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ON SET–VALUED NON–BOOLEAN FUNCTIONS

Ratko Tošić[1], Lidija Čomić[2]

Abstract. In the set of functions $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ the subset of Boolean functions is not complete. We study one way of partitioning the definition domain of a set–valued function $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ into equivalence classes with respect to an equivalence relation generated by F so that on these classes exists a Boolean function f equal to F, and investigate this equivalence relation for some values of n and r.

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1. Introduction

Let $\mathbf{r} = \{0, 1, ..., r-1\}$, $r \geq 1$, and let $\mathcal{P}(\mathbf{r})$ be the set of subsets of $\mathbf{r}$. Then $(\mathcal{P}(\mathbf{r}), \emptyset, \mathbf{r}, \cup, \cap, ^-)$ is Boolean algebra. There are $2^{r2^{rn}}$ set-valued functions $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$, and only 2^{r2^n} of them are Boolean.

Let $\oplus$ denote the symmetric difference over $\mathcal{P}(\mathbf{r})$. It is well known that a function $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ is Boolean if and only if it can be represented in the form

$$F(X_1, ..., X_n) = A_0 \oplus \sum_{m=1}^{n} \sum_{i_1,...,i_m}^{1,2,...,n} A_{i_1...i_m} X_{i_1} ... X_{i_m}$$

for all $X_1, ..., X_n \in \mathcal{P}(\mathbf{r})$, where A_0 and $A_{i_1...i_m}$ are constants of $\mathcal{P}(\mathbf{r})$, and the sum is extended over all $\binom{n}{m}$ subsets $\{i_1, ..., i_m\}$ of m distinct indices from the set $\{1, ..., n\}$. The coefficients A_0 and $A_{i_1...i_m}$ are uniquely determined by F.

The following property of Boolean functions, given in [7], is the generalization of the results of McKinsey and Scognamiglio.

Theorem 1.1. *If $f : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ is a Boolean function then*

$$f(X_1, ..., X_n) \oplus f(Y_1, ..., Y_n) \subseteq \bigcup_{i=1}^{n} (X_i \oplus Y_i)$$

for all $X = (X_1, ..., X_n)$, $Y = (Y_1, ..., Y_n) \in \mathcal{P}^n(\mathbf{r})$.

[1]Institute of Mathematics, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Yugoslavia
[2]Faculty of Engineering, University of Novi Sad, Trg D. Obradovića 6, 21000 Novi Sad, Yugoslavia

Proof. Since f is Boolean, it can be represented in the form

$$f(X_1, X_2, ..., X_n) = A_0 \oplus \sum_{m=1}^{n} \sum_{i_1,...,i_m}^{1,...,n} A_{i_1...i_m} X_{i_1} ... X_{i_m},$$

so that

$$f(X_1, X_2, ..., X_n) \oplus f(Y_1, Y_2, ..., Y_n) = \sum_{m=1}^{n} \sum_{i_1,...,i_m}^{1,...,n} A_{i_1...i_m} (X_{i_1} ... X_{i_m} \oplus Y_{i_1} ... Y_{i_m}).$$

For an arbitrary term of this sum we have

$$A_{i_1...i_m}(X_{i_1}...X_{i_m} \oplus Y_{i_1}...Y_{i_m}) \subseteq X_{i_1}...X_{i_m} \oplus Y_{i_1}...Y_{i_m} =$$

$$X_{i_1}...X_{i_m}\overline{Y_{i_1}...Y_{i_m}} \cup \overline{X_{i_1}...X_{i_m}}Y_{i_1}...Y_{i_m} =$$

$$X_{i_1}...X_{i_m}(\overline{Y_{i_1}} \cup ... \cup \overline{Y_{i_m}}) \cup (\overline{X_{i_1}} \cup ... \cup \overline{X_{i_m}})Y_{i_1}...Y_{i_m} =$$

