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# IDENTIFICATION OF THE SUPPORT FOR THE GENERALIZED FUNCTIONS [1]

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**Abstract.** Let $T$ be a distribution with compact support and $T_\lambda$ be a regularizing sequence of $T$. If we define for each $\lambda > 0$, $M_\lambda(\eta) = \sup_\xi |\hat{T}_\lambda(\xi + i\eta)|, \eta \in \mathbb{R}^n$ then $H(\eta) = \lim_{\lambda\to 0} \lim_{t\to\infty} \frac{\log M_\lambda(t\eta) - \log M_\lambda(0)}{t}$ is the supporting function of the smallest compact convex set supporting $T$.

Moreover, an analog for ultradistributions with compact support is also given.

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## 1. Introduction

In the Schwartz distribution theory it is well known that if $F(\zeta)$ is the Fourier-Laplace transform of a distribution with support in $K$, then $F(\zeta)$ is an entire function satisfying

$$|F(\zeta)| \le C(1 + |\zeta|)^N e^{H_K(\eta)}, \quad \zeta = \xi + i\eta \in \mathbb{C}^n$$

for some constants $N > 0$ and $C > 0$. Here $H_K(\eta)$ is the supporting function of a compact convex subset $K$ of $\mathbb{R}^n$, which gives information about the location of a convex hull of the support.

The supporting function $H_K(\eta)$ is defined on $\mathbb{R}^n$ for a compact subset $K$ and is a convex function which is positively homogeneous on $\mathbb{R}^n$. Conversely, a function $H(\eta)$ on $\mathbb{R}^n$ which is convex and positively homogeneous is given by $H(\eta) = H_K(\eta)$ for some compact convex set $K$. Thus, if we can describe a supporting function using $F(z)$ we can identify a support of generalized functions whose Fourier-Laplace transform is given by $F(z)$.

For example, a problem in this direction was studied in [3] as for a (Radon) measure with a compact support as follows:

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**Theorem 1.** *Let $T$ be a (Radon) measure with a compact support whose Fourier-Laplace transform is $F(z)$ and $M(\eta) = \sup_\xi |F(\xi + i\eta)|$. Then*

$$H(\eta) = \lim_{t\to\infty} \frac{\log M(t\eta) - \log M(0)}{t}$$

*is the supporting function of the smallest compact convex set supporting $T$.*

In this paper we will generalize this theorem to the case of distributions and ultradistributions. The function $M(\eta)$ above can be defined only for measures with compact support, since its Fourier-Laplace transform $F(\zeta)$ satisfies an inequality $|F(\zeta)| \le Ce^{A|\eta|}$ for all $\zeta = \xi + i\eta \in \mathbb{C}^n$. But $M(\eta)$ does not work, in general, for the distributions or the ultradistributions. Hence a point of this paper is to develope a new function which substitutes $M(\eta)$ and $H(\eta)$ above. To overcome this difficulty we use the regularizations of the generalized functions via some sequence of cut off functions.

## 2. The supporting function for the distributions

Throughout this paper we refer the reader to [1] and [4] for the notations and details of distributions.

We use the multi-index notations such as $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$ and $\partial_j = \frac{\partial}{\partial x_j}$ for $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of all non-negative integers.

For a compact set $E$ in $\mathbb{R}^n$, a supporting function $H_E$ of $E$ on $\mathbb{R}^n$ is defined by

$$H_E(\eta) = \sup_{x\in E}\langle x, \eta\rangle, \quad \eta \in \mathbb{R}^n.$$

Since it is the supremum of linear functions, we have

(*i*) $\quad H_E(\xi + \eta) \le H_E(\xi) + H_E(\eta), \quad \xi, \eta \in \mathbb{R}^n$ (i.e subadditive)

(*ii*) $\quad H_E(t\xi) = tH_E(\xi), \quad \xi \in \mathbb{R}^n, \ t \ge 0$ (i.e positively homogeneous)

Also, we have $H_E(\xi) = H_{chE}(\xi)$ for any compact $E$ in $\mathbb{R}^n$ where $chE$ denotes the smallest closed convex set containing $E$. The following theorem will be very useful later.

