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(2m)-TH MEAN BEHAVIOR OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS UNDER PARAMETRIC PERTURBATIONS[1]

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Abstract. We consider the perturbed stochastic differential equation of the Ito type depending on a small parameter. We give conditions of the closeness in the $(2m)$-th mean between the solution of this perturbed equation and the solution of the corresponding unperturbed equation of the equal type.

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1. Introduction

Stochastic differential equations depending on deterministic and random perturbations have been extensively investigated both theoretically and experimentally over a long period of time. Mathematical models in mechanics and engineering (see [4, 13], for example) and recently in financial mathematics (see [13, 14], for example) are represented by these equations. The researcher's interest is focused on exploring the bifurcational behavior and on conditions of stability or instability of the solutions of these equations under deterministic and, especially, stochastic excitations of a Gaussiam white noise type. Having in mind that a Gaussian white noise is an abstraction and not a physical process, at least mathematically described as a formal derivative of a Brownian motion process, all such problems are essentially based on stochastic differential equations of the Itô type [6] in the form

$$dx_t = a(t, x_t)\,dt + b(t, x_t)\,dw_t, \quad t \in [0, T], \quad x_0 = \eta. \tag{1.1}$$

Here $w = (w_t, t \geq 0)$ is an R^k-valued normalized Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with a natural filtration $\{\mathcal{F}_t, t \geq 0\}$ of nondecreasing sub σ-algebras of $\mathcal{F}$, the functions $a : [0, T] \times R^n \to R^n$ and $b : [0, T] \times R^n \to R^n \times R^k$ are assumed to be Borel measurable on their domains, the initial condition η is a random variable defined on the same probalility

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space and independent of w, and $x = (x_t, t \in [0,T])$ is an R^n–valued stochastic process. The process x is a strong solution of Eq. (1.1) if it is adapted to $\{\mathcal{F}_t, t \geq 0\}$, $\int_0^T |a(t,x_t)|\,dt < \infty$ a.s., $\int_0^T |b(t,x_t)|^2\,dt < \infty$ a.s. (under these conditions the Lebesgue and Itô integrals in the integral form of Eq. (1.1) are well defined), $x_0 = \eta$ and Eq. (1.1) holds a.s. for all $t \in [0,T]$.

We shall note that the problems concerning stochastic perturbed equations of the Ito type have been studied by a number of authors, the results being presented in papers and books [4, 5, 8 – 16].

Furthermore, we should mention that for notational simplicity we shall restrict ourselves here to scalar–valued processes. The extension to multidimensional case is analogous and is not difficult in itself.

On the basis of classical theory of stochastic differential equations of the Itô type (see [1, 3, 5, 7, 12], for example) one can prove that if the functions $a(t,x)$ and $b(t,x)$ satisfy the global Lipschitz condition and usual linear growth condition on the last argument, i.e. if there exists a constant $L > 0$ such that

$$(1.2)\qquad |a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq L|x-y|,$$

$$(1.3)\qquad |a(t,x)|^2 + |b(t,x)|^2 \leq L\left(1 + |x|^2\right),$$

for all $x, y \in R$, $t \in [0,T]$, and if $E|\eta|^{2m} < \infty$ for any fixed natural number m, then there exists a unique a.s. continuous strong solution $x = (x_t, t \in [0,T])$ of Eq. (1.1) satisfying $E\{\sup_{t\in[0,T]} |x_t|^{2m}\} < \infty$. Moreover,

$$(1.4)\qquad E|x_t|^{2m} \leq \left(1 + E|\eta|^{2m}\right) e^{c_1 t} - 1, \quad t \in [0,T],$$

where $c_1 > 0$ is a constant independent on T (see ([12]), for example).

In the present paper we consider the stohastic differential equation of the Itô type with perturbations depending on a small parameter by comparing it in the $(2m)$-th moment sense with an appropriate unperturbed equation of the equal type. We give a new form of perturbations, partially motivated by the ones from [15], and also from [8, 9].

