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A NOTE ON ULTRAPOLYNOMIALS AND THE WIGNER DISTRIBUTION

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Abstract. In this short note, following Komatsu's approach to ultradistributions, we give new estimates on a class of ultrapolynomials. Our second result is an estimate on the growth of the Wigner Distribution on a space of Gelfand-Shilov type. The obtained results are important for theorems on boundedness of pseudodifferential operators on ultramodulation spaces quoted in the paper.

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1. Motivation

Our approach to the study of pseudodifferential operators on spaces of ultradistributions starts with modulation spaces introduced by H. G. Feichtinger [9] and studied by many authors cf. [34], [10], [15], [11], [17], [27]. A class of modulation spaces includes a number of classical spaces of functions and distributions, e.g. L^2, certain Sobolev spaces, the Bessel potential spaces, Feichtinger's algebra S_0. It is also proved that the space of rapidly decreasing functions, test function space for the space of tempered distributions, is projective limit of a class of modulation spaces [12]. Using this fact, K. Tachizawa [35], showed boundedness of a class of pseudodifferential operators on modulation spaces.

In most of applications, modulation spaces are defined via polynomial weights. Since the test function spaces for the spaces of tempered ultradistributions consist of ultradifferential functions of ultrapolynomial growth, they can not be expressed via such modulation spaces. To overcome this problem, in [31] are introduced ultramodulation spaces. Projective limit of a class of ultramodulation spaces is a test function space for a space of tempered ultradistributions of Beurling type (see [31]).

Boundedness of a class of pseudodifferential operators on ultramodulation spaces has been proved in [32]. In the proof, unlike Tachizawa's approach, it is necessary to apply ultradifferential operators and a new estimate on the Wigner distribution. This was a basic motivation for our study.

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The paper is organized as follows. In Section 2 we give definitions of the space $\mathcal{S}^{(\gamma)}$ and ultrapolynomials, and prove Theorem 2, which is a modification of Komatsu's more general result. We obtain sharp estimates by carring out explicit calculations. In Section 3 we define the Wigner distribution and prove Theorem 3. Finally, in Section 4, as an application of obtained results, we state, without the proof, a theorem on boundedness of pseudodifferential operators on ultramodulation spaces. Another consequence of our arguments is that some ultradifferential operators belong to the observed class of pseudodifferential operators.

2. Ultradistributions and ultrapolynomials

The spaces of ultradistributions were introduced in [3], [33], [4] as a natural generalization of some well known spaces of distributions. Since then they have been studied by many authors cf. [22]- [24], [18], [8], [5], [6]. Spaces of tempered ultradistributions, a generalization of the space of tempered distributions, are studied in [18], [30], [26], [7], [21], [31].

Following Komatsu's approach [22], test function spaces for the spaces of ultradistributions consist of ultradifferential functions introduced via the sequences $(M_p)_{p\in\mathbf{N}_0}$ of positive numbers satisfying some convexity, stability and non-quasi-analyticity conditions $(M.1)-(M.3)$, see [22, page 26]. Instead of a general (M_p) sequence, we observe the special case $M_p = p!^{1/\gamma}$, $\gamma \in (0,1)$. It satisfies all the conditions $(M.1)-(M.3)$.

We now define the space $\mathcal{S}'^{(\gamma)}$, a space of tempered ultradistributions of Beurling type.

Definition 1. *Let there be given $\gamma \in (0,1)$. Then $\mathcal{S}_s^\gamma$, $s \geq 0$, is the space of smooth functions $f \in C^\infty(\mathbf{R})$ such that*

$$\sup_{x\in\mathbf{R}}\sup_{\alpha,\beta\in\mathbf{N}_0} \frac{s^{\alpha+\beta}|x^\alpha f^{(\beta)}(x)|}{\alpha!^{1/\gamma}\beta!^{1/\gamma}} < \infty,$$

and $\mathcal{S}^{(\gamma)} = \operatorname{proj}\lim_{s\to\infty} \mathcal{S}_s^\gamma$.