$$X_{i_1}...X_{i_m}\overline{Y_{i_1}} \cup ... \cup X_{i_1}...X_{i_m}\overline{Y_{i_m}} \cup \overline{X_{i_1}}Y_{i_1}...Y_{i_m} \cup ... \cup \overline{X_{i_m}}Y_{i_1}...Y_{i_m} \subseteq$$

$$X_{i_1}\overline{Y_{i_1}} \cup ... \cup X_{i_m}\overline{Y_{i_m}} \cup \overline{X_{i_1}}Y_{i_1} \cup ... \cup \overline{X_{i_m}}Y_{i_m} = (X_{i_1} \oplus Y_{i_1}) \cup ... \cup (X_{i_m} \oplus Y_{i_m}) \subseteq$$

$$(X_1 \oplus Y_1) \cup ... \cup (X_n \oplus Y_n) = \bigcup_{i=1}^{n} (X_i \oplus Y_i).$$

Every term is contained in $\bigcup_{i=1}^{n} (X_i \oplus Y_i)$, so

$$f(X_1, ..., X_n) \oplus f(Y_1, ..., Y_n) \subseteq \bigcup_{i=1}^{n} (X_i \oplus Y_i).$$ □

2. The equivalence relation generated by a set-valued function

Definition 2.1. *Let* $X = (X_1, ..., X_n)$, $Y = (Y_1, ..., Y_n) \in \mathcal{P}^n(\mathbf{r})$. *Then* $(X_1, ..., X_n) \sim_F (Y_1, ..., Y_n)$ *if*

$$F(X_1, ..., X_n) \oplus F(W_1, ..., W_n) \subseteq \bigcup_{i=1}^{n} (X_i \oplus W_i)$$

is equivalent to

$$F(Y_1, ..., Y_n) \oplus F(W_1, ..., W_n) \subseteq \bigcup_{i=1}^{n} (Y_i \oplus W_i)$$

for every $W = (W_1, ..., W_n) \in \mathcal{P}^n(\mathbf{r})$, *and* $[X]_F$ *denotes the set of all* Y *such that* $X \sim_F Y$.

Relation $\sim_F$ is obviousely an equivalence relation on $\mathcal{P}^n(\mathbf{r})$.

Theorem 2.1. *Let $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$. For every element $X = (X_1, ..., X_n) \in \mathcal{P}^n(\mathbf{r})$ there exists a Boolean function $f_X : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$ which coincides with F on $[X]_F$. This Boolean function is given by*

$$f_X(U_1, ..., U_n) = \bigcup_{Y \in [X]_F} F(Y_1, ..., Y_n) \bigcap_{i=1}^{n} (Y_i \oplus U_i \oplus \mathbf{r})$$

for every $U = (U_1, ..., U_n) \in \mathcal{P}^n(\mathbf{r})$.

Proof. Let $U = (U_1, ..., U_n) \in [X]_F$. Then

$$f_X(U_1, ..., U_n) = F(U_1, ..., U_n) \cup \bigcup_{\substack{Y \in [X]_F \\ Y \neq U}} F(Y_1, ..., Y_n) \bigcap_{i=1}^{n} (Y_i \oplus U_i \oplus \mathbf{r}).$$

Since $(U_1, ..., U_n) \sim_F (Y_1, ..., Y_n)$, for every $(Y_1, ..., Y_n) \in [X]_F$ we have

$$F(U_1, ..., U_n) \oplus F(Y_1, ..., Y_n) \subseteq \bigcup_{i=1}^{n} (U_i \oplus Y_i),$$

i.e.

$$\bigcap_{i=1}^{n} (U_i \oplus Y_i \oplus \mathbf{r}) \subseteq F(U_1, ..., U_n) \oplus F(Y_1, ..., Y_n) \oplus \mathbf{r},$$

