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RELATIVE COMPLETENESS WITH RESPECT TO TRANSPOSITIONS

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Abstract. One of the most important results in the theory of clones of operations is the fact that the number of clones is a continuum for $k \geq 3$, while the corresponding set for $k = 2$ is countable. This shows a sharp difference when we go from the binary to the ternary case. This paper discusses the relative completeness with respect to the clone generated by two unary functions and show the sharp difference when we go from the four-valued logic to k-valued logic for $k > 4$, as well. The number of maximal clones over a finite set is finite and increases with increase in k. However, there are two relative maximal clones if $k = 3, 4$ and there is one relative maximal clone if $k > 4$.

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1. Notation and preliminaries

Denote by $\mathbf{N}$ the set $\{1, 2, \ldots\}$ of positive integers. For $k, n \in \mathbf{N}$ Let $E_k = \{0, 1, \ldots, k-1\}$, denote by $P_k^{(n)}$ the set of all maps $E_k^n \to E_k$, and $P_k = \bigcup_{n \in \mathbf{N}} P_k^{(n)}$. We say that f is an *i-th projection of arity* n $(1 \leq i \leq n)$ if $f \in P_k^{(n)}$ and f satisfies the identity $f(x_1, \ldots, x_n) \approx x_i$. We say that $f \in P_k^{(n)}$ is *essential* if it depends on at least two variables and it takes all values from E_k. Let π_i^n denote the i-th projection of arity n, and let Π_k denote the set of all the projections over E_k. For $n, m \geq 1$, $f \in P_k^{(n)}$ and $g_1, \ldots, g_n \in P_k^{(m)}$ *the superposition of* f and $g_1, \ldots, g_n$, denoted by $f(g_1, \ldots, g_n)$,is defined by $f(g_1, \ldots, g_n)(a_1, \ldots, a_m) = f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$ for all $(a_1, \ldots, a_m) \in E_k^m$. A set $F \subseteq P_k$ is *a clone of operations on* E_k (or *clone* for short) if $\Pi_k \subseteq F$ and F is closed with respect to superposition. For $F \subseteq P_k$, $\langle F \rangle_{\mathrm{CL}}$ stands for the clone generated by F. We say that the clone F is *maximal* if there is no clone G such that $F \subset G \subset P_k$. $F \subseteq P_k$ is *complete* iff $\langle F \rangle_{\mathrm{CL}} = P_k$.

Let $\varrho \subseteq E_k^h$ be an h-ary relation and $f \in P_k^{(n)}$. We say that f *preserves* ϱ if for all h-tuples $(a_{11}, \ldots, a_{1h}), \ldots, (a_{n1}, \ldots, a_{nh})$ from ϱ we have

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$(f(a_{11},\ldots,a_{n1}),\ldots, f(a_{1h},\ldots,a_{nh})) \in \varrho$. $\mathrm{Pol}\,\varrho$ is the set of all $f \in P_k$ which preserve ϱ. For $F \subseteq P_k$, $\mathrm{Inv}\,F$ denotes the set of all the relations preserved by each $f \in F$.

It is interesting to consider the following problem: What are the maximal clones on a finite universe not containing a given clone C; or, equivalently, what operations are to be added to C to make it complete (or primal).

The following concept of relative completeness was introduced in [6].

Let C be a clone on E_k and $F \subseteq P_k$. F is *complete relative* to C (or *C-complete*) if $\langle F \cup C\rangle_{\mathrm{CL}} = P_k$.

The following theorem gives a necessary and sufficient condition for F to be C-complete. It is analogous to the Post completeness criterion.

Theorem 1.1. [6] *Let C be a clone on E_k. $F \subseteq P_k$ is complete relative to C if and only if $F \setminus M_i \neq \emptyset$ for every maximal clone containing C.*$\square$

Therefore, the problem of determining whether a set F is complete relative to C reduces to determining all the maximal clones that contain C.

This paper heavily depends upon the famous Rosenberg characterization of maximal clones. The following special sets of relations are considered:

R_1 - the set of all bounded partial orders on E_k;

R_2 - the set of self-dual relations, i.e. relations of the form $\{(x, s(x)) : x \in E_k\}$, where s is a fixed point free permutation of prime order (i.e. $s^p = \mathrm{id}$ for some prime p);

R_3 - the set of affine relations, i.e. relations of the form $\{(a, b, c, d) \in E_k^4 : a * b = c * d\}$, where $(E_k, *)$ is a p-elementary Abelian group (p prime);

R_4 - the set of all nontrivial equivalence relations on E_k;

R_5 - the set of all central relations on E_k;

R_6 - the set of all h-regular relations on E_k $(h \geq 3)$.