The space $\mathcal{S}_s^\gamma$, $s \geq 0$, $\gamma \in (0,1)$, is a Banach space with the norm

$$\sigma_{\gamma,\infty}(f) = \sup_{\alpha,\beta\in\mathbf{N}_0} \frac{s^{\alpha+\beta}}{\alpha!^{1/\gamma}\beta!^{1/\gamma}} \|x^\alpha f^{(\beta)}(x)\|_\infty, \quad f \in \mathcal{S}_s^\gamma,$$

and its projective limit $\mathcal{S}^{(\gamma)}$ is therefore a Frechet space with the topology defined by a countable family of norms. Moreover, it is a (FN) space, i.e., locally convex topological vector space which is a projective limit of countable compact nuclear specter of spaces. The space $\mathcal{S}^{(\gamma)}$ is a Gelfand-Shilov type space

[14], and its dual $\mathcal{S}'^{(\gamma)}$ is a space of tempered ultradistributions of Beurling type, which is also (FN) space. The following inclusions are dense and continuous:

$$\mathcal{S}^{(\gamma)} \hookrightarrow \mathcal{S}, \quad \mathcal{S}' \hookrightarrow \mathcal{S}'^{(\gamma)}.$$

Moreover, the Fourier transform is an isomorphism of $\mathcal{S}^{(\gamma)}$ into itself. The following theorem is proved in [21], [7], [31].

Theorem 1. *Let there be given $\gamma \in (0,1)$. The following conditions are equivalent:*

a) $f \in \mathcal{S}^{(\gamma)}$;

b) $\sup_x |f(x)| e^{s|x|^\gamma} < \infty$ *and* $\sup_\xi |\hat{f}(\xi)| e^{s|\xi|^\gamma} < \infty$, *for every* $s \geq 0$;

c) $\int_{\mathbf{R}} |\hat{f}(t)|^2 e^{2s|t|^\gamma} dt < \infty$ *and* $\int_{\mathbf{R}} |f(x)|^2 e^{2s|x|^\gamma} dx < \infty$, *for every* $s \geq 0$.

Ultradifferential operators, a generalization of partial differential operators of finite order, play an essential role in the ultradistribution theory. They can be defined by means of certain entire functions called ultrapolynomials. In such way the properties of ultrapolynomials can be transferred into the properties of the corresponding ultradifferential operators. In this paper we restate Komatsu's results on ultrapolynomials [22, Propositions 4.4-4.6], and calculate explicitly some constants appearing in the original proofs.

Let there be given $L > 0$, and let $m_p = \dfrac{M_p}{M_{p-1}} = p^{1/\gamma}$. We observe the ultrapolynomial

$$P_L(\zeta) = \prod_{p=1}^{\infty} \left(1 + \frac{L^2\zeta^2}{p^{2/\gamma}}\right) = \prod_{p=1}^{\infty} \left(1 + \frac{iL\zeta}{p^{1/\gamma}}\right) \cdot \left(1 - \frac{iL\zeta}{p^{1/\gamma}}\right), \quad \zeta \in \mathbf{C}.$$

The associated function for a general $(M_p)_{p\in\mathbf{N}_0}$ sequence is $M(\rho) = \sup_p \ln \dfrac{\rho^p}{M_p}$, $\rho > 0$. If $m(\lambda)$ is the number of p such that $p^{1/\gamma} \leq \lambda$ then $M(\rho) = \displaystyle\int_0^\rho \frac{m(\lambda)}{\lambda} d\lambda$, $\rho > 0$. The non–quasi–analyticity condition (M.3)' is equivalent to

$$\int_0^\infty \frac{m(\lambda)}{\lambda^2} d\lambda < \infty \quad \Longleftrightarrow \quad \int_0^\infty \frac{M(\rho)}{\rho^2} d\rho < \infty.$$

For $M_p = p!^{1/\gamma}$, $p \in \mathbf{N}_0$, we have $M(\rho) \sim \rho^\gamma$, $\rho \to \infty$, see [29].
The following theorem is the main result of this section.

Theorem 2. *Let there be given the sequence $(M_p)_{p\in\mathbf{N}_0}$, such that $M_p = p!^{1/\gamma}$, $p \in \mathbf{N}_0$, $\gamma \in (0,1)$, and let $M(\rho)$ be its associated function. Let*

$$P_L(\zeta) = \prod_{p=1}^{\infty}\left(1+\frac{L^2\zeta^2}{p^{2/\gamma}}\right) = \sum_{p=0}^{\infty} a_p\zeta^p, \quad \zeta \in \mathbf{C}, \quad \Re\zeta \geq 0.$$

