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APPROXIMATIONS TO MILD SOLUTIONS OF STOCHASTIC SEMILINIAR EQUATIONS

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Abstract. Using a Picard type method of approximation and the Hausdorff measure of noncompactness, we shall investigate the global existence of mild solutions for a class of Ito type stochastic differential equations whose coefficients satisfy more general than the Lipschitz and linear growth conditions.

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1. Introduction

Let us consider a stochastic differential equation of Ito type,

$$(1)\qquad \begin{cases} dX(t) &= (AX(t)+F(t,X(t)))dt+B(t,X(t))dW(t) \\ X(0) &= \xi \end{cases} .$$

We will assume that a probability space $(\Omega, \mathcal{F}, P)$ together with a complete right continuous filtration $\mathcal{F}_t$, $t \geq 0$ are given. We denote by $\mathcal{P}_T$ the predictable σ-fields on $\Omega_T = [0,T] \times \Omega$.

Let U and H be two separable Hilbert spaces and W a Wiener process on U with the covariance operator Q, positive, linear and bounded on U with $TrQ < \infty$.

Let $U_0 = Q^{\frac{1}{2}}(U)$ with the induced norm $||u||_0 = ||Q^{-\frac{1}{2}}u||$. Denote by L_2^0 the separable Hilbert space of all the Hilbert-Schmidt operators from U_0 to H equipped with the norm

$$||D||_{L_2^0} = [\sum_{j=1}^{\infty} ||DQ^{\frac{1}{2}}e_j||^2]^{\frac{1}{2}}, \quad D \in L_2^0$$

where $\{e_j\}$ is a complete orthonormal basis on U. The spaces H and L_2^0 are equipped with Borel σ-fields $\mathcal{B}(H)$ and $\mathcal{B}(L_2^0)$. Moreover, ξ is a H-valued random variable, $\mathcal{F}_0$-measurable.

We fix $T > 0$ and impose the following conditions on the coefficients A, F and B of the equation (1):

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(i) A is the infinitesimal generator of a strongly continuous semigroup $S(t), t \geq 0$ in H.

(ii) The mapping $F : [0,T] \times \Omega \times H \to H,\ (t,\omega,x) \to F(t,\omega,x)$ is measurable from $(\Omega_T \times H, \mathcal{P}_T \times \mathcal{B}(H))$ into $(H, \mathcal{B}(H))$.

(iii) The mapping $B : [0,T] \times \Omega \times H \to L_2^0,\ (t,\omega,x) \to B(t,\omega,x)$ is measurable from $(\Omega_T \times H, \mathcal{P}_T \times \mathcal{B}(H))$ into $(L_2^0, \mathcal{B}(L_2^0))$.

A mapping $X : [0,T] \times \Omega \to H$ which is measurable from $(\Omega_T, \mathcal{P}_T)$ into $(H, \mathcal{B}(H))$, is said to be a *mild solution* of (1), if for arbitrary $t \in [0,T]$, we have

$$P\left(\int_0^t (||S(t-s)F(s,X(s))|| + ||S(t-s)B(s,X(s)||_{L_2^0})ds < +\infty\right) = 1$$

and

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(s,X(s))ds + \int_0^t S(t-s)B(s,X(s)dW(s) \quad P \text{ a.s..}$$

Existence and uniqueness theorems for the solutions of equation (1) under Lipschitz conditions on the coefficients are studied in [5] (Th 7.7.4 and Th 7.7.6).

Stochastic evolution equations in infinite dimensions are natural generalisations of stochastic ordinary differential equations and their theory has motivations coming both from mathematics and natural sciences: physics, chemistry and biology (cf. [5]).

In the present paper we shall present a global existence and uniqueness theorem for solutions of the above mentioned equation under more general conditions. Similar results in finite dimensional case can be found in [1], [11]. For the deterministic case see [2], [6], [10]. For the existence we shall use the Picard approximations, some compactness results (see [1],[6], [9],[10]) and the following proposition ([5], P 7.7.3).