(by De Morgan laws, $A \oplus \mathbf{r} = \overline{A}$ and $A \subseteq B \Rightarrow \overline{B} \subseteq \overline{A}$). By intersecting both sides of this inequality with $F(Y_1, ..., Y_n)$ for every $Y \in [X]_F, Y \neq U$ we get

$$F(Y_1, ..., Y_n) \bigcap_{i=1}^{n} (U_i \oplus Y_i \oplus \mathbf{r}) \subseteq$$

$$F(Y_1, ..., Y_n)(F(U_1, ..., U_n) \oplus F(Y_1, ..., Y_n) \oplus \mathbf{r}) =$$

$$F(Y_1, ..., Y_n)F(U_1, ..., U_n) \oplus F(Y_1, ..., Y_n) \oplus F(Y_1, ..., Y_n) =$$

$$F(Y_1, ..., Y_n)F(U_1, ..., U_n) \subseteq F(U_1, ..., U_n),$$

so that

$$\bigcup_{\substack{Y \in [X]_F \\ Y \neq U}} F(Y_1, ..., Y_n) \bigcap_{i=1}^{n} (Y_i \oplus U_i \oplus \mathbf{r}) \subseteq F(U_1, ..., U_n),$$

and $f_X(U_1, ..., U_n) = F(U_1, ..., U_n)$ for every $(U_1, ..., U_n) \in [X]_F$. □

For $X = (X_1, ..., X_n) \in \mathcal{P}^n(\mathbf{r})$ we introduce the collection of sets

$$Q_F(X) = \{(W_1, ..., W_n) \in \mathcal{P}^n(\mathbf{r}) | F(X_1, ..., X_n) \oplus F(W_1, ..., W_n) \subseteq \bigcup_{i=1}^{n} (X_i \oplus W_i)\}.$$

Then $X \sim_F Y$ if and only if $Q_F(X) = Q_F(Y)$. A function F is Boolean if the relation $\sim_F$ has one equivalence class.

Further next we investigate this relation for some values of n and r.

case $r = 1$

In this case the set $\mathcal{P}(\mathbf{r})$ is isomorphic to the two-element Boolean algebra $\mathbf{B_2}$, $\mathcal{P}^n(\mathbf{r})$ is isomorphic to $\mathbf{B_2^n}$ so that every function $F : \mathcal{P}^n(\mathbf{1}) \to \mathcal{P}(\mathbf{1})$ is Boolean and has one equivalence class.

case $n = 1$, $r = 2$

This case is studied in [6], and the following results are obtained:

number of classes	number of functions with n classes
1	16
2	16
3	128
4	96

case $n = 1$, $r = 3$

By using the program given in Appendix we obtain the following results:

number of classes	number of functions with n classes
1	64
2	1024
3	5504
4	34880
5	165888
6	779520
7	3386880
8	12403456

case $n = 2$, $r = 2$

Let the element (X_1, X_2) from $\mathcal{P}(\mathbf{2}) \times \mathcal{P}(\mathbf{2}) = \mathcal{P}(\{0,1\}) \times \mathcal{P}(\{0,1\})$ be represented by an integer j between 0 and 15 such that in the binary representation of j the first two digits correspond to the characteristic vector of the set X_1, and the last two digits correspond to the characteristic vector of the set X_2. So, for example, the binary representation of the number 11 is $1011 = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$. The first two digits, 10, determine the characteristic vector $(1, 0)$ of the set $\{1\}$, and the last two digits, 11, determine the characteristic vector $(1, 1)$ of the set $\{0, 1\}$, and $j = 11$ corresponds to the ordered pair $(\{1\}, \{0, 1\})$.

There are 2^{32} functions $F : \mathcal{P}(\mathbf{2}) \times \mathcal{P}(\mathbf{2}) \to \mathcal{P}(\mathbf{2})$, and among them only 256 generate the equivalence with one class.