Then we have

a) $e^{2M(L|\zeta|)} \leq |P_L(\zeta)|$, *and, for a sufficiently large* $\rho \in \mathbf{R}$, $\exists C > 0$ *such that*

$$|P_L(\zeta)| \leq Ce^{\frac{4}{1-\gamma^2}M(L|\zeta|)}, \quad |\zeta| = \rho;$$

b) $(\exists C > 0)\ (\exists L > 0)\ (|a_p| \leq C\frac{L^p}{p!^{1/\gamma}},\ p \in \mathbf{N}_0) \Rightarrow$

$$(\exists s > 1)(\exists C_s > 0)(|P_L(\zeta)| \leq C_s e^{M(sL|\zeta|)}, \zeta \in \mathbf{C});$$

c) $(\exists C > 0)\ (\exists L > 0)\ (|P_L(\zeta)| \leq Ce^{M(L|\zeta|)},\ \zeta \in \mathbf{C}) \Rightarrow$

$$|a_p| \leq C\frac{L^p}{p!^{1/\gamma}}, p \in \mathbf{N}_0.$$

Proof. a) Let $\zeta \in \mathbf{C}$. Then

$$\begin{aligned} |P_L(\zeta)| &\geq \sup_p \prod_{q=1}^{p}\left|1+\frac{L^2\zeta^2}{q^{2/\gamma}}\right| \geq \sup_p \prod_{q=1}^{p}\left|\frac{L^2\zeta^2}{q^{2/\gamma}}\right| \\ &\geq \left(\sup_p \prod_{q=1}^{p}\frac{L|\zeta|}{q^{1/\gamma}}\right)^2 = \left(\sup_p \frac{L^p|\zeta|^p}{p!^{1/\gamma}}\right)^2 = e^{2M(L|\zeta|)}, \quad \Re\zeta \geq 0. \end{aligned}$$

Let $n(\lambda) = 2\{p;\ \ p^{1/\gamma}/L \leq \lambda\}$ be the number of zeroes of P_L less than or equal to λ, $\lambda > 0$ and $\tilde{n}(\lambda) = n(\lambda)/2$. For $\tilde{n}_p = p^{1/\gamma}/L$ or, equivalently, $\tilde{N}_p = p!^{1/\gamma}/L$, we have $\tilde{N}(\rho) = \sup_p \ln\frac{\rho^p}{\tilde{N}_p} = M(L\rho)$, $\rho > 0$. Hence

$$N(\rho) = \int_0^{\rho}\frac{n(\lambda)}{\lambda}d\lambda = 2\int_0^{\rho}\frac{\tilde{n}(\lambda)}{\lambda}d\lambda = 2\tilde{N}(\rho) = 2M(L\rho), \ \rho > 0.$$

Since $1 + (\rho/\tilde{n}_p)^2 \leq (1+\rho/\tilde{n}_p)^2$, we have

$$\ln\sup_{|\zeta|=\rho}|P_L(\zeta)| \leq 2\sum_{p=0}^{\infty}\ln\left(1+\frac{L\rho}{p^{1/\gamma}}\right) = 2\int_0^{\infty}\ln\left(1+\frac{\rho}{\lambda}\right)d\tilde{n}(\lambda).$$

Applying two times integration by parts we obtain

$$\begin{aligned}\ln \sup_{|\zeta|=\rho} |P_L(\zeta)| &\leq 2 \lim_{\sigma\to\infty} \left(\ln\left(1+\frac{\rho}{\sigma}\right)\tilde{n}(\sigma) + \frac{\rho\tilde{N}(\sigma)}{\sigma+\rho} + \int_0^\sigma \frac{\rho\tilde{N}(\lambda)}{(\lambda+\rho)^2}d\lambda \right) \\ &\leq 2\int_0^\infty \frac{\rho M(L\lambda)}{(\lambda+\rho)^2}d\lambda \leq 2\left(\frac{1}{L\rho}\int_0^{L\rho} M(\lambda)d\lambda + L\rho\int_{L\rho}^\infty \frac{M(\lambda)}{\lambda^2}d\lambda\right).\end{aligned}$$

Since $M(\lambda)\sim\lambda^\gamma$, $\lambda\to\infty$, calculating both integrals for $\rho\in\mathbf{R}$ large enough, we obtain

$$\begin{aligned}\ln \sup_{|\zeta|=\rho} |P_L(\zeta)| &\leq 2\left(\frac{1}{1+\gamma}(L\rho)^\gamma + \frac{1}{1-\gamma}(L\rho)^\gamma + C_1\right) \\ &= \frac{4}{1-\gamma^2}(L\rho)^\gamma + 2C_1,\end{aligned}$$