The paper is organized as follows: In the next section we define the problem and give an auxiliary result, important for the future investigation. In fact, we give the global estimation for the $(2m)$–th moment closeness of the solutions of the perturbed and unperturbed equation. After that we give our main results, the conditions under which these solutions are close in the $(2m)$-th moment sense on finite time–intervals or on the intervals whose length tends to infinity as the small parameter tends to zero. We also give some remarks and point out the possible applications of the preceding considerations.

2. Formulation of the problem and main results

Together with (1.1) in its integral form,

$$(2.1)\qquad x_t = \eta + \int_0^t a(s,x_s)\,ds + \int_0^t b(s,x_s)\,dw_s, \quad t \in [0,T],$$

we consider the following equation

$$(2.2)\quad x_t^\varepsilon = \eta^\varepsilon + \int_0^t \left[\alpha_1(s, x_s^\varepsilon, \varepsilon)\, a(s, x_s^\varepsilon) + \alpha_2(s, x_s^\varepsilon, \varepsilon)\right] ds + \int_0^t \left[\beta_1(s, x_s^\varepsilon, \varepsilon)\, b(s, x_s^\varepsilon) + \beta_2(s, x_s^\varepsilon, \varepsilon)\right] dw_s, \quad t \in [0, T],$$

in which ε is *a small parameter* from the interval $(0,1)$, the initial value η^ε satisfying $E|\eta^\varepsilon|^{2m} < \infty$ is independent on the same Brownian motion w, and $\alpha_i : [0,T] \times R \to R$ and $\beta_i : [0,T] \times R \to R$, $i = 1,2$ are given functions depending on ε.

There are various, essentially different conditions for the existence and uniqueness of solutions of the equations (2.1) and (2.2). Furthermore, we shall assume without emphasizing that there exist a.s. continuous solutions of these equations, satisfying $E \sup_{t\in[0,T]} |x_t|^{2m} < \infty$ and $E \sup_{t\in[0,T]} |x_t^\varepsilon|^{2m} < \infty$, and we shall emphasize only the conditions immediately used in our discussion.

We shall suppose that there exist a non–random value $\delta_0(\cdot)$, such that

$$(2.3)\qquad E|\eta^\varepsilon - \eta|^{2m} \le \delta_0(\varepsilon),$$

and the continuous functions $\delta_i(\cdot)$ and $\gamma_i(\cdot)$, $i = 1,2$, defined on $[0,T]$ and depending on ε, such that

$$(2.4)\qquad \begin{aligned} &\sup_{x\in R} |\alpha_1(t,x,\varepsilon) - 1| \le \delta_1(t,\varepsilon), \quad && \sup_{x\in R} |\alpha_2(t,x,\varepsilon)| \le \delta_2(t,\varepsilon),\\ &\sup_{x\in R} |\beta_1(t,x,\varepsilon_2) - 1| \le \gamma_1(t,\varepsilon), \quad && \sup_{x\in R} |\beta_2(t,x,\varepsilon)| \le \gamma_2(t,\varepsilon). \end{aligned}$$

Obviously, if the values $\delta_0(\varepsilon), \delta_i(t,\varepsilon), \gamma_i(t,\varepsilon)$ are small for small ε, then we could expect that the solutions x_t and x_t^ε are close in any reasonable sense. In accordance with [8, 9] and first of all with [15], the functions $\alpha_i(\cdot)$ and $\beta_i(\cdot)$ are called *the perturbations*, while Eq. (2.2) is logically called *the perturbed equation* with respect to *the unperturbed equation* (2.1).

In fact, the problem of perturbations considered here is a generalization of the one from paper [15] for $\alpha_1(\cdot) \equiv 1$, $\beta_1(\cdot) \equiv 1$. If we take $\alpha_1(t,x,\varepsilon) = 1+\nu(t,x,\varepsilon)$, then $|\nu(t,x,\varepsilon)\, a(t,x)| \le \delta_1(t,\varepsilon)\, |a(t,x)|$, which need not be bounded with respect to $x \in R$. Clearly, our problem could be treated as the one from [15] only if $\sup_{x\in R} |\nu(t,x,\varepsilon)\, a(t,x)| \le \delta_3(t,\varepsilon)$ for any continuous function $\delta_3(\cdot)$, which is a very strong assumption with respect to the linear growth condition (1.3). Obviously, similar reasoning is valid for $\beta_1(\cdot)$.