**Theorem 2.1. ([4,Theorem 4.3.2])** *For every convex positively homogeneous function $H$ in $\mathbb{R}^n$ there is precisely one convex compact set $K$ such that $H = H_K$, in fact, $K = \{x : \langle x, \xi\rangle \le H(\xi), \xi \in \mathbb{R}^n\}$.*

We now introduce a regularization of distributions. For an infinitely differentiable function $\phi$ with support in the closed unit ball $B$ centered at the origin, satisfying $\int \phi(x)dx = 1$ we set $\phi_\lambda(x) = \frac{1}{\lambda^n}\phi(\frac{x}{\lambda})$ for any positive number $\lambda > 0$.

Then it is easy to see that $\phi_\lambda \in C_0^\infty(\mathbf{\Omega})$, $\text{supp}\phi_\lambda = B_\lambda = \{x \in \mathbb{R}^n \mid |x| \le \lambda\}$ and $\int \phi_\lambda(x)\,dx = 1$. For any distribution $T$, the function $T*\phi_\lambda$ converges weakly to $T$ as $\lambda \to 0$. In view of this fact a sequence of functions $T_\lambda = T*\phi_\lambda$ is called a *regularizing sequence* or a *regularization* of $T$.

Now we introduce a result which identifies the support of Radon measures as follows:

**Theorem 2.2.** **([3])** *Let a distribution $T$ be a measure in $\mathbb{R}^n$ with compact support and $M(\eta) = \sup_\xi |\hat{T}(\xi+i\eta)|$ where $\hat{T}$ is the Fourier (-Laplace) transform of $T$. Then*

$$H(\eta) = \lim_{t\to\infty} \frac{\log M(t\eta) - \log M(0)}{t}$$

*is the supporting function of the smallest closed convex set supporting $T$.*

As done in the case of the measure with compact support we give here a supporting function for distributions with compact support by use of but a regularizing sequence as follows:

**Theorem 2.3.** *Let $T$ be a nonzero distribution with compact support and $T_\lambda$ be a regularizing sequence of $T$. If we define for each $\lambda > 0$*

$$M_\lambda(\eta) = \sup_\xi |\hat{T}_\lambda(\xi + i\eta)|, \quad \eta \in \mathbb{R}^n,$$

*then*

$$H(\eta) = \lim_{\lambda\to 0}\lim_{t\to\infty} \frac{\log M_\lambda(t\eta) - \log M_\lambda(0)}{t}$$

*is the supporting function of the smallest compact convex set supporting $T$.*

*Proof.* Considering $T_\lambda$ as a distribution of order 0 with compact support the Paley-Wiener theorem implies that there are constants $C_\lambda > 0$ and $A_\lambda > 0$ such that

$$|\hat{T}_\lambda(\zeta)| \le C_\lambda \exp A_\lambda |\text{Im}\zeta|, \qquad \zeta \in \mathbb{C}^n.$$

Therefore, $M_\lambda(\eta)$ satisfies the same inequality and, in particular, is always finite. Moreover, $M_\lambda(\eta)$ is never 0 since the vanishing of $\hat{T}_\lambda(\xi + i\eta)$ for some $\eta$ and all real $\xi$ implies that $\hat{T}_\lambda(\xi)$ is identically 0.