In the sequel, we make use of Table 1, in which 1 in row i and column j denotes that the element (X_1, X_2) corresponding to the row i belongs to the collection Q_F of the element (Y_1, Y_2) that corresponds to column j and vice versa for any function $F : \mathcal{P}(\mathbf{2}) \times \mathcal{P}(\mathbf{2}) \to \mathcal{P}(\mathbf{2})$, because for those elements we have $(X_1 \oplus Y_1) \cup (X_2 \oplus Y_2) = \mathbf{2} = \{0,1\}$, and $F(X_1, X_2) \oplus F(Y_1, Y_2) \subseteq \bigcup_{i=1}^{2} (X_i \oplus Y_i)$ regardless of the values of function F.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	0	0	1	0	0	1	1	0	1	0	1	1	1	1	1
1	0	1	1	0	0	0	1	1	1	0	1	0	1	1	1	1
2	0	1	1	0	1	1	0	0	0	1	0	1	1	1	1	1
3	1	0	1	1	1	0	0	1	0	1	0	1	1	1	1	1
4	0	0	1	1	1	0	0	1	1	1	1	1	0	1	0	1
5	0	0	1	1	0	1	1	0	1	1	1	1	1	0	1	0
6	1	1	0	0	0	1	1	0	1	1	1	1	0	1	0	1
7	1	1	0	0	1	0	0	1	1	1	1	1	1	0	1	0
8	0	1	0	1	1	1	1	1	1	0	0	1	0	0	1	1
9	1	0	1	0	1	1	1	1	0	1	1	0	0	0	1	1
10	0	1	0	1	1	1	1	1	0	1	1	0	1	1	0	0
11	1	0	1	0	1	1	1	1	1	0	0	1	1	1	0	0
12	1	1	1	1	0	1	0	1	0	0	1	1	1	0	0	1
13	1	1	1	1	1	0	1	0	0	0	1	1	0	1	1	0
14	1	1	1	1	0	1	0	1	1	1	0	0	0	1	1	0
15	1	1	1	1	1	0	1	0	1	1	0	0	1	0	0	1

Table 1

Theorem 2.2. *There is no function $F : \mathcal{P}^2(\mathbf{2}) \to \mathcal{P}(\mathbf{2})$ whose equivalence $\sim_F$ has four classes.*

Proof. Let us suppose that there exists a function $F : \mathcal{P}(\mathbf{2}) \times \mathcal{P}(\mathbf{2}) \to \mathcal{P}(\mathbf{2})$ that generates the equivalence $\sim_F$ on $\mathcal{P}(\mathbf{2}) \times \mathcal{P}(\mathbf{2})$ with four classes, K_1, K_2, K_3 and K_4. Then to all elements $(X_1, X_2) \in \mathcal{P}(\{0,1\}) \times \mathcal{P}(\{0,1\})$ from the same class K_i corresponds the same collection $Q_F^i(X_1, X_2)$, $i = 1..4$. Let us represent this situation by a table similar to Table 1, where 1 in the row i and the column j denotes that the element j belongs to the collection of element i and vice versa. Obviously, this new table must have 1 in all the places where there is 1 in Table 1, and some of the 0's are replaced by 1 so that the table remains symmetric with respect to the diagonal, and there are four different rows, each of them representing a collection of the element j.

We show first that there are different integers i, j and k between 0 and 15 such that in this table there are 0's in the places (i,j), (i,k) and (j,k). Since the collections Q_F^i are different, at most one of them can have 1 in each place, so at least three of them have at least one 0. Let there be 0 in the row i and the column j, and let Q_F^1 denote the collection of the element i. Then, because of symmetry, there is 0 in the row j and the column i, and the rows i and j are not equal because the row j has 0 and row i has 1 in the column i (every element belongs to the collection attached to it). Denote the row j by Q_F^2. Let Q_F^3 denote the row k, different from the rows i and j, that has at least one 0.

case 1 The row k has 1 in the columns i and j. Since it has at least one 0, let it be in the column l. Then there is also 0 in the row l, column k. The row l is different from the row k, by the same argument as for the rows i and j. It also differs from the rows i and j in the column k, and let denote the row l by Q_F^4.

		i		j	l			k
	1							
i		1		0				1
			1					
j		0		1				1
l					1			0
						1		
							1	
k		1		1	0			1

Let m be an integer between 0 and 15, not equal to i, j, k or l. The row m must be equal to one of the rows i, j, k or l, say to the row i. Then, there is 0 in the place (m, j) and, by symmetry, in the place (j, m), so the row j has at least two 0's, in the columns i and m.