First, we shall present an auxiliary result, the global estimation of the $(2m)$–th moment closeness for the solutions x and x^ε. Note that the line of the proof is partially similar to [15] and [9], but different from [8].

Proposition 1. *Let x and x^ε be the solutions of the equations* (2.1) *and* (2.2) *respectively, defined on a finite interval* $[0,T]$ *and let the conditions* (1.2), (1.3), (2.3) *and* (2.4) *be satisfied. Then, for every* $t\in[0,T]$,

$$(2.5)\qquad E|x_t^\varepsilon - x_t|^{2m} \le \Big[\big(\nu(T)\big)^{1/m}\exp\Big\{\frac{1}{m}\int_0^t \xi(s)\,ds\Big\} + \frac{1}{m}\int_0^t \theta(s)\exp\Big\{\frac{1}{m}\int_s^t \xi(r)\,dr\Big\}ds\Big]^m,$$

where

$$\begin{aligned} \nu(t) &= \delta_0(\varepsilon) + c\int_0^t [\delta_1(s,\varepsilon) + 3(2m-1)\,\gamma_1^2(s,\varepsilon)]\,e^{c_1 s}\,ds \\ (2.6)\quad \xi(t) &= (2m-1)\,\delta_1(t,\varepsilon) + 2m\,\delta_2(t,\varepsilon) + 3(m-1)(2m-1)\,\gamma_1^2(t,\varepsilon) \\ &\quad +2mL + 3m(2m-1)\,L^2 \\ \theta(t) &= 2m\,\delta_2(t,\varepsilon) + 3m(2m-1)\,\gamma_2^2(t,\varepsilon), \end{aligned}$$

c and c_1 are some generic positive constants independent on ε and T.

Proof. Let us denote that

$$z_t^\varepsilon = x_t^\varepsilon - x_t, \quad \Delta_t^\varepsilon = E|z_t^\varepsilon|^{2m},$$

after that let us subtract the equations (2.1) and (2.2) and then apply the Itô differential formula to $(z_t^\varepsilon)^{2m}$. Thus,

$$(z_t^\varepsilon)^{2m} = (z_0^\varepsilon)^{2m} + 2mI_1(t) + m(2m-1)I_2(t) + 2mI_3(t),$$

where

$$\begin{aligned} I_1(t) &= \int_0^t \big[\alpha_1(s,x_s^\varepsilon,\varepsilon)\,a(s,x_s^\varepsilon) + \alpha_2(s,x_s^\varepsilon,\varepsilon)\big]\,(z_s^\varepsilon)^{2m-1}\,ds, \\ I_2(t) &= \int_0^t \big[\beta_1(s,x_s^\varepsilon,\varepsilon)\,b(s,x_s^\varepsilon) + \beta_2(s,x_s^\varepsilon,\varepsilon)\big]^2\,(z_s^\varepsilon)^{2m-2}\,ds, \\ I_3(t) &= \int_0^t \big[\beta_1(s,x_s^\varepsilon,\varepsilon)\,b(s,x_s^\varepsilon) + \beta_2(s,x_s^\varepsilon,\varepsilon)\big]\,(z_s^\varepsilon)^{2m-1}\,dw_s. \end{aligned}$$

Because $EI_3(t) = 0$ for $t\in[0,T]$, then

$$(2.7)\qquad \Delta_t^\varepsilon = \Delta_0^\varepsilon + 2mEI_1(t) + m(2m-1)EI_2(t), \quad t\in[0,T].$$

