* S, $r(Q) = Q$.

Theorem 4.8. *Let* S *be a commutative* Γ*-semiring and* L *be its left operator semiring.* S *is a regular* Γ*-semiring if and only if* L *is a multiplicatively regular semiring.*

Proof. Let the commutative Γ-semiring S be regular and let P be a proper ideal of L. Then P^+ is a proper ideal of S(Proposition 6.1 [2]). So by Theorem 4.7, $r(P^+) = P^+$. Hence $r(P^+)^{+'} = (P^+)^{+'}$. So by Theorem 3.15, $r(P^{++'}) = P$ i.e. $r(P) = P$. So by Proposition 2([6]), L is a regular semiring.

The converse follows by reversing the above argument. □

We know that every prime ideal of a Γ-semiring is strongly irreducible and every strongly irreducible ideal is irreducible whence every prime ideal of a Γ-semiring is irreducible. But we have a restricted converse in the following theorem:

Theorem 4.9. *A commutative* Γ*-semiring* S *is regular if and only if every irreducible ideal of* S *is prime.*

Proof. Let the commutative Γ-semiring S be regular and P be an irreducible ideal of S. Then L is regular (Theorem 4.8) and $P^{+'}$ is irreducible in L (Proposition 3.29). Again, L is commutative (Proposition 9.8 [2]). So $P^{+'}$ is prime in L (Proposition 6.27 [4]). Hence $P = (P^{+'})^+$ is prime in S (Lemma 3.2, Theorem 3.17[3]). The converse follows by reversing the above argument. □

Combining Theorems 3.33 and 4.9 we have the following theorem:

Theorem 4.10. *In a commutative regular* Γ*-semiring* S *any proper ideal* P *is the intersection of all prime ideals of* S *containing it.*

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Received by the editors January 31, 2000.