Proposition 1. *Let $p > 2, T > 0$ and let Φ be an L_2^0-valued, predictable process, such that $E(\int_0^T ||\Phi(s)||_{L_2^0}^p) < +\infty$. Then there exists a constant C_T such that*

$$E\left(\sup_{t\in[0,T]} \left|\left| \int_0^t S(t-s)\Phi(s)dW(s)\right|\right|^p\right) \leq C_T E\left(\int_0^T ||\Phi(s)||_{L_2^0}^p ds\right).$$

Moreover $W_A^\Phi(t) = \int_0^t S(t-s)\Phi(s)dW(s)$ has a continuous modification.

Remark 1.1 i) If A generates a contraction semigroup, then Proposition 1.1 is true for $p \geq 2$ (see [12]).

ii) A generalization of Proposition 1.1 for evolution systems can be found in [8].

2. Existence and uniqueness of solution

Let us fix a real number p, $p > 2$ and denote by B_T the space of all H-valued predictable processes $X(t,\omega)$ defined on $[0,T]\times\Omega$ which are continuous in t for a.e. fixed $\omega\in\Omega$ and for which

$$||X(\cdot,\cdot)||_{B_T} \stackrel{\text{def}}{=} \{E(\sup_{0\le t\le T}||X(t,\omega)||^p)\}^{\frac{1}{p}} < \infty.$$

The space B_T is a Banach space (see [1] for $p=2$, the case $p>2$ has a similar proof).

In the following we shall impose Taniguchi conditions on F and B (see [11]) that is:

(a1) The functions $F(t,\omega,x)$ and $B(t,\omega,x)$ are continuous in x for each fixed $(t,\omega)\in\Omega_T$ and there exists a function $H:[0,\infty)\times[0,\infty)\to[0,\infty)$, $(t,u)\to H(t,u)$ such that

$$E(||F(t,X)||^p) + E(||B(t,X)||^p_{L_2^0}) \le H(t, E(||X||)^p)$$

for all $t\in[0,T]$ and all $X\in L^p(\Omega,\mathcal{F},H)$.

(a2) $H(t,u)$ is locally integrable in t for each fixed $u\in[0,\infty)$, it is continuous and nondecreasing in u, for each fixed $t\in[0,T]$ and for all $\alpha>0$, $u_0\ge 0$ the integral equation $u(t)=u_0+\alpha\int_0^t H(s,u(s))$ has a global solution on $[0,T]$.

(a3) There exists a function $K:[0,\infty)\times[0,\infty)\to[0,\infty)$ which is locally integrable in t for each fixed $u\in[0,\infty)$ and continuous, monotone nondecreasing in u, for each fixed $t\in[0,\infty)$. Moreover, $K(t,0)\equiv 0$ and

$$E(||F(t,X)-F(t,Y)||^p)+E(||B(t,X)-B(t,Y)||^p_{L_2^0}) \le K(t,E(||X-Y||^p)$$

for all $t\in[0,T]$ and $X,Y\in L^p(\Omega,\mathcal{F},H)$.

(a4) If a non-negative, continuous function z satisfies

$$\begin{cases} z(t) & \le & \alpha\int_{t_0}^t K(s,z(s))ds, \quad t\in[0,T] \\ z(0) & = & 0 \end{cases}$$

for some $\alpha>0$, then $z(t)=0$ for all $t\in[0,T]$.

In the following we shall consider the Picard type approximations to (1):

$$\begin{cases} X_0(t) & = & S(t)\xi \\ X_{n+1}(t) & = & S(t)\xi+\int_0^t S(t-s)F(s,X_n(s))ds+ \\ & & \int_0^t S(t-s)B(s,X_n(s))dW(s), \quad t\in[0,T],\ n\ge 0. \end{cases}$$

The main result of this paper is the following.

