NOVI SAD J. MATH.
VOL. 30, NO. 1, 2000, 15-21

ON THE NON-COMMUTATIVE NEUTRIX PRODUCT OF x_+^λ AND $x_+^{-\lambda-1}$

Brian Fisher[1], **Fatma Al-Sirehy**[2]

Abstract. The non-commutative neutrix product of the distributions x_+^λ and $x_+^{-\lambda-1}$ is evaluated for $\lambda \neq 0, \pm1, \pm2 \dots$

AMS Mathematics Subject Classification (1991): 46F10

Key words and phrases: distribution, delta-function, neutrix, neutrix limit, neutrix product.

In the following, we let $\rho(x)$ be an infinitely differentiable function having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2].

Definition 1. *Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a, b), f is the k-th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is*

[1] Institute of Simulation Sciences, SERC, Hawthorn Building, De Montfort University, Leicester, LE1 9BH, England
and Department of Mathematics and Computer Science, Leicester University, Leicester, LE1 7RH, England. e-mail: fbr@le.ac.uk

[2] P.O. Box 32477, Jeddah, Saudi Arabia, e-mail: gcp3128@kaau.edu.sa

a locally summable function in $L^q(a,b)$ *with* $1/p + 1/q = 1$. *Then the product* $fg = gf$ *of* f *and* g *is defined on the interval* (a,b) *by*

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [Fg^{(i)}]^{(k-i)}.$$

The following definition for the neutrix product of two distributions, given in [4], generalizes Definition 1.

Definition 2. *Let* f *and* g *be distributions in* $\mathcal{D}'$ *and let* $g_n(x) = (g * \delta_n)(x)$. *We say that the neutrix product* $f \circ g$ *of* f *and* g *exists and is equal to the distribution* h *on the interval* (a,b) *if*

$$\mathop{\mathrm{N-lim}}_{n\to\infty} \langle f(x) g_n(x), \varphi(x)\rangle = \langle h(x), \varphi(x)\rangle$$

for all functions φ *in* $\mathcal{D}$ *with support contained in the interval* (a,b), *where* N *is the neutrix, see van der Corput [1], having domain* $N' = \{1, 2, \ldots, n, \ldots\}$ *and range the real numbers, with negligible functions finite linear sums of the functions*

$$n^\lambda \ln^{r-1} n, \quad \ln^r n: \qquad \lambda > 0, \quad r = 1, 2, \ldots$$

and all functions which converge to zero in the normal sense as n *tends to infinity.*

Note that if

$$\lim_{n\to\infty} \langle f(x) g_n(x), \varphi(x)\rangle = \langle h(x), \varphi(x)\rangle,$$

we simply say that the *product* $f.g$ exists and equals h, see [3].

It is obvious that if the product $f.g$ exists, then the neutrix product $f \circ g$ exists and the two are equal. Further, it was proved in [3] that if the product fg exists by Definition 1, then the product f.g exists by Definition 2 and the two are equal.

The following theorem was proved in [6].

Theorem 1. *Let* f *and* g *be distributions in* $\mathcal{D}'$ *and suppose that the neutrix products* $f \circ g^{(i)}$ *(or* $f^{(i)} \circ g$*) exist on the interval* (a,b) *for* $i = 0, 1, 2, \ldots, r$. *Then the neutrix product* $f^{(r)} \circ g$ *(or* $f \circ g^{(r)}$*) exists on the interval* (a,b) *and*

$$f^{(r)} \circ g = \sum_{i=0}^{r} \binom{r}{i} (-1)^i [f \circ g^{(i)}]^{(r-i)}$$

or

$$f \circ g^{(r)} = \sum_{i=0}^{r} \binom{r}{i} (-1)^i [f^{(i)} \circ g]^{(r-i)}$$

on the interval (a,b).

The next theorem was proved in [4].

Theorem 2. *The neutrix product* $x_+^{r-1/2} \circ x_+^{-r-1/2}$ *exists and*

$$x_+^{r-1/2} \circ x_+^{-r-1/2} = x_+^{-1} + a_r \delta(x) \tag{1}$$

for $r = 0, \pm 1, \pm 2, \ldots$, *where*

$$\begin{aligned} a_0 &= 2[\ln 2 - c(\rho)], \\ a_r = a_{-r} &= 2\Big[\ln 2 - c(\rho) - \sum_{i=1}^{r} \frac{1}{2i-1}\Big] \end{aligned}$$

for $r = 1, 2, \ldots$ *and*

$$c(\rho) = \int_0^1 \ln t \rho(t)\, dt.$$

Before proving our main result we need the following definition of the Beta function given in [5].