Here we notice that $M_\lambda(\eta)$ is a logarithmically convex function of $\eta$. To show this, we consider three real vectors $\xi, \eta', \eta''$ and one-dimensional complex variable $z = t + is$; the point $\xi + i\{z\eta' + (1-z)\eta''\}$ depends analytically on $z$ and may be written as $\xi - s(\eta' - \eta'') + i\{t\eta' + (1-t)\eta''\}$. Then the function

$$f_\lambda(z) = \hat{T}_\lambda(\{\xi - s(\eta' - \eta'')\} + i\{t\eta' + (1-t)\eta''\})$$

is an entire function of $z$. By Three Line Theorem (see [2]) the supremum

$$\begin{aligned}\mu_\lambda(t) &= \sup_s |f_\lambda(t+is)| \\ &= \sup_s |\hat{T}_\lambda(\{\xi - s(\eta' - \eta'')\} + i\{t\eta' + (1-t)\eta''\})|\end{aligned}$$

is a logarithmically convex function of $t$. Taking the supremum again over $\xi$ we find that $\sup_\xi \sup_s |\hat{T}_\lambda(\{\xi - s(\eta' - \eta'')\} + i\{t\eta' + (1-t)\eta''\})|$ is also a logarithmically convex function in $t$. This implies that $M_\lambda(t\eta' + (1-t)\eta'')$ is logarithmically convex in $t$. Since the vectors $\eta'$ and $\eta''$ were arbitrary, it follows that $M_\lambda(\eta)$ is logarithmically convex.

In view of this fact, the difference quotient

$$\frac{\log M_\lambda(t\eta) - \log M_\lambda(0)}{t}$$

is convex in $\eta$ for a fixed positive $t$, and being the difference quotient of a convex function, it increases with $t$.
Since $M_\lambda(t\eta) \le C_\lambda \exp A_\lambda |t\eta|$,

$$\begin{aligned}\frac{\log M_\lambda(t\eta) - \log M_\lambda(0)}{t} &\le \frac{\log C_\lambda + A_\lambda|t\eta| - \log M_\lambda(0)}{t} \\ &= \frac{\log C_\lambda - \log M_\lambda(0)}{t} + A_\lambda|\eta|\end{aligned}$$

for all $t > 0$.

It follows that there exists a limit as $t$ approaches infinity. Thus

$$\begin{aligned}H_\lambda(\eta) &= \lim_{t\to\infty} \frac{\log M_\lambda(t\eta) - \log M_\lambda(0)}{t} \\ &= \sup_{t>0} \frac{\log M_\lambda(t\eta) - \log M_\lambda(0)}{t}\end{aligned}$$

is convex in $\eta$ and finite everywhere. Then it follows that for $s > 0$

$$\begin{aligned}H_\lambda(s\eta) &= \sup_{t>0} \frac{\log M_\lambda(ts\eta) - \log M_\lambda(0)}{t} \\ &= s \sup_{r>0} \frac{\log M_\lambda(r\eta) - \log M_\lambda(0)}{r} \\ &= sH_\lambda(\eta).\end{aligned}$$

Thus $H_\lambda(\eta)$ is positively homogeneous. It follows from Theorem 2.1 that $H_\lambda(\eta)$ is a supporting function of a compact convex set $K(\lambda)$ in $\mathbb{R}^n$.

Evidently, $H_\lambda(\eta)$ is at least as large as the quotient for $t = 1$, whence $H_\lambda(\eta) \ge \log M_\lambda(\eta) - \log M_\lambda(0)$, that is $M_\lambda(\eta) \le M_\lambda(0) \exp H_\lambda(\eta)$. From the Paley-Wiener theorem, it is clear that $K(\lambda)$ is a support for the measure $T_\lambda$, and it suffices to show that it is exactly the smallest convex set supporting $T_\lambda$.