Theorem 1.2. [3] *A clone $\mathcal{M}$ is maximal iff there is a $\varrho \in R_1 \cup \ldots \cup R_6$ such that $\mathcal{M} = \mathrm{Pol}\,\varrho$* $\square$

2. Relative completeness with respect to transpositions

Consider the following transpositions ([1], *Theorem* 8, *p.* 54) on E_k:

$$g_i(x) = \begin{cases} i & , x = 0 \\ 0 & , x = i \\ x & , otherwise. \end{cases} \quad \text{and the clone generated by them: } \mathcal{C} = \langle \bigcup_{i=1}^{k-1} g_i \rangle_{\mathrm{CL}}.$$

Lemma 2.1. *For each $\varrho \in R_1$ there is $f \in \mathcal{C}$ such that f does not preserve ϱ.*

Proof. Pick $\varrho \in R_1$ and let a and b denote the smallest and the greatest element in E_k with respect to ϱ, respectivelly. Suppose that g_i preserves ϱ for each $i \in \{1, ..., k-1\}$. We have two cases:
$a = 0 : (0, b) \in \varrho$ implies $(g_b(0), g_b(b)) = (b, 0) \in \varrho$. Contradiction.
$a \neq 0 : (a, 0) \in \varrho$ implies $(g_a(a), g_a(0)) = (0, a) \in \varrho$. Contradiction. □

Lemma 2.2. *For each $\varrho \in R_2$ there is $f \in C$ such that f does not preserve ϱ.*

Proof. Pick $\varrho \in R_2$ and let $\varrho = \{(x, s(x)) : x \in E_k\}$ for some regular permutation s. Denote by p a prime number such that $s^p = id_{E_k}$ and let $(0, a_1, ..., a_{p-1})$ be a cycle of s which contains zero. Suppose that g_i preserves ϱ for each $i \in \{1, ..., k-1\}$. We have $(0, a_1) \in \varrho$ and $(g_{a_1}(0), g_{a_1}(a_1)) = (a_1, 0) \in \varrho$. Therefore, $p = 2$. From $k > 2$ it follows that there exists $a_j \in \{1, ..., k-1\} \setminus \{a_1\}$. Applying g_{a_j} on $(0, a_1)$ we conclude that $(g_{a_j}(0), g_{a_j}(a_1)) = (a_j, a_1) \in \varrho$, which is a contradiction with $(0, a_1) \in \varrho$ and $p = 2$. □

Lemma 2.3.

(a) *If $k > 4$ then for each $\varrho \in R_4$ there is $f \in C$ such that f does not preserve ϱ.*

(b) *If $k \in \{3, 4\}$ then g_i preserves ϱ for each $i \in \{1, ..., k-1\}$.*

Proof. (a) Let ϱ be an affine relation with the corresponding Abelian group $(E_k, *, e)$ and suppose g_i preserves ϱ for each $i \in \{1, ..., k-1\}$. Consider the following two cases:
$e \neq 0$: Choose $i, m, n \in \{1, ..., k-1\} \setminus \{e\}$ such that $i = m*n$. Then $(i, e, m, n) \in \varrho$. Since g_e preserves ϱ we have $(g_e(i), g_e(e), g_e(m), g_e(n)) = (i, 0, m, n) \in \varrho$. $i * e = m * n$ and $i * 0 = m * n$ implies $e = 0$. Contradiction.
$e = 0$: There are $i, j, m, n \in \{1, ..., k-1\}$ for which $i = m * n, j \notin \{m, n, i\}$. This implies $(i, 0, m, n) \in \varrho$ because 0 is the identity element. Sinse g_j preserves ϱ we have $(g_j(i), g_j(0), g_j(m), g_j(n)) = (i, j, m, n) \in \varrho$, i.e. $i * j = m * m$ and $i * 0 = m * n$ implies $j = 0$. Contradiction.
(b) If $k = 3$ then the class R_3 contains only one maximal set:
$Pol\varrho = Pol(\{(a, b, c)^T \in E_3^3 = 2(a+b)\})$ = Operations from C are permutations on E_3 and obviously they preserve ϱ.

For $k = 4$, the only maximal set contained in R_3 is $Pol\varrho = Pol(\{(a, b, c, d) \in E_4^4 | a * b = c * d\})$, where $(E_4, *, e)$ is a 2-elementary Abelian group.

It can be shown in a straightforward way that the function $g_i, i \in \{1, ..., k-1\}$ preserves ϱ. □

Lemma 2.4. *For each $\varrho \in R_4$ there is $f \in C$ such that f does not preserve ϱ.*

Proof. Choose any $\varrho \in R_4$ and suppose that g_i preserves ϱ for each $i \in \{1, ..., k-1\}$. We consider two possible cases:
$card(0/\varrho) = 1$: Suppose that $card(i/\varrho) > 1$ for some $i \in \{1, ..., k-1\}$. Thus, there exists $j \in \{1, ..., k-1\} \setminus \{i\}$ such that $j \in i/\varrho$. Since g_i preserves ϱ, $(i, j) \in \varrho$

implies $(g_i(i), g_i(j)) = (0, j) \in \varrho$ and we get a contradiction with the assumption that $card(0/\varrho) = 1$. Therefore, $card(i/\varrho) = 1$ for each $i \in \{1, ..., k-1\}$.
$card(0/\varrho) > 1$: Suppose that there exists $i \in \{1, ..., k-1\}$ such that $i \notin 0/\varrho$. Choose an arbitrary $j \in 0/\varrho, j \neq 0$. Since g_i preserves ϱ, $(0, j) \in \varrho$ implies $(g_i(0), g_i(j)) = (i, j) \in \varrho$, *i.e.*, $i \in j/\varrho = 0/\varrho$. However, this is a contradiction with the assumption that $i \notin 0/\varrho$, and it follows that $card(0/\varrho) = k$. Since all equivalence relations from R_4 are nontrivial we get a contradiction in both cases. □