$C_1>0$, i.e.,

$$|P_L(\zeta)| \leq e^{\frac{4}{1-\gamma^2}(L|\zeta|)^\gamma + 2C_1} \leq Ce^{\frac{4}{1-\gamma^2}M(L|\zeta|)}, \quad |\zeta| = \rho.$$

b)

$$\begin{aligned}|P_L(\zeta)| &\leq \sum_{p=0}^\infty |a_p||\zeta^p| \leq C\sum_{p=0}^\infty \frac{L^p|\zeta|^p}{M_p} \\ &\leq C\sum_{p=0}^\infty \frac{1}{s^p}\sup_p \frac{(sL|\zeta|)^p}{M_p} \leq C_s e^{M(sL|\zeta|)}, \zeta\in\mathbf{C}.\end{aligned}$$

c) Let $p\in\mathbf{N}_0$. By the Cauchy integral formula we obtain

$$|a_p| = \frac{1}{2\pi}\int_{|\zeta|=\rho} \frac{P_L(\zeta)}{\zeta^{p+1}}d\zeta \leq \inf_{0<\rho<\infty} \frac{Ce^{M(L\rho)}}{\rho^p} \leq CL^p \inf_{0<\rho<\infty}\frac{e^{M(L|\zeta|)}}{(L\rho)^p} = C\frac{L^p}{p!^{1/\gamma}},$$

where Proposition 3.2. [22] has been used. □

We say that $P(x,D) = \sum_{|\alpha|=0}^\infty a_\alpha(x)D^\alpha$ is an ultradifferential operator of class (γ) on an open set $\Omega\subset\mathbf{R}^d$ if $\forall K\subset\Omega$, (K is a compact set)

$$(\exists L)(\forall k)(\exists C)\ \sup_{x\in K} |D^\beta a_\alpha(x)| \leq Ck^{|\beta|}|\beta|!^{1/\gamma}\frac{L^{|\alpha|}}{|\alpha|!^{1/\gamma}} \quad \alpha,\beta\in\mathbf{N}_0^d.$$

Theorem 2 gives the necessary and sufficient conditions for the differential operator of infinite order

$$P(D) = \sum_{|\alpha|=0}^\infty a_\alpha D^\alpha$$

with constant coefficients to be an ultradifferential operator. Recall, the Fourier transform of an integrable function f is defined by

$$\mathcal{F}f(\xi)=\hat{f}(\xi)=\int_{\mathbf{R}^d} e^{-2\pi i x\xi} f(x)dx, \ \ \xi\in\mathbf{R}^d.$$

For a rapidly decreasing function f we have

$$\mathcal{F}(P(D)f)(\xi)=P(\xi)\mathcal{F}f(\xi)=\sum_{|\alpha|=0}^{\infty} a_\alpha \xi^\alpha \hat{f}(\xi).$$

It follows that

(1) $$P_L(D)=\prod_{p=1}^{\infty}\left(1+\frac{L^2D^2}{p^{2/\gamma}}\right)$$

is an ultradifferential operator of class (γ).

3. The Wigner distribution

The Wigner distribution was introduced in [38] as a new concept in quantum mechanics. It is now an important tool in many fields of applied mathematics, e.g., signal analysis [37], [28], radar analysis [1], optics [2], stochastic process [19]. In [39] it is treated as a symbol of certain pseudodifferential operators. The Wigner distribution on Gelfand-Shilov type spaces is studied in [20] and [21]. In the following theorem we give a new estimate of the Wigner distribution acting on the elements of the space $\mathcal{S}^{(\gamma)}$.

We introduce the Wigner distribution as the Fourier transform of the Fourier–Wigner transform. Recall, the Fourier–Wigner transform $A(f,g)$, also known as ambiguity function, is defined by

$$A(f,g)(x,\xi)=\int_{\mathbf{R}} e^{-2\pi i\xi t} f(t+\frac{x}{2})\overline{g(t-\frac{x}{2})}dt, \ \ f,g\in L^2(\mathbf{R}).$$

For its applications in harmonic analysis and the analysis of radar signals we refer the reader to [1], [13], [39], [17], [16]. The Fourier transform of the function $A(f,g)$ is the Wigner distribution $W(f,g)$ given by

$$W(f,g)(x,\xi)=\int_{\mathbf{R}} e^{-2\pi i\xi t} f(x+\frac{t}{2})\overline{g(x-\frac{t}{2})}dt, \ \ f,g\in L^2(\mathbf{R}).$$