If $K(\lambda)^*$ is a compact, convex support for $T_\lambda$ with the supporting function $H_\lambda{}^*(\eta)$, then $|\hat{T}_\lambda(\zeta)| \le C_\lambda \exp H_\lambda{}^*(\eta)$ and therefore $M_\lambda(\eta) \le C_\lambda \exp H_\lambda{}^*(\eta)$, whence, for $t > 0$

$$\begin{aligned}\log M_\lambda(t\eta) &\le \log C_\lambda + H_\lambda{}^*(t\eta) \\ \log M_\lambda(t\eta) - \log M_\lambda(0) &\le \log C_\lambda + tH_\lambda{}^*(\eta) - \log M_\lambda(0)\end{aligned}$$

then

$$\frac{\log M_\lambda(t\eta) - \log M_\lambda(0)}{t} \le H_\lambda{}^*(\eta) + \frac{\log C_\lambda - \log M_\lambda(0)}{t}$$

and therefore $H_\lambda(\eta) \le H_\lambda{}^*(\eta)$. It follows that $K(\lambda)$ is contained in $K(\lambda)^*$. Hence, for each $\lambda > 0$, $H_\lambda(\eta) = \lim_{t\to 0} \frac{\log M_\lambda(t\eta) - \log M_\lambda(0)}{t}$ is the supporting function of the smallest closed convex set supporting $T_\lambda$.

Now we show that $H_\lambda(\eta)$ approaches the supporting function of the smallest closed convex set supporting $T$ as $\lambda \to 0$.

Since $T_\lambda = T * \phi_\lambda$ and $\phi_\lambda$ has a support $B_\lambda$, we can see that

$$ch\mathrm{supp}T_\lambda = chK + B_\lambda = K_\lambda$$

where $K_\lambda = \{x \in \mathbb{R}^n \mid dist(x, K) \le \lambda\}$ and $chA$ denotes a convex hull of a set $A$.

Thus $H_{K_\lambda}$ is also the supporting function of the smallest closed convex set supporting $T_\lambda$, that is, we have $H_{K_\lambda}(\eta) = H_\lambda(\eta)$. From this fact and Theorem 2.1 we can see that $K(\lambda) = K_\lambda$.

For $\lambda < \lambda'$, we have

$$K \subset K_\lambda \subset K_{\lambda'}.$$

Then it follows that

$$H_K(\eta) \le H_\lambda(\eta) \le H_{\lambda'}(\eta). \tag{2.1}$$

In view of (2.1) the supporting function $H_\lambda(\eta)$ is a monotone function of $\lambda$ and bounded from below.

Hence we obtain the limit of $H_\lambda(\eta)$ as $\lambda$ approaches zero, say $H(\eta)$. Since $H_\lambda(\eta)$ is convex in $\eta$ and positively homogeneous of degree 1 for all $\lambda$, $H(\eta)$ is also convex in $\eta$ and positively homogeneous of degree 1. Thus, by Theorem 2.1 there is precisely one convex compact set $E$ in $\mathbb{R}^n$ such that

$$H(\eta) = H_E(\eta) \tag{2.2}$$

On the other hand, it is clear from (2.1) that

$$H_K(\eta) \le H(\eta). \tag{2.3}$$

By the definition of the supporting function, we have

$$\begin{aligned} H_\lambda(\eta) &= \sup\nolimits_{\xi\in K_\lambda}\langle \xi, \eta\rangle \\ &\le \sup\nolimits_{\xi_1\in K}\langle \xi_1, \eta\rangle + \sup\nolimits_{\xi_2\in B_\lambda}\langle \xi_2, \eta\rangle \\ &= H_K(\eta) + |\eta|\lambda. \end{aligned}$$

It follows that

$$H(\eta) \le H_K(\eta), \tag{2.4}$$

as $\lambda \to 0$.

Thus we obtain from (2.2), (2.3) and (2.4) that

$$H_E(\eta) = H(\eta) = H_K(\eta).$$

Now we show that the above supporting function is independent of the choice of a regularizing sequence. To show this we take any $C^\infty$-function $\phi$ and $\psi$ with support in the unit ball $B$ and $\int \phi(x)dx = \int \psi(x)dx = 1$. For any positive number $\lambda$ we set

$$\phi_\lambda(x) = \frac{1}{\lambda}\phi(\frac{x}{\lambda}), \quad \psi_\lambda(x) = \frac{1}{\lambda}\psi(\frac{x}{\lambda}).$$