Theorem 2.1 *Under the conditions (a1) to (a4), assume that* $\xi \in L^p(\Omega, \mathcal{F}_0, P)$. *Then the sequence* $\{X_n\}_{n\geq 0}$ *converge in* B_T *to the unique solution of (1) in* B_T.

For the proof of the theorem we shall state some lemmas.

Let $\mathcal{M}[0,T]$ denote the partially ordered linear space of real monotone non-decreasing function defined on $[0,T]$ and let us consider the following function

$$\Psi : B_T \to \mathcal{M}[0,T], \quad [\Psi(O)](t) = \chi_t[O_t]$$

where χ_t is the Hausdorff measure of noncompactness (see [1]) on the space B_t, O is a bounded subset of B_T and $O_t = \{x_{|[0,t]} : x \in O\} \subset B_t$.

Lemma 2.1. *Under the conditions (a1) and (a2) the operator* $G : B_T \to B_T$

$$GX(t) = S(t)\xi + \int_0^t S(t-s)F(s,X(s))ds + \int_0^t S(t-s)B(s,X(s))dW(s),$$

$t \in [0,T]$ *is well defined and continuous.*

Proof. If $X \in B_T$ then $E(||X(s)||^p) \leq E(\sup_{0\leq t\leq T} ||X(s)||^p) = ||X||^p_{B_T}$. We have

$$E(\sup_{0\leq t\leq T} ||GX(t)||^p) \leq 3^p E(\sup_{t\in[0,T]} ||S(t)\xi||^p) +$$

$$+3^p E(\sup_{t\in[0,T]} || \int_0^t S(t-s)F(s,X(s))ds||^p)$$

$$+3^p E(\sup_{t\in[0,T]} || \int_0^t S(t-s)B(s,X(s))dW(s)||^p)$$

$$\leq 3^p M^p E(||\xi||^p) + 3^p M^p T^{p-1} \int_0^T E(||F(s,X(s))||^p)ds$$

$$+3^p C_T \int_0^T E(||B(s,X(s))||^p_{L_2^0})ds$$

$$\leq 3^p M^p E(||\xi||^p) + C_T^{'} \int_0^T H(s, E(||X||^p_{B_T}))ds < \infty.$$

We have denoted $M = \sup_{t\in[0,T]} ||S(t)||_{L(H)}$, $C_T^{'} = 3^p M^p T^{p-1} + 3^p C_T$ and we applied the Hölder inequality for the first integral and Proposition 1.1 for the second integral.

The continuity of the operator G follows easily. In fact, for $X, X_1, \cdots$ in B_T we have

$$||GX - GX_n||^p_{B_T} = E(|| \sup_{t\in[0,T]} ||GX(t) - GX_n(t)||^p)$$

$$\leq 2^p M^p T^{p-1} \int_0^T E(||F(s, X(s)) - F(s, X_n(s))||^p) ds$$

$$+2^p C_T \int_0^T E(||B(s, X(s)) - B(s, X_N(s))||^p_{L_2^0}) ds$$

$$\leq C_T' \int_0^T K(s, E(||X(s) - X_n(s)||^p)) ds \leq C_T' \int_0^T K(s, ||X - X_n||^p_{B_T}) ds$$

from which we get $||GX - GX_n||^p_{B_T} \to 0$ as $||X - X_n||_{B_T} \to 0$. □

Lemma 2.2. *Under the condition (a1) to (a4), there exists $C_T'' > 0$ such that, if O is a bounded subset of the space B_T and $\Psi(O) \leq \Psi(GO)$, where G is defined above, then*

$$[\Psi(O)(t)]^p \leq C_T'' \int_0^t K(s, [(\Psi(O)(s)]^p ds$$

for each $t \in [0, T]$.

For the proof see [3] Theorem 2.3, which is an extension to infinite dimensional case of a similar results from [1].