From Table 1 we get that for any two fixed columns exactly two rows have 0 in those columns, so collection Q_F^2 can correspond to at most two elements from $\mathcal{P}(\mathbf{2}) \times \mathcal{P}(\mathbf{2})$. If there is another row, apart from the row m, equal to the row i, then row j has at least three 0's, and from Table 1 we get that then no row can be equal to the row j. Also, since there are six 0's in each column of Table 1, at most six elements from $P(\mathbf{2}) \times P(\mathbf{2})$ may correspond to the collection Q_F^1. In any case, there remain at least nine rows different from Q_F^1 and Q_F^2, so they must be equal to either Q_F^3 or Q_F^4. By a similar argument we get a contradiction.

case 2 The row k has 1 in the column i and 0 in the column j. Then, there is 0 in the row j column k, so Q_F^2 has at least two 0's. Further, since the rows i and k are not equal, there must be a column $l \neq j$ in which they differ.

a. If there is 0 in the row i and 1 in the row k, then there is 0 in the place (l, i) and 1 in the place (l, k).

		i		j			l		k
	1								
i		1		0			0		1
			1						
j		0		1					0
					1				
						1			
l		0					1		1
								1	
k		1		0			1		1

The row l differs from the rows i, j and k in the columns i, k and i respectively, and we denote the row l by Q_F^4. Then, there are at least two rows, Q_F^1

and Q_F^2 with at least two 0's and this is impossible by the argument analogous to *case 1*.

b. If there is 0 in the row k and 1 in the row i, then there is 0 in the place (l, k) and 1 in the place (l, i).

		i		j			l		k
	1								
i		1		0			1		1
			1						
j		0		1					0
					1				
						1			
l		1					1		0
								1	
k		1		0			0		1

The row l differs from the rows i, j and k in the columns k, i and k respectively, and we denote the row l by Q_F^4. Then, there are at least two rows, Q_F^1 and Q_F^2 with at least two 0's and this is impossible by the argument analogous to *case 1*.

case 3 The row k has 1 in the column j and 0 in the column i. This case is analogous to *case 2*, and is also impossible.

case 4 The row k has 0 in both columns i and j.

		i		j				k
	1							
i		1		0				0
			1					
j		0		1				0
					1			
						1		
							1	
k		0		0				1

Then i, j and k are integers between 0 and 15 such that there are 0's in the places (i, j), (i, k) and (j, k).

Let the integers i, j and k correspond to the elements (X_1, X_2), (Y_1, Y_2) and (Z_1, Z_2) from $\mathcal{P}(\mathbf{2}) \times \mathcal{P}(\mathbf{2})$ respectively. Then $(X_1 \oplus Y_1) \cup (X_2 \oplus Y_2)$ is not $\{0, 1\} = 3$ (because it would then contain $F(X_1, X_2) \oplus F(Y_1, Y_2)$ regardless of the values of the function F), or $\emptyset = 0$ (the elements (X_1, X_2) and (Y_1, Y_2) are different since i and j are different). So $(X_1 \oplus Y_1) \cup (X_2 \oplus Y_2)$ can be either $\{0\} = 1$ or $\{1\} = 2$, and $F(X_1, X_2) \oplus F(Y_1, Y_2) = A \oplus B$ can be $\{1\} = 2$ or $\{0, 1\} = 3$, or $\{0\} = 1$ or $\{0, 1\} = 3$ respectively. The same holds for $F(X_1, X_2) \oplus F(Z_1, Z_2) = A \oplus C$ and $F(Y_1, Y_2) \oplus F(Z_1, Z_2) = B \oplus C$.

If $(X_1 \oplus Y_1) \cup (X_2 \oplus Y_2) = (X_1 \oplus Z_1) \cup (X_2 \oplus Z_2) = (Y_1 \oplus Z_1) \cup (Y_2 \oplus Z_2) = \alpha$, $\alpha = 1$ or $\alpha = 2$, then it is easy to see that the corresponding system of equations

$$A \oplus B = \beta_1,$$

$$A \oplus C = \beta_2,$$

$$B \oplus C = \beta_3,$$

$\beta_i = \overline{\alpha}$ or $\beta_i = 3$, $i = 1, 2, 3$, has no solution.