Then $T * \phi_\lambda$ and $T * \psi_\lambda$ are regularizing sequences of $T$. Let

$$M_\lambda(\eta) = \sup_\xi |(T * \phi_\lambda)\hat{}(\xi + i\eta)| = \sup_\xi |\hat{T}(\xi + i\eta)\hat{\phi}_\lambda(\xi + i\eta)|$$

and

$$N_\lambda(\eta) = \sup_\xi |(T * \psi_\lambda)\hat{}(\xi + i\eta)| = \sup_\xi |\hat{T}(\xi + i\eta)\hat{\psi}_\lambda(\xi + i\eta)|.$$

It follows from Theorem 2.2 that

$$\lim_{t\to\infty} \frac{\log M_\lambda(t\eta) - \log M_\lambda(0)}{t} \quad \text{and} \quad \lim_{t\to\infty} \frac{\log N_\lambda(t\eta) - \log N_\lambda(0)}{t}$$

are the supporting functions of the smallest closed convex set supporting $T * \phi_\lambda$ and $T * \psi_\lambda$ respectively.

Since $ch(T * \phi_\lambda) = chT + B_\lambda = ch(T * \psi_\lambda)$, we have

$$\lim_{t\to\infty} \frac{\log M_\lambda(t\eta) - \log M_\lambda(0)}{t} = \lim_{t\to\infty} \frac{\log N_\lambda(t\eta) - \log N_\lambda(0)}{t}.$$

Hence we have

$$\lim_{\lambda\to 0}\lim_{t\to\infty} \frac{\log M_\lambda(t\eta) - \log M_\lambda(0)}{t} = \lim_{\lambda\to 0}\lim_{t\to\infty} \frac{\log N_\lambda(t\eta) - \log N_\lambda(0)}{t}.$$

This fact implies that the supporting function $H(\eta)$ can be obtained independently of the choice of the regularizing sequence. This completes the proof. □

**Remark.** In fact, we can see that $H(\eta)$ in Theorem 2.3 is reduced to $H(\eta)$ in Theorem 2.2 when $T$ is a measure. To see this let us consider a measure $T$ with compact support whose regularizing sequence is $T_\lambda$. Let

$$M(\eta) = \sup_\xi |\hat{T}(\xi + i\eta)|$$

and

$$M_\lambda(\eta) = \sup_\xi |\hat{T}_\lambda(\xi + i\eta)|$$

for each $\lambda > 0$. Then we have

$$\lim_{t\to\infty} \frac{\log M(t\eta) - \log M(0)}{t} - \lim_{\lambda\to 0}\lim_{t\to\infty} \frac{\log M_\lambda(t\eta) - \log M_\lambda(0)}{t} = H_K(\eta) - \lim_{\lambda\to 0} H_{K_\lambda}(\eta).$$

It follows from the definition of the supporting function that

$$\begin{aligned} H_{K_\lambda}(\eta) &= \sup_{\xi\in K_\lambda}\langle\xi,\eta\rangle \\ &\le \sup_{\xi_1\in K}\langle\xi_1,\eta\rangle + \sup_{\xi_2\in B_\lambda}\langle\xi_2,\eta\rangle \\ &= H_K(\eta) + |\eta|\lambda, \end{aligned}$$

and

$$\lim_{\lambda\to 0} H_{K_\lambda}(\eta) \le H_K(\eta).$$

On the other hand, since $H_K(\eta) \le H_{K_\lambda}(\eta)$ for all $\lambda > 0$, we have $H_K(\eta) \le \lim_{\lambda\to 0} H_{K_\lambda}(\eta)$. Thus we can easily see that

$$\lim_{t\to\infty} \frac{\log M(t\eta) - \log M(0)}{t} = \lim_{\lambda\to 0}\lim_{t\to\infty} \frac{\log M_\lambda(t\eta) - \log M_\lambda(0)}{t}.$$