Theorem 3. *a) The Wigner distribution and the Fourier–Wigner transform are mappings from $\mathcal{S}^{(\gamma)}(\mathbf{R}^d)$ into $\mathcal{S}^{(\gamma)}(\mathbf{R}^d\times\mathbf{R}^d)$.*

b) Let $\varphi\in\mathcal{S}^{(\gamma)}(\mathbf{R}^d)$, and let $W(\varphi,\varphi)$ be its Wigner distribution. Then we have

(2) $$\sup_{\alpha,\beta\in\mathbf{N}_0^d}\sup_{x,\xi\in\mathbf{R}^d}\frac{s^{\alpha+\beta}}{\alpha!^{1/\gamma}\beta!^{1/\gamma}}|D_x^\alpha D_\xi^\beta W(\varphi,\varphi)(x,\xi)e^{s(|x|^\gamma+|\xi|^\gamma)}|<\infty$$

for every $s\geq 0$. The same holds for $A(\varphi,\varphi)$.

Proof. According [21, Theorem 2.1], $\varphi \in \mathcal{S}^{(\gamma)}$ if and only if

$$W(\varphi,\varphi)(x,\xi) = \mathcal{O}(e^{-\lambda(|x|^\gamma+|\xi|^\gamma)})$$

holds for every $\lambda > 0$. The same estimate holds for $A(\varphi,\varphi)$, since the Fourier–Wigner transform and the Wigner distribution are the Fourier transforms of one another. Actually,

$$|W(f,g)(x,\xi)| = 2|A(f,\check{g})(2x,2\xi)|,\ \forall f,g \in L^2,$$

where $\check{g}(x) = g(-x)$ ([16, Lemma 4.3.1], [13, page 56]).

Then, by Theorem 1 *b)* we have

$$W(\varphi,\varphi)(\cdot,\cdot) \in \mathcal{S}^{(\gamma)}(\mathbf{R}^d \times \mathbf{R}^d)$$

as well as

$$A(\varphi,\varphi)(\cdot,\cdot) \in \mathcal{S}^{(\gamma)}(\mathbf{R}^d \times \mathbf{R}^d).$$

b) In [21, Theorem 1.2] equivalence between various norms in $\mathcal{S}^{(\gamma)}$ is proved. Particularly, $F \in \mathcal{S}^{(\gamma)}$ if and only if

$$\sup_{\alpha\in\mathbf{N}_0^d} \frac{q^\alpha}{\alpha!^{1/\gamma}} \|e^{q|\cdot|^\gamma} D^\alpha F\|_\infty < \infty,$$

for all $q \geq 0$. This, together with the part *a)*, implies (2). □

4. A class of pseudodifferential operators

For the given $L_1, L_2 \geq 0$, $\lambda,\tau \in \mathbf{R}$, and $\gamma \in (0,1)$, we define the symbol class

$$S^{\lambda,\tau,\gamma}_{L_1,L_2}(\mathbf{R}^{2d}) = \{\sigma \in C^\infty(\mathbf{R}^{2d}) \mid |\frac{L_1^\alpha}{\alpha!^{1/\gamma}}\frac{L_2^\beta}{\beta!^{1/\gamma}} D_x^\alpha D_\xi^\beta \sigma(x,\xi)| \leq Ce^{\lambda|x|^\gamma+\tau|\xi|^\gamma}\},$$

where the above inequality holds for every $\alpha,\beta \in \mathbf{N}_0^d$, $x,\xi \in \mathbf{R}^d$ and some positive constant C. The infimum of such constants will be denoted by $\|\sigma\|_{L_1,L_2}$, that is

$$\|\sigma\|_{L_1,L_2} = \sup_{\alpha,\beta\in\mathbf{N}_0} \left| \frac{L_1^\alpha}{\alpha!^{1/\gamma}}\frac{L_2^\beta}{\beta!^{1/\gamma}} D_x^\alpha D_\xi^\beta \sigma(x,\xi) e^{-\lambda|x|^\gamma-\tau|\xi|^\gamma}\right|.$$

We associate the pseudodifferential operator $\sigma(x,D)$ to the symbol $\sigma(x,\xi) \in S^{\lambda,\tau,\gamma}_{L_1,L_2}(\mathbf{R}^{2d})$ through the Weyl correspondence

$$\sigma(x,D)f(x) = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \sigma(\frac{x+y}{2},\xi) e^{2\pi i(x-y)\xi} f(y)dyd\xi,\ f \in \mathcal{S}^{(\gamma)}(\mathbf{R}^d).$$

It is easy to show that the ultradifferential operator given by (1) is a pseudodifferential operator. This is a consequence of Theorem 2 and the arguments given in the proof of Theorem 3.