Definition 3. *The Beta function* $B(\lambda, \mu)$ *is defined for all* λ, μ *by*

$$B(\lambda, \mu) = \operatorname*{N-lim}_{n \to \infty} \int_{1/n}^{1-1/n} t^{\lambda-1}(1-t)^{\mu-1}\, dt.$$

It was proved that if $\lambda, \mu \neq 0, \pm 1, \pm 2, \ldots$, then the above definition is in agreement with the standard definition of the Beta function.

In particular, it was proved in [5] that

$$B(0, \mu) = -\gamma - \psi(\mu)$$

for $\mu \neq 0, \pm 1, \pm 2, \ldots$, where γ denotes Euler's constant and

$$\psi(\mu) = \frac{\Gamma'(\mu)}{\Gamma(\mu)}.$$

We now generalize Theorem 2.

Theorem 3. *The neutrix product* $x_+^\lambda \circ x_+^{-\lambda-1}$ *exists and*

$$x_+^\lambda \circ x_+^{-\lambda-1} = x_+^{-1} - [\gamma + \tfrac{1}{2}\psi(-\lambda) + \tfrac{1}{2}\psi(\lambda+1) + 2c(\rho)]\delta(x) \tag{2}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

Proof. We first of all suppose that $-1 < \lambda < 0$ and put

$$
\begin{aligned}
(x_+^{-\lambda-1})_n &= x_+^{-\lambda-1} * \delta_n(x) \\
&= \begin{cases} \displaystyle\int_{-1/n}^{1/n} (x-t)^{-\lambda-1}\delta_n(t)\,dt, & x > 1/n, \\ \displaystyle\int_{-1/n}^{x} (x-t)^{-\lambda-1}\delta_n(t)\,dt, & -1/n \le x \le 1/n, \\ 0, & x < -1/n. \end{cases}
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_{-1}^{1} x_+^{\lambda}(x_+^{-\lambda-1})_n\,dx &= \int_0^{1/n} x^{\lambda} \int_{-1/n}^{x} (x-t)^{-\lambda-1}\delta_n(t)\,dt\,dx + \\
&\quad + \int_{1/n}^{1} x^{\lambda} \int_{-1/n}^{1/n} (x-t)^{-\lambda-1}\delta_n(t)\,dt\,dx \\
&= \int_0^{1/n} \delta_n(t) \int_t^1 x^{\lambda}(x-t)^{-\lambda-1}\,dx\,dt + \\
&\quad + \int_{-1/n}^{0} \delta_n(t) \int_0^1 x^{\lambda}(x-t)^{-\lambda-1}\,dx\,dt \\
&= \int_0^1 \rho(v) \int_v^n u^{\lambda}(u-v)^{-\lambda-1}\,du\,dv + \\
&\quad + \int_0^1 \rho(v) \int_0^n u^{\lambda}(u+v)^{-\lambda-1}\,du\,dv, \qquad (3)
\end{aligned}
$$

where the substitutions $nt = v$ and $nx = u$ have been made in the first integral and $nt = -v$ and $nx = u$ in the second integral.

Making the substitution $u = v/y$, we have

$$
\begin{aligned}
\int_v^n u^{\lambda}(u-v)^{-\lambda-1}\,du &= \int_{v/n}^{1} y^{-1}(1-y)^{-\lambda-1}\,dy \\
&= \int_{v/n}^{1} y^{-1}[(1-y)^{-\lambda-1} - 1]\,dy - \ln v + \ln n
\end{aligned}
$$

and it follows that

$$
\begin{aligned}
\operatorname*{N-lim}_{n\to\infty} \int_v^n u^{\lambda}(u-v)^{-\lambda-1}\,du &= \int_0^1 y^{-1}[(1-y)^{-\lambda-1} - 1]\,dy - \ln v \\
&= B(0,-\lambda) - \ln v \\
&= -\gamma - \psi(-\lambda) - \ln v. \qquad (4)
\end{aligned}
$$