## 3. The supporting functions for the ultradistributions

First, we introduce an ultradistribution. Let $M_p$, $p = 0, 1, 2, \cdots$, be a sequence of positive numbers and let $\Omega$ be an open subset of $\mathbb{R}^n$. An infinitely differentiable function $\phi$ on $\Omega$ is called an ultradifferentiable function of class $(M_p)$ if for any compact set $K$ of $\Omega$ and for each $h > 0$

$$|\phi|_{M_p,K,h} = \sup_{\substack{x\in K\\ \alpha\in\mathbb{N}_0^n}} \frac{|\partial^\alpha\phi(x)|}{h^{|\alpha|}M_{|\alpha|}}$$

is finite.

We impose the following conditions on $M_p$ :

(M.1) $M_p^2 \le M_{p-1}M_{p+1}$, $p = 1, 2, \cdots$.

(M.2)$'$ There are constants $A > 0$ and $H > 0$ such that

$$M_{p+1} \le AH^pM_p \; p = 0, 1, \cdots.$$

(M.3)$'$ $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$.

For example. the sequence $M_p = p!^s$ $(s > 1)$ satisfies all above conditions.

We denote by $\mathcal{E}_{(M_p)}(\Omega)$ the space of all ultradifferentiable functions of class $(M_p)$ on $\Omega$ $\mathcal{E}_{(M_p)}(\Omega)$. The topology of such space is defined as follows:

A sequence $\phi_j \to 0$ in $\mathcal{E}_{(M_p)}(\Omega)$ if for any compact set $K$ of $\Omega$ and for every $h > 0$ we have

$$\sup_{\substack{x\in K\\ \alpha\in\mathbb{N}_0^n}} \frac{|\partial^\alpha \phi_j(x)|}{h^{|\alpha|}M_{|\alpha|}} \to 0 \quad \text{as} \quad j\to\infty.$$

In addition, we denote by $\mathcal{D}_{(M_p)}(\Omega)$ the space of all ultradifferentiable functions of class $(M_p)$ on $\Omega$ with compact support.

We denote by $\mathcal{E}'_{(M_p)}(\Omega)$ the strong dual space of $\mathcal{E}_{(M_p)}(\Omega)$ and we call its elements ultradistributions of type $(M_p)$ with compact support in $\Omega$. The space $\mathcal{D}'_{(M_p)}(\Omega)$ is also defined similarly as in the distributions $\mathcal{D}'(\Omega)$. For more details on the ultradistributions $\mathcal{E}'_{(M_p)}(\Omega)$ and $\mathcal{D}'_{(M_p)}(\Omega)$ we refer the reader to [5]. If $u$ is a ultradistribution of type $(M_p)$ with compact support and $\phi \in \mathcal{E}_{(M_p)}(\Omega)$, then we define the convolution $u * \phi$ by

$$f * \phi(x) = f_y(\phi(x-y)).$$

Moreover, a convolution $u * v$ of two ultradistributions $u$ and $v$ with compact support is defined by

$$(u*v)(\phi) = u_x(v_y(\phi(x+y)))$$

for every $\phi \in \mathcal{D}_{(M_p)}$.

We introduce the regularizing sequences of ultradistributions.

It follows from (M.3)$'$ that there is $\gamma(x) \in \mathcal{D}_{(M_p)}$ with support in the unit ball $B$ such that $\gamma(x) \geq 0$ and $\int \gamma(x)dx = 1$. Let $\gamma_\lambda(x) = \frac{1}{\lambda^n}\gamma(\frac{x}{\lambda})$ for any positive number $\lambda > 0$. Then $\gamma_\lambda \in \mathcal{D}_{(M_p)}$ with support in $B_\lambda$ and $\int \gamma_\lambda(x)dx = 1$. As done in the case of $C^\infty$ functions we can easily show that for each $\phi \in \mathcal{D}_{(M_p)}$, $\gamma_\lambda * \phi$ converges to $\phi$ in $\mathcal{D}_{(M_p)}$ as $\lambda$ goes to 0. Thus for each $T \in \mathcal{D}'_{(M_p)}(\Omega)$ one can see that $T_\lambda = T * \gamma_\lambda$ belongs to $\mathcal{E}_{(M_p)}(\Omega)$ and $T_\lambda$ converges to $T$ as $\lambda \to 0$ in the dual space of $\mathcal{D}_{(M_p)}(\Omega)$. We call $T_\lambda$ a *regularization* of $T$.