If $(X_1 \oplus Y_1) \cup (X_2 \oplus Y_2) = (X_1 \oplus Z_1) \cup (X_2 \oplus Z_2) = \alpha$, $(Y_1 \oplus Z_1) \cup (Y_2 \oplus Z_2) = \beta$, $\beta = \overline{\alpha}$, then $Y_1 \oplus Z_1$ or $Y_2 \oplus Z_2$, say $Y_1 \oplus Z_1$, is equal to β, and we have two cases:

case 1 $Y_1 = 0$, $Z_1 = \beta$. Then $X_1 \oplus Y_1 = 0$ ($\Rightarrow X_1 = 0 \Rightarrow X_1 \oplus Z_1 = 0 \oplus \beta = \beta \Rightarrow (X_1 \oplus Z_1) \cup (X_2 \oplus Z_2) \neq \alpha$, a contradiction), or $X_1 \oplus Y_1 = \alpha$ ($\Rightarrow X_1 = \alpha \Rightarrow X_1 \oplus Z_1 = \alpha \oplus \beta = 3 \Rightarrow (X_1 \oplus Z_1) \cup (X_2 \oplus Z_2) \neq \alpha$, a contradiction).

case 2 $Y_1 = \alpha$, $Z_1 = 3$. This case can be treated in the same way.

So, no function $F : \mathcal{P}(\mathbf{2}) \times \mathcal{P}(\mathbf{2}) \to \mathcal{P}(\mathbf{2})$ generates an equvalence on $\mathcal{P}(\mathbf{2}) \times \mathcal{P}(\mathbf{2})$ with four equivalence classes. □

It is easy to see that if $F : \mathcal{P}(\mathbf{2}) \times \mathcal{P}(\mathbf{2}) \to \mathcal{P}(\mathbf{2})$ and the relation $\sim_F$ has three equivalence classes, then there exists a partition of $\mathcal{P}(\mathbf{2}) \times \mathcal{P}(\mathbf{2})$ in two classes such that on these classes exists a Boolean function f equal to F.

case $n \geq 2$, $r \geq 2$

Theorem 2.3. *There is no function $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$, $n \geq 2$ whose equivalence has two classes.*

Proof. Suppose, on the contrary, that there is a function F whose equivalence has two classes K_1 and K_2. First we show that the collection Q_F^i of the element $X \in K_i$ is equal to K_i, $i = 1, 2$.

If $Y \in Q_F(X)$ then $X \in Q_F(Y)$. The collection Q_F^i contains every element Y from K_i, because if $X \sim_F Y$ then $Y \in Q_F(X)$. (The converse is not true. The collection $Q_F(X)$ may contain elements that are not in $[X]_F$.) If $Y \in K_2$ then Y does not belong to Q_F^1. Indeed, if we suppose thet $Y \in Q_F^1$, than every X from K_1 (since it belongs to Q_F^1) must belong to the collection Q_F^2 that corresponds to Y. Then the collection Q_F^2 contains all the elements X from $\mathcal{P}(\mathbf{2}) \times \mathcal{P}(\mathbf{2})$. Further, by a similar argument, we get that Q_F^1 also contains every X from $\mathcal{P}(\mathbf{2}) \times \mathcal{P}(\mathbf{2})$. (Since every X from K_1 belongs to Q_F^2, then also every Y from K_2 belongs to Q_F^1) But then Q_F^1 and Q_F^2 are equal, and so are K_1 and K_2, and there is only one class in the equivalence generated by F, a contradiction. So Q_F^1 is equal to K_1, and Q_F^2 to K_2.