Further, making the substitution $u = v(y^{-1} - 1)$, we have

$$
\int_0^n u^{\lambda}(u+v)^{-\lambda-1}\,du = \int_{v/(n+v)}^{1} y^{-1}(1-y)^{\lambda}\,dy
$$

$$= \int_{v/(n+v)}^1 y^{-1}[(1-y)^\lambda - 1]\,dy - \ln v + \ln n$$

and it follows that

$$\begin{aligned} \operatorname*{N-lim}_{n\to\infty} \int_0^n u^\lambda (u+v)^{-\lambda-1}\,du &= \int_0^1 y^{-1}[(1-y)^\lambda - 1]\,dy - \ln v \\ &= B(0,\lambda+1) - \ln v \\ &= -\gamma - \psi(\lambda+1) - \ln v. \end{aligned} \tag{5}$$

It now follows from equations (3), (4) and (5) that

$$\operatorname*{N-lim}_{n\to\infty} \int_{-1}^1 x_+^\lambda (x_+^{-\lambda-1})_n\,dx = -\gamma - \tfrac{1}{2}\psi(-\lambda) - \tfrac{1}{2}\psi(\lambda+1) - 2c(\rho). \tag{6}$$

Now let φ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1,1]$. By the mean value theorem

$$\varphi(x) = \varphi(0) + x\varphi'(\xi x),$$

where $0 < \xi < 1$ and so

$$\begin{aligned} \langle x_+^\lambda (x_+^{-\lambda-1})_n, \varphi(x)\rangle &= \int_0^1 x^\lambda (x_+^{-\lambda-1})_n \varphi(x)\,dx \\ &= \varphi(0)\int_0^1 x^\lambda (x_+^{-\lambda-1})_n\,dx + \int_0^1 x^\lambda [x(x_+^{-\lambda-1})_n]\varphi'(\xi x)\,dx. \end{aligned}$$

Since the sequence of continuous functions $\{x(x_+^{-\lambda-1})_n\}$ converges uniformly to the continuous function $x^{-\lambda}$ on the closed interval $[0,1]$, it follows on using equation (6) that

$$\begin{aligned} \operatorname*{N-lim}_{n\to\infty} \langle x_+^\lambda (x_+^{-\lambda-1})_n, \varphi(x)\rangle &= \operatorname*{N-lim}_{n\to\infty} \varphi(0)\int_0^1 x^\lambda (x_+^{-\lambda-1})_n\,dx + \\ &\quad + \lim_{n\to\infty}\int_0^1 x^\lambda [x(x_+^{-\lambda-1})_n]\varphi'(\xi x)\,dx \\ &= -[\gamma + \tfrac{1}{2}\psi(-\lambda) + \tfrac{1}{2}\psi(\lambda+1) + 2c(\rho)]\varphi(0) + \int_0^1 \varphi'(\xi x)\,dx \\ &= -[\gamma + \tfrac{1}{2}\psi(-\lambda) + \tfrac{1}{2}\psi(\lambda+1) + 2c(\rho)]\varphi(0) + \\ &\quad + \int_0^1 x^{-1}[\varphi(x) - \varphi(0)]\,dx \\ &= -[\gamma + \tfrac{1}{2}\psi(-\lambda) + \tfrac{1}{2}\psi(\lambda+1) + 2c(\rho)]\varphi(0) + \langle x_+^{-1}, \varphi(x)\rangle, \end{aligned}$$

giving equation (2) on the interval $[-1,1]$ when $-1 < \lambda < 0$. However, since $x_+^\lambda . x_+^{-\lambda-1} = x_+^{-1}$ on any closed interval not containing the origin, equation (2) holds on the real line when $-1 < \lambda < 0$.

Now suppose that equation (2) holds when $-k < \lambda < -k+1$. If $-k-1 < \lambda < -k$, the product $x_+^{\lambda+1} x_+^{-\lambda-1}$ exists by Definition 1 and is equal to $H(x)$. By Theorem 1 we have

$$(\lambda+1)x_+^\lambda \circ x_+^{-\lambda-1} - (\lambda+1)x_+^{\lambda+1} \circ x_+^{-\lambda-2} = \delta(x)$$

and it follows from our assumption that

$$x_+^\lambda \circ x_+^{-\lambda-1} = x_+^{-1} - [\gamma + \tfrac{1}{2}\,\psi(-\lambda-1) + \tfrac{1}{2}\,\psi(\lambda+2) - (\lambda+1)^{-1} + 2c(\rho)]\delta(x). \tag{7}$$

Since $\Gamma(x+1) = x\Gamma(x)$, it follows that

$$\psi(x+1) = x^{-1} + \psi(x).$$

Equation (7) reduces to

$$x_+^\lambda \circ x_+^{-\lambda-1} = x_+^{-1} - [\gamma + \tfrac{1}{2}\,\psi(-\lambda) + \tfrac{1}{2}\,\psi(\lambda+1)^{-1} + 2c(\rho)]\delta(x)$$

and equation (2) follows by induction for negative $\lambda \neq -1, -2, \ldots$.