It remains to estimate the terms $EI_1(t)$ and $EI_2(t)$. So, by using the assumptions (2.4) and the Lipschitz condition (1.2), we find

$$EI_1(t) \le E\int_0^t |\alpha_1(s,x_s^\varepsilon,\varepsilon) - 1|\cdot|a(s,x_s^\varepsilon)|\cdot|z_s^\varepsilon|^{2m-1}ds + L\int_0^t \Delta_s^\varepsilon\,ds$$

$$
\begin{aligned}
&+E\int_0^t |\alpha_2(s,x_s^\varepsilon,\varepsilon)|\cdot|z_s^\varepsilon|^{2m-1}ds\\
\leq\ & \int_0^t \delta_1(s,\varepsilon)\,E\{|a(s,x_s^\varepsilon)|\cdot|z_s^\varepsilon|^{2m-1}\}ds + L\int_0^t \Delta_s^\varepsilon\,ds\\
&+\int_0^t \delta_2(s,\varepsilon)\,E|z_s^\varepsilon|^{2m-1}ds.
\end{aligned}
$$

By applying the elementary inequality $a^{1/p}b^{1/q} \leq a/p + b/q,\ \ p > 1, 1/p + 1/q = 1, a, b \geq 0$ and Hölder's inequality on the third term and taking that $p = 2m/(2m-1)$ in both of them, we obtain

$$
\begin{aligned}
EI_1(t) \ \leq\ & \int_0^t \delta_1(s,\varepsilon)\left(\frac{2m-1}{2m}\Delta_s^\varepsilon + \frac{1}{2m}E|a(s,x_s^\varepsilon)|^{2m}\right)ds + L\int_0^t \Delta_s^\varepsilon\,ds\\
&+\int_0^t \delta_2(s,\varepsilon)\,(\Delta_s^\varepsilon)^{\frac{2m-1}{2m}}ds.
\end{aligned}
$$

From (1.3) and (1.4) it follows that

$$
\begin{aligned}
E|a(s,x_s^\varepsilon)|^{2m} \ &\leq\ L^{2m}E\big(1+|x_s^\varepsilon|^2\big)^m\\
&\leq\ L^{2m}2^{m-1}\big(1+E|\eta^\varepsilon|^{2m}\big)\,e^{c_1 s}.
\end{aligned}
$$

By taking $L^{2m}2^{m-1}(1+E|\eta^\varepsilon|^{2m}) \leq c$ for any constant c independent on ε and T, we find

$$
\begin{aligned}
(2.8)\ EI_1(t) \ \leq\ & \frac{c}{2m}\int_0^t \delta_1(s,\varepsilon)\,e^{c_1 s}ds + \int_0^t \left(\frac{2m-1}{2m}\delta_1(s,\varepsilon)+L\right)\Delta_s^\varepsilon\,ds\\
&+\int_0^t \delta_2(s,\varepsilon)\,(\Delta_s^\varepsilon)^{\frac{2m-1}{2m}}ds.
\end{aligned}
$$

Similarly, in order to estimate $EI_2(t)$ we shall employ the procedure used above and Hölder's inequality for $p = m/(m-1)$. Thus we get

$$
\begin{aligned}
(2.9) EI_2(t) \ \leq\ & 3E\int_0^t |\beta_1(s,x_s^\varepsilon,\varepsilon)-1|^2\cdot|b(s,x_s^\varepsilon)|^2\cdot|z_s^\varepsilon|^{2m-2}ds\\
&+3L^2\int_0^t \Delta_s^\varepsilon\,ds + 3E\int_0^t |\beta_2(s,x_s^\varepsilon,\varepsilon)|^2\cdot|z_s^\varepsilon|^{2m-2}ds\\
\leq\ & \frac{3c}{m}\int_0^t \gamma_1^2(s,\varepsilon)\,e^{c_1 s}ds + 3\int_0^t\left(\frac{m-1}{m}\gamma_1^2(s,\varepsilon)+L^2\right)\Delta_s^\varepsilon\,ds\\
&+3\int_0^t \gamma_2^2(s,\varepsilon)\,(\Delta_s^\varepsilon)^{\frac{m-1}{m}}ds.
\end{aligned}
$$