Let K_1 denote the equivalence class $[(\emptyset, \emptyset, ..., \emptyset)]$. Then every element of the form $(X_1, \overline{X_1}, ..., X_n)$ belongs to K_1, (since $(X_1 \oplus \emptyset) \cup (\overline{X_1} \oplus \emptyset) \cup ... \cup (X_n \oplus \emptyset) = \mathbf{r}$), and $(X_1, X_2, ..., X_n) \sim_F (\overline{X_1}, X_2', ..., X_n')$, (since $(X_1 \oplus \overline{X_1}) \cup (X_2 \oplus X_2') \cup ... \cup (X_n \oplus X_n') = \mathbf{r}$), regardless of the values of the function F.

Let $(X_1, X_2, ..., X_n) \in \mathcal{P}^n(\mathbf{r})$.

a) If $X_1 \cap X_2 = \emptyset$ then, because of $\overline{X_1} \cup \overline{X_2} = \mathbf{r}$, $(\emptyset, \emptyset, ..., \emptyset) \sim_F (\overline{X_1}, \overline{X_2}, ..., X_n) \sim_F (\overline{X_1}, X_2, ..., X_n) \sim_F (X_1, X_2, ..., X_n)$, and $(X_1, X_2, ..., X_n) \in K_1$.

b) If $X_1 \subseteq X_2$ then, because of $\overline{X_1} \cup X_2 = \mathbf{r}$, $(\emptyset, \emptyset, ..., \emptyset) \sim_F (\overline{X_1}, X_2, ..., X_n) \sim_F (X_1, X_2, ..., X_n)$, and $(X_1, X_2, ..., X_n) \in K_1$.

c) If $X_1 \cap X_2 \neq \emptyset$ and neither $X_1 \subseteq X_2$ nor $X_2 \subseteq X_1$ then, because of $X_1 \cap X_2 \subseteq X_1$, $(\emptyset, \emptyset, ..., \emptyset) \sim_F (X_1, X_1 \cap X_2, ..., X_n) \sim_F (\overline{X_1}, X_2, ..., X_n) \sim_F (X_1, X_2, ..., X_n)$, and $(X_1, X_2, ..., X_n) \in K_1$.

This implies that $K_1 = \mathcal{P}^n(\mathbf{r})$, so there is no function $F : \mathcal{P}^n(\mathbf{r}) \to \mathcal{P}(\mathbf{r})$, $n \geq 2$ whose equivalence has two classes. □

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3 Appendix

```
program n1r3;
    type skup = set of 0..7;
    var n,broj,i,j, i1, i2, i3, i4, i5, i6, i7, i8 : integer;
```

```
        x, f : array [0..7] of 0..7;
        g : array [0..7] of skup;
        novaklasa : boolean;
        brklasa : array [1..8] of real;
        iz : text;
    begin
    n:=8;
    assign(iz,'izn1r3.txt');
    rewrite(iz);
    for i:=1 to n do begin
                        brklasa[i]:=0;
                        x[i-1]:=i-1;
                    end;
    for i1:=0 to n-1 do begin f[0]:=i1;
    for i2:=0 to n-1 do begin f[1]:=i2;
    for i3:=0 to n-1 do begin f[2]:=i3;
    for i4:=0 to n-1 do begin f[3]:=i4;
    for i5:=0 to n-1 do begin f[4]:=i5;
    for i6:=0 to n-1 do begin f[5]:=i6;
    for i7:=0 to n-1 do begin f[6]:=i7;
    for i8:=0 to n-1 do begin f[7]:=i8;
    for i:=0 to n-1 do begin
        g[i]:=[];
        for j:=0 to n-1 do begin
            if (((f[i] xor f[j]) and (x[i] xor x[j])) = (f[i] xor f[j]))
                then g[i] :=g[i] + [x[j]];
        end;
    end;
    broj := 1;
    for i:=1 to n-1 do begin
        novaklasa := true;
        for j:=0 to i-1 do begin
            if g[j] = g[i] then novaklasa := false;
        end;
        if novaklasa then broj := broj + 1;
    end;
    brklasa[broj]:=brklasa[broj]+1;
    end; end; end; end; end; end; end; end;
    for i:= 1 to n do
    writeln (iz,' ',i,' ',brklasa[i]);
    close(iz);
    end.
```

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