For the next theorem we need the notion of ultramodulation space.

Definition 2. *[31] Let there be given $\gamma \in (0,1)$. A strictly positive and continuous function $w_{s,exp}$ on $\mathbf{R}^d \times \mathbf{R}^d$ is called an exp–type weight if there exist $s \geq 0, C > 0$ such that*

$$w_{s,exp}(x+y,\xi+\eta) \leq Ce^{s(|x|^\gamma+|\xi|^\gamma)} w_{s,exp}(y,\eta)$$

for all $x, y, \xi, \eta \in \mathbf{R}^d$, and if for every $\varepsilon \in \{-1,1\}^d$ we have

$$w_{s,exp}(x,\xi(\varepsilon)) = w_{s,exp}(x,\xi),$$

where $\xi(\varepsilon) = (\varepsilon_1 \cdot \xi_1, \ldots, \varepsilon_d \cdot \xi_d)$ for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)$.

(See also [36, page 512].) A typical example of exp–type weight is

$$w_{s_0,exp}(x,\xi) = e^{s_1|x|^\gamma + s_2|\xi|^\gamma}, \quad x, \xi \in \mathbf{R}^d, \ s_1, s_2 \geq 0,$$

where $s_0 = \max\{s_1, s_2\}$.

Let there be given $\lambda, \tau \in \mathbf{R}$, $\gamma \in (0,1)$ an exp–type weight $w_{s,exp}(x,\xi)$, and let

$$\tilde{w}_{s_0,exp}(x,\xi) = w_{s,exp}(x,\xi)e^{-\lambda|x|^\gamma - \tau|\xi|^\gamma}.$$

The function $\tilde{w}_{s_0,exp}$ is also exp–type weight, with $s_0 = \max\{s + |\lambda|, s + |\tau|\}$.

Definition 3. *[31] Let there be given $g \in \mathcal{S}$, $g \not\equiv 0$, $s \geq 0$ and an exp-type weight $w_{s,exp}(x,\xi)$. The Banach space*

$$M_{p,q}^{w_{s,exp}} = \{f \in \mathcal{S}' : \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |\langle \overline{T_x M_y g}, f \rangle|^p \, w_{s,exp}^p(x,\xi) dx \right)^{q/p} dy < \infty \},$$

$0 \not\equiv g \in \mathcal{S}^{(m)}$, with the norm

$$\|f\|_{M_{p,q}^{w_{s,exp}}} = \left[\int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |\langle \overline{T_x M_y g}, f \rangle|^p \, w_{s,exp}^p(x,\xi) dx \right)^{q/p} dy \right]^{1/q}$$

is called ultramodulation space, where $T_x f(\cdot) = f(\cdot - x)$ and $M_y f(\cdot) = e^{2\pi i y \cdot} f(\cdot)$.

Theorem 4. *[32] Let there be given exp–type weight $w_{s,exp}$ and the Weyl symbol $\sigma(x,\xi) \in S_{A,B}^{\lambda,\tau,\gamma}$, $\lambda, \tau \in \mathbf{R}$, $\gamma \in (0,1)$, where*

$$A = (\frac{4}{1-\gamma^2})^{1/\gamma} L_1 E, \quad B = (\frac{4}{1-\gamma^2})^{1/\gamma} L_2 F,$$

$$E > 2, \quad F > 2, \quad L_1 > \left(s + |\tau| + \frac{\tau}{2^\gamma}\right)^{1/\gamma}, \quad L_2 > \left(|\lambda| + \frac{s+\lambda}{2^\gamma}\right)^{1/\gamma},$$

and $L_1 \geq \frac{-\tau}{2^{\gamma+1/\gamma}}$ if $\tau < 0$.

The corresponding operator $\sigma(x, D)$ *is a bounded linear operator from* $M_{p,q}^{w_{s,exp}}$ *to* $M_{p,q}^{\tilde{w}_{s_0,exp}}$, $1 \leq p, q < \infty$, *where*

$$\tilde{w}_{s_0,exp}(x,\xi) = w_{s,exp}(x,\xi)e^{-2^{\gamma}\lambda|x|^{\gamma}-\tau|\xi|^{\gamma}},$$

i.e., there exists a positive constant C *such that*

$$\|\sigma(x,D)f\|_{M_{p,q}^{\tilde{w}_{s_0,exp}}} \leq C\|\sigma\|_{A,B}\|f\|_{M_{p,q}^{w_{s,exp}}}.$$

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