Now, the relation (2.7) together with (2.3), (2.8) and (2.9), implies that

$$
\Delta_t^\varepsilon \ \leq\ \delta_0(\varepsilon) + c\int_0^t \big[\delta_1(s,\varepsilon)+3(2m-1)\gamma_1^2(s,\varepsilon)\big]\,e^{c_1 s}ds
$$

$$+\int_0^t \left[(2m-1)\delta_1(s,\varepsilon)+3(m-1)(2m-1)\gamma_1^2(s,\varepsilon)+2mL\right.$$
$$\left.+3m(2m-1)L^2\right]\Delta_s^\varepsilon\,ds+2m\int_0^t \delta_2(s,\varepsilon)\,(\Delta_s^\varepsilon)^{\frac{2m-1}{2m}}ds$$
$$+3m(2m-1)\int_0^t \gamma_2^2(s,\varepsilon)\,(\Delta_s^\varepsilon)^{\frac{m-1}{m}}ds.$$

Since $v^{r_2}\le v^{r_1}+v$ for any non-negative number v and $0<r_1\le r_2<1$, by taking $v=\Delta_s^\varepsilon$, $r_1=(m-1)/m$, $r_2=(2m-1)/2m$ it follows that $(\Delta_s^\varepsilon)^{(2m-1)/2m}\le(\Delta_s^\varepsilon)^{(m-1)/m}+\Delta_s^\varepsilon$. Thus the last inequality becomes

$$(2.10)\quad \Delta_t^\varepsilon\le\nu(t)+\int_0^t \xi(s)\,\Delta_s^\varepsilon\,ds+\int_0^t \theta(s)\,(\Delta_s^\varepsilon)^{(m-1)/m}ds,\quad t\in[0;T],$$

where the functions $\nu(t)$, $\xi(t)$ and $\theta(t)$ are defined as in (2.6).

To estimate Δ_t^ε from this integral inequality, we shall apply the following version of the well–known Gronwall–Bellman lemma [2, p. 39]: Let $u(t)$, $a(t)$ and $b(t)$ be non-negative continuous functions in $[0,T]$ and let $c>0$, $0\le\gamma<1$ be constants. If, for every $t\in[0,T]$,

$$u(t)\le c+\int_0^t a(s)u(s)\,ds+\int_0^t b(s)\,u^\gamma(s)\,ds,$$

then

$$u(t)\le\left(c^{1-\gamma}\,e^{(1-\gamma)\int_0^t a(s)\,ds}+(1-\gamma)\int_0^t b(s)\,e^{(1-\gamma)\int_s^t a(r)\,dr}\,ds\right)^{1/(1-\gamma)}.$$

Since $\nu(t)$ is increasing in $t\in[0,T]$, by taking $\nu(T)$ instead of $\nu(t)$ in (2.10) and also $u(t)=\Delta_t^\varepsilon$, $\gamma=(m-1)/m$, and then by applying the above lemma, we immediately obtain the estimation (2.5), which completes the proof. □

Having in mind that the size of perturbations is limited in the sense (2.3) and (2.4), then if we require that $\delta_0(\cdot),\delta_i(\cdot),\gamma_i(\cdot), i=1,2$ tend to zero as $\varepsilon\to0$, we could expect that the solutions x and x^ε are close in the $(2m)$–th mean. Remember that in [15] similar problems are considered for special types of perturbations $\alpha_2(\cdot),\beta_2(\cdot)$, specially chosen η^ε and for $\alpha_1(\cdot)\equiv1,\beta_1(\cdot)\equiv1$. The next considerations and conclusions are based on the ones from [8, 9]. Because of that, we shall briefly present the following results.