Lemma 2.3. *Under the conditions (a1) and (a2) the sequence $\{X_n\}_{n \geq 0}$ is bounded.*

Proof. For $n \geq 0$, by the same argument as in Lemma 2.1, we have

(2) $$||X_{n+1}||^p_{B_t} \leq k_1 + k_2 \int_0^t H(s, ||X_n||^p_{B_s}) ds$$

where $k_1,\ k_2$ are some positive constants independent of n. Let $u(t),\ t \in [0, T]$, be a global solution of the equation

$$u(t) = u_0 + k_2 \int_0^t H(s, u(s)) ds,\ t \in [0, T]$$

with the initial condition $u_0 > \max(k_1, M^p E(||\xi||^p))$. We shall prove by mathematical induction that

(3) $$||X_n(t)||^p_{B_t} \leq u(t),\ \text{for } t \in [0, T].$$

For $n = 0$ the inequality (3) holds by the definition of u. Let us suppose that

$$||X_n(t)||^p_{B_t} \leq u(t),\ \text{for } t \in [0, T].$$

Then by (2) we obtain that

$$u(t) - ||X_{n+1}||^p_{B_t} \geq k_2 \int_0^t (H(s, u(s)) - H(s, ||X_n||^p_{B_s})) ds \geq 0.$$

The inequalities follow from the assumption of the mathematical induction and (a2). □

Lemma 2.4. *Under the conditions (a1) to (a4) the sequence $\{X_n\}_{n\geq 0}$ satisfies*

$$||X_{n+1} - X_n||_{B_T} \overset{n\to\infty}{\to} 0.$$

Proof. Let

$$r_n(t) = \sup_{m\geq n} (||X_{m+1} - X_m||^p_{B_t}), \ t \in [0,T] \ n \geq 0.$$

By Lemma 2.5 these functions are uniformly bounded and, evidently, monotone nondecreasing. Since $\{r_n(t)\}_{n\geq 0}$ is a monotone nonincreasing sequence for each $t \in [0,T]$, then there exists a monotone nondecreasing function r such that

$$\lim_{n\to\infty} r_n(t) = r(t). \tag{4}$$

By an argument similar to that in Lemma 2.1, we find

$$||X_{m+1} - X_m||^p_{B_t} \leq k \int_0^t K(s, ||X_m - X_{m-1}||^p_{B_s})ds$$

for some positive constant k, from which it follows that

$$r(t) \leq r_n(t) \leq k \int_0^t K(s, r_{n-1}(s))ds.$$

Taking into account (4) and the Lebesgue convergence theorem, we obtain

$$r(t) \leq k \int_0^t K(s, r(s))ds.$$

Now from (a4) follows $r \equiv 0$. But $||X_{n+1} - X_n||^p_{B_T} \leq r_n(T)$ and therefore $||X_{n+1} - X_n||_{B_T} \overset{n\to\infty}{\to} 0$, which completes the proof of Lemma. □

Lemma 2.5. *Equation (1) has at most one solution in B_T.*

Proof. If $X, Y \in B_{T'}$ were two fixed points of G, then we would have

$$E(\sup_{0\leq s\leq t} ||X(s) - Y(s)||^p) \leq 2^p M^p t^{p-1} E(\int_0^t ||F(s, X(s)) - F(s, Y(s))||^p ds)$$

$$+2^p C_T E(\int_0^t ||B(s, X(s)) - B(s, Y(s))||^p_{L_2^0} ds)$$

$$\leq (2^p M^p t^{p-1} + 2^p C_T) \int_0^t K(s, E(||X(s) - Y(s)||^p))ds.$$

Therefore

$$||X - Y||^p_{B_t} \leq (2^p M^p T^{p-1} + 2^p C_T) \int_0^t K(s, ||X - Y||^p_{B_t})ds.$$

From the condition (a4) it follows that $||X - Y||^p_{B_T} \equiv 0$, that is $X \equiv Y$. □