Lemma 2.5. *For each $\varrho \in R_5$ there is $f \in C$ such that f does not preserve ϱ.*

Proof. It is well known that every central relation can be represented in the form $\varrho = E_k^h - P_{a_1a_2...a_h} - ...$ for some $a_1, a_2, ..., a_h, ... \in E_k$, where $P_{a_1a_2...a_h} = \{(a_{\pi(1)}, ..., a_{\pi(h)}) : \pi \text{ is a permutation of } \{1, 2, ..., h\}\}$. Suppose that g_i preserves ϱ for each $i \in \{1, ..., k-1\}$.
Case 1: 0 is a central element. Let $\varrho = E_k^h - P_{ja_2...a_h} - ...$ for some $j, a_2, ...a_h, ... \in E_k$. $(0, a_2, ..., a_h) \in \varrho$ because 0 is a central element. However, $(g_j(0), g_j(a_2), ..., g_j(a_h)) = (j, a_2, ..., a_h) \notin \varrho$ gives a contradiction.
Case 2: 0 is not a central element. Then we have $\varrho = E_k^h - P_{a_1a_2....a_h} - ...$ for some $a_2, ..., a_h, ... \in E_k$. Let $j \in \{1, ..., k-1\}$ be a central element of ϱ. Since g_j preserves ϱ, $(j, a_2, ..., a_h) \in \varrho$ implies $(g_j(j), g_j(a_2), ..., g_j(a_h)) = (0, a_2, ..., a_h) \in \varrho$, which is a contradiction. □

Lemma 2.6.

(a) For each $\varrho \in R_6, 2 < h < k$ there is $f \in C$ such that f does not preserve ϱ.

(b) The k-ary relation $\varrho \in R_6$ is preserved by g_i for each $i \in \{1, ..., k-1\}$.

Proof. (*a*) Let ϱ be a h-regular relation, $2 < h < k$ determined by a h-regular family of equivalence relations $\tau = \{q_1, ..., q_m\}$. Consider the following cases:
$card(0/q_1) > 1$: Denote by $C_i, 1 \leq i \leq h$ the classes of q_1, where $C_1 = 0/q_1$. There exists $a_2 \in C_1, a_2 \in E_k \setminus \{0\}$. Since τ is h-regular family of equivalence relations there exist $a_{j+1} \in C_j \cap \bigcap_{l=0}^{\lceil \frac{m-1}{h-2} \rceil - 1} a_j/q_{i_{l(h-2)+j}}$ for each $j \in \{2, ..., h-1\}, a_{j+1} \in E_k \setminus \{a_2, ..., a_j, 0\}$ (If $l(h-2)+j \notin \{2, ..., m\}$, suppose that $a_j/q_{i_{l(h-2)+j}} = \emptyset$).

It follows from the definition of the h-regular relation and the previous construction that $(0, a_2, ..., a_h) \in \varrho$ (For each $j, 1 \leq j \leq m$, at least two elements among $\{a_1, ..., a_h\}$ are q_j-equivalent)

Since every equivalence relation has exactly h equivalence classes there exists $j \in E_k \setminus \{0, a_2, ..., a_h\}$ such that $j \in C_h$.

From $(0, a_2, ..., a_h) \in \varrho$ it follows that $(g_j(0), g_j(a_2), ..., g_j(a_h)) = (j, a_2, ..., a_h) \in \varrho$, which is a contradiction with the definition of the h-regular relation because there are no two q_1-equivalent elements among $j, a_2, ..., a_h$.

$card(0/q_1) = 1$: Since $h < k$ there exists $a_1 \in E_k \setminus \{0\}$ such that $card(a_1/q_1) > 1$. If we consider a_1 instead of 0 in the previous case and if we do not choose elements from the class $0/q_1$ we get $(a_1, ..., a_h) \in \varrho$ implies $(g_{a_1}(a_1), ..., g_{a_1}(a_h)) = (0, a_2, ..., a_h) \in \varrho$ which is again a contradiction with the definition of the h-regular relation.
(b) Since $Pol\varrho, \varrho = E_k^k - P_{01...k-1}$ is Slupecki clone, it contains all essential unary functions. □

Theorem 2.1.

(a) *If* $k > 4$ *then there is exactly* 1 *relative maximal clone with respect to* $\mathcal{C}$.

(b) *If* $k \in \{3, 4\}$ *then there are exactly* 2 *relative maximal clones with respect to* $\mathcal{C}$.

Corollary 2.1. *If* $k > 4$, *then* $\mathcal{F}$ *is relatively complete with respect to* $\mathcal{C}$ *iff it contains an essential function.*

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