We need the following Lemma which is just a variation of Theorem 4.3.3 in [4], and can be proved similarly.

**Lemma 3.3.** *Let $u_1$ and $u_2$ be ultradistributions of type $(M_p)$ with compact support. Then we have*

$$chsupp(u_1 * u_2) = chsuppu_1 + chsuppu_2.$$

As in the case of the distributions we give here the supporting functions for the ultradistributions:

**Theorem 3.4.** *Let $T$ be an ultradistribution of type $(M_p)$ with compact support and $T_\lambda$ be its regularization. If $M_\lambda(\eta) = \sup_\xi |\hat{T}_\lambda(\xi + i\eta)|$, $\eta \in \mathbb{R}^n$, then*

$$H(\eta) = \lim_{\lambda\to 0}\lim_{t\to\infty} \frac{\log M_\lambda(t\eta) - \log M_\lambda(0)}{t}$$

*is the supporting function of the smallest closed convex set supporting $T$.*

*Proof.* Since $T_\lambda$ is the measure with compact support, we can see that $H_\lambda(\eta) = \lim_{t\to\infty} \dfrac{\log M_\lambda(\eta) - \log M_\lambda(0)}{t}$ is the supporting function of the smallest closed convex set supporting $T_\lambda$. It follows from Lemma 3.3 that $H_\lambda(\eta)$ is the supporting function of $K_\lambda$, where $K_\lambda = ch\mathrm{supp}T + B_\lambda$. For $\lambda < \lambda'$ we have

$$K \subset K_\lambda \subset K_{\lambda'}.$$

Then it follows that

$$H_K(\eta) \le H_\lambda(\eta) \le H_{\lambda'}(\eta). \tag{3.1}$$

In view of (3.1) the supporting function $H_\lambda(\eta)$ is a monotone function of $\lambda$ and bounded from below.

Hence we obtain the limit of $H_\lambda(\eta)$ as $\lambda$ approaches zero, say $H(\eta)$. Since $H_\lambda(\eta)$ is convex in $\eta$ and positively homogeneous of degree 1 for all $\lambda$, $H(\eta)$ is also convex in $\eta$ and positively homogeneous of degree 1. Therefore, there is precisely one convex compact set $E$ in $\mathbb{R}^n$ such that

$$H(\eta) = H_E(\eta).$$

On the other hand, it is clear from (3.1) that

$$H_K(\eta) \le H(\eta). \tag{3.2}$$

By the definition of the supporting function, we have

$$\begin{aligned} H_\lambda(\eta) \quad &\le \sup\nolimits_{\xi_1\in K}\langle \xi_1, \eta\rangle + \sup\nolimits_{\xi_2\in B_\lambda}\langle \xi_2, \eta\rangle \\ &= H_K(\eta) + |\eta|\lambda. \end{aligned}$$

Then it follows that

$$H(\eta) \le H_K(\eta), \tag{3.3}$$

as $\lambda \to 0$. It follows from (3.2) and (3.3) that

$$H_E(\eta) = H_K(\eta) = H(\eta).$$

Since such a compact convex set $E$ is unique the set $E$ is the convex hull of supp $T$. □

**Remark.** It will be quite interesting if one consider a hyperfunctional analog of this result. But, in general, since the hyperfunctions do not have cutoff functions as test functions, the method used in this paper does not work any longer for the hyperfunctions.

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