A similar proof shows that equation (2) holds for positive $\lambda \neq 1, 2, \ldots$. This completes the proof of the theorem. □

Comparing Theorems 2 and 3 when $\lambda = -\frac{1}{2}$ we note that we have proved that

$$2\ln 2 = -\gamma - \psi(\tfrac{1}{2}).$$

Corollary 3.1. *The neutrix product $x_-^\lambda \circ x_-^{-\lambda-1}$ exists and*

$$x_-^\lambda \circ x_-^{-\lambda-1} = x_-^{-1} - [\gamma + \tfrac{1}{2}\,\psi(-\lambda) + \tfrac{1}{2}\,\psi(\lambda+1) + 2c(\rho)]\delta(x) \tag{8}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

Proof. Equation (8) follows immediately on replacing x by $-x$ in equation (2). □

In the next corollary, the distribution $(x+i0)^\lambda$ is defined by

$$(x+i0)^\lambda = x_+^\lambda + e^{i\lambda\pi} x_-^\lambda$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and

$$(x+i0)^{-1} = x^{-1} - i\pi\delta(x).$$

Corollary 3.2. *The neutrix product $(x+i0)^\lambda \circ (x+i0)^{-\lambda-1}$ exists and*

$$(x+i0)^\lambda \circ (x+i0)^{-\lambda-1} = (x+i0)^{-1} \tag{9}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

Proof. The neutrix product is distributive with respect to addition and so

$$(x+i0)^\lambda \circ (x+i0)^{-\lambda-1} = x_+^\lambda \circ x_+^{-\lambda-1} - x_-^\lambda \circ x_-^{-\lambda-1} + -e^{-i\lambda\pi} x_+^\lambda \circ x_-^{-\lambda-1} + e^{i\lambda\pi} x_-^\lambda \circ x_+^{-\lambda-1}. \tag{10}$$

Further, it was proved in [3] that

$$x_+^\lambda \circ x_-^{-\lambda-1} = x_-^\lambda \circ x_+^{-\lambda-1} = -\tfrac{1}{2}\pi \operatorname{cosec}(\pi\lambda)\delta(x) \tag{11}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$. It follows from equations (2), (8), (10) and (11) that

$$(x+i0)^\lambda \circ (x+i0)^{-\lambda-1} = x^{-1} - i\pi\delta(x) = (x+i0)^{-1},$$

proving equation (9). □

We finally note that the following results can be proved similarly.

$$\begin{aligned}
|x|^\lambda \circ (\operatorname{sgn} x|x|^{-\lambda-1}) &= (\operatorname{sgn} x|x|^\lambda) \circ |x|^{-\lambda-1} = x^{-1}, \\
|x|^\lambda \circ |x|^{-\lambda-1} &= |x|^{-1} - [2\gamma + \psi(-\lambda) + \psi(\lambda+1) + 4c(\rho) + \\
&\qquad +\pi \operatorname{cosec}(\pi\lambda)]\delta(x), \\
(\operatorname{sgn} x|x|^\lambda) \circ (\operatorname{sgn} x|x|^{-\lambda-1}) &= |x|^{-1} - [2\gamma + \psi(-\lambda) + \psi(\lambda+1) + 4c(\rho) + \\
&\qquad -\pi \operatorname{cosec}(\pi\lambda)]\delta(x)
\end{aligned}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$.

References

[1] van der Corput, J.C., Introduction to the neutrix calculus, J. Analyse Math. 7(1959-60), 291-398.

[2] Fisher, B., The product of distributions, Quart. J. Math. Oxford (2), 22(1971), 291-298.

[3] Fisher, B., A non-commutative neutrix product of distributions, Math. Nachr. 108(1982), 117-127.

[4] Fisher, B., Kuribayashi, Y., A theorem on the neutrix product of distributions, Dem. Math. 17(1984), 785-794.

[5] Fisher, B., Kuribayashi, Y., Neutrices and the Beta function, Rostock. Math. Kolloq. 32(1987), 56-66.

[6] Fisher, B., Savaş, E., Pehlivan, S., Özçağ,E., Results on the non-commutative neutrix product of distributions, Math. Balkanica 7(1993), 347-356.

[7] Gel'fand, I.M., Shilov, G.E., Generalized Functions, Vol. I, Academic Press, 1964.

Received by the editors May 4, 1998.