Theorem 1. *Let the conditions of Proposition* 1 *be satisfied and let* $\delta_0(\cdot)$, $\delta_i(\cdot)$, $\gamma_i(\cdot)$, $i=1,2$ *monotonously tend to zero as* $\varepsilon\to0$, *uniformly in* $[0,T]$. *Then*

$$\sup_{t\in[0,T]} E|x_t^\varepsilon-x_t|^{2m}\to0\quad\text{as}\quad\varepsilon\to0.$$

Proof. Let us denote that

$$\overline{\delta}_i(\varepsilon)\ \sup_{t\in[0,T]}\delta_i(t,\varepsilon),\quad \overline{\gamma}_i(\varepsilon)=\sup_{t\in[0,T]}\gamma_i(t,\varepsilon),\quad i=1,2$$

and

$$\phi(\varepsilon)=\max\{\delta_0(\varepsilon),\,\overline{\delta}_i(\varepsilon),\,\overline{\gamma}_i^2(\varepsilon),\,i=1,2\}. \tag{2.11}$$

Then, from (2.5) and (2.6) it follows that

$$(\Delta_t^\varepsilon)^{1/m} \le (\phi(\varepsilon))^{1/m}\left[1+c(6m-2)\frac{e^{c_1t}-1}{c_1}\right]^{1/m}\cdot e^{c_2t} + \phi(\varepsilon)\,(6m-1)\,\frac{e^{c_2t}-1}{c_2}, \tag{2.12}$$

where $c_2=[4m-1+2L+2(2m-1)L^2+3(m-1)(2m-1)\rho]/m$ and ρ is a constant for which $\phi(\varepsilon)\le\rho$ for $\varepsilon\in(0,1)$. In view of the fact that T is finite and $\phi(\varepsilon)\to 0$ as $\varepsilon\to 0$, it follows immediately that $\sup_{t\in[0,T]}\Delta_t^\varepsilon\to 0$ as $\varepsilon\to 0$. □

If we assume that there exist the unique solutions x and x^ε of the equations (2.1) and (2.2) respectively, defined on $[0,\infty)$, then the previous assertion is generally not valid. Our intention is to construct finite time–intervals which depend on ε and whose length goes to infinity as ε goes to zero, such that the solutions x_t^ε and x_t are close in the $(2m)$-th moment sense on these intervals.

Theorem 2. *Let the conditions of Theorem* 1 *be satisfied for* $t\in[0,\infty)$ *and the functions* $\delta_i(\cdot),\gamma_i(\cdot),i=1,2$ *be bounded on* $[0,\infty)$. *Then, for an arbitrary number* $r\in(0,1)$ *and* ε *sufficiently small, there exists a number* $T(\varepsilon)>0$, *determined by*

$$T(\varepsilon)=-\frac{r}{c_1+mc_2}\ln\phi(\varepsilon), \tag{2.13}$$

where $\phi(\varepsilon)$ *is given by* (2.11), *and* c_1,c_2 *are some generic positive constants, such that*

$$\sup_{t\in[0,T(\varepsilon)]}E|x_t^\varepsilon-x_t|^{2m}\to 0\quad\text{as}\quad\varepsilon\to 0.$$

Proof. The main idea in this proof is to take $T=T(\varepsilon)$ and define $T(\varepsilon)$ from (2.12) such that $\sup_{t\in[0,T(\varepsilon)]}(\Delta_t^\varepsilon)^{1/m}\to 0$ as $\varepsilon\to 0$.

First, let us remember that since $\phi(\varepsilon)\to 0$ as $\varepsilon\to 0$, then there exists $\overline{\varepsilon}\in(0,1)$ such that $\phi(\varepsilon)<1$ for $\varepsilon<\overline{\varepsilon}$. Because $[0,T(\varepsilon)]$ is a finite time–interval, we can apply Theorem 1 to estimate the closeness of the solutions x and x^ε on this interval. By using the elementary inequality $|a+b|^\nu\le(2^{\nu-1}\vee 1)(|a|^\nu+|b|^\nu)$, $\nu>1$, from (2.12), it follows that

$$\sup_{t\in[0,T(\varepsilon)]}(\Delta_t^\varepsilon)^{1/m}\le(\phi(\varepsilon))^{1/m}\left[q_1+q_2e^{c_1/m\cdot T(\varepsilon)}\right]\cdot e^{c_2T(\varepsilon)}+\phi(\varepsilon)\left[q_3+q_4\,e^{c_2T(\varepsilon)}\right],$$

where $q_i, i = \overline{1,4}$ are generic positive constants independent on ε and $T(\varepsilon)$. Now, let us determine $T(\varepsilon)$ with respect to $\phi(\varepsilon)$ such that the greatest term in the right side of the previous inequality tends to zero as $\varepsilon \to 0$. Indeed, by taking

$$(c_1/m + c_2)T(\varepsilon) = -r/m \cdot \ln \phi(\varepsilon)$$

for any number $r \in (0,1)$ and $\varepsilon < \overline{\varepsilon}$, we obtain $T(\varepsilon)$ in the form (2.13).