Proof of Theorem 2.1. Put $V = \{X_n : n = 0, 1, \cdots\}$. It is clear from Lemma 2.3 and Lemma 2.4 that the sets V and GV are bounded in B_T and

$$\Psi(V) = \Psi(GV).$$

From Lemma 2.2 it results that

$$[\Psi(V)(t)]^p \leq C''_T \int_0^t K(s, [(\Psi(V)(s)]^p ds$$

for all $t \in [0, T]$. Using (a4) we deduce that $\Psi(V) \equiv 0$. In fact $\Psi(V)$ may not be continuous but even in this case we have $\Psi(V) \equiv 0$ (see [3], Lemma 2.2). Consequently, V is relatively compact in B_T. Then there exists $v \in B_T$ and a subsequence k_n such that $u_{k_n} \overset{B_T}{\to} v$. From the continuity of G and Lemma 2.4 we deduce that $G(v) = v$. That is v is a solution for (1).

The uniqueness of solution follows from Lemma 2.5 □

Remark 2.1 For the existence of mild solutions to equation (1) in the conditions (a1) to (a4), the assumption $E(|\xi|^p) < \infty$ can be removed. Indeed it can be shown that if ξ and η are two initial conditions satisfying $E(|\xi|^p) < \infty$, $E(|\eta|^p) < \infty$ and $X, Y \in B_T$ are the corresponding solutions of equation (1) then

$$I_\Gamma X = I_\Gamma Y \quad P \text{ a.s.}$$

where $\Gamma = \{\omega \in \Omega : \xi(\omega) = \eta(\omega)\}$. The argument is the same as in [5], Th. 7.7.4. Now if $E(|\xi|^p) = \infty$ then we define, for $n = 1, 2, \cdots$

$$\xi_n = \begin{cases} \xi, & \text{if } |\xi| \leq n \\ 0, & \text{if } |\xi| > n \end{cases}$$

and denote by $X_n \in B_T$ the corresponding solution of (2). By the previous argument we have

$$X_n(t) = X_{n+1}(t) \ \text{ on } \{\omega \in \Omega : |\xi| \leq n\}.$$

Therefore, the process

$$X(t) = \lim_{n \to \infty} X_n(t)$$

is P-a.s. well defined and satisfies equation (1).

The following corollary is an immediate consequence of our Theorem 2.1 and Remark 2.1 from [3].

Corollary 2.1. *For the stochastic differential equation (1), suppose that the following conditions are satisfied:*

(i) $||F(t,x) - F(t,y)||^p + ||B(t,x) - B(t,y)||^p_{L_2^0} \le \lambda(t)\alpha(||X - Y||^p)$

(ii) $E(||F(t,0)||)$, $E(||B(t,0)||_{L_2^0}) \in L^p_{loc}([0,\infty), R^+)$
for all $t \in [0,\infty)$ *and* $x, y \in H$, *where* $\lambda(t) \ge 0$ *is locally integrable and* $\alpha : R_+ \to R_+$ *is a continuous, monotone nondecreasing and concave function with* $\alpha(0) = 0$ *and* $\int_{0+} \frac{1}{\alpha(u)} du = \infty$.

Let $E(||\xi||^p) < \infty$. *Then, on any finite interval* $[0,T]$ *equation* (1) *has a unique solution which can be found by the Picard approximations given in Theorems 2.1.*

Remark 2.2 i) If $\lambda(t) \equiv L$ $(L > 0)$ and $\alpha(u) = u$, $u \ge 0$ condition (a3) implies a global Lipschitz condition.

ii) Another example is: $\alpha(u) = u\ln(\frac{1}{u})$ for $0 < u < u_0$ (u_0 sufficiently small), $\alpha(0) = 0$ and $\alpha(u) = (au + b)$ for $u \ge u_0$, where $au + b$ is the tangent line of the function $u\ln(\frac{1}{u})$ at point u_0.

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