For a $T(\varepsilon)$ chosen in that way it is easy to conclude that $T(\varepsilon) \to \infty$ as $\varepsilon \to 0$ and

$$(2.14) \sup_{t\in[0,T(\varepsilon)]} (\Delta_t^\varepsilon)^{1/m} \leq q_1\,(\phi(\varepsilon))^{1/m} + q_2\,(\phi(\varepsilon))^{(1-r)/m} + q_3\,\phi(\varepsilon) + q_4\,(\phi(\varepsilon))^{(c_1+c_2(m-r))/(c_1+mc_2)} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

which completes the proof. □

Example: Let us consider the following perturbed equation

$$(2.15) dx_t^\varepsilon = \left(ax_t^\varepsilon + \varepsilon \sin x_t^\varepsilon\right) dt + bx_t^\varepsilon\, e^{\varepsilon/(1+t+|x_t^\varepsilon|)} dw_t, \quad x_0^\varepsilon = \eta + \varepsilon, \quad t \geq 0,$$

in which a, b are non-random constants and $E|\eta|^{2m} < \infty$, by comparing its solution with the one of the corresponding linear equation

$$(2.16) \qquad dx_t = ax_t\, dt + bx_t\, dw_t, \quad x_0 = \eta, \quad t \geq 0.$$

Since $E|x_0^\varepsilon - x_0|^{2m} = \varepsilon^{2m}$, $|\varepsilon \sin x| < \varepsilon$, $\left|e^{\varepsilon/(1+t+|x|)} - 1\right| \leq e^\varepsilon - 1$, all the conditions of Theorem 2 are satisfied for $\phi(\varepsilon) = \max\{\varepsilon^{2m}, \varepsilon, (e^\varepsilon - 1)^2\} = \varepsilon$, where $\varepsilon < \varepsilon_0, (e^{\varepsilon_0} - 1)^2 = \varepsilon_0$, and $\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$. In accordance with (2.13),

$$T(\varepsilon) = -\frac{2r}{c_1 + mc_2} \ln \varepsilon,$$

where c_1 and c_2 are easily obtained generic constants and, therefore, it follows that $\sup_{t\in[0,T(\varepsilon)]} E|x_t^\varepsilon - x_t|^{2m} \to 0$ as $\varepsilon \to 0$.

Remember also that $E|x_t|^{2m} = |\eta|^{2m} e^{[a+(2m-1)/2\, b^2]\,t}, t \geq 0$ (see [1, 7, 13]), i.e. the solution x_t of the linear equation (2.16) is exponentially stable if and only if $a + (2m-1)/2\, b^2 < 0$. Therefore, under this condition the solution x_t^ε of the perturbed equation (2.15) behaves as the solution x_t of the corresponding linear equation, approximately in the $(2m)$–th moment sense, when the small parameter ε goes to zero.

Let us give some remarks: The inequality (2.14) describes an important result, the size of the closeness of the solutions x and x^ε for a fixed small parameter ε on the time–interval $[0, T(\varepsilon)]$.

The initial condition and the perturbations $\alpha_i(\cdot), \beta_i(\cdot), i = 1,2$ could depend on different small parameters $\varepsilon_i, i = \overline{0,4}$. Then, all the assertions remain to be valid if we take $\varepsilon = \max\{\varepsilon_i, i = \overline{0,4}\}$.

The results of this paper could be used to study stability properties in the $(2m)$–th mean for the solution of the perturbed equation, by studying stability properties in the same sense as for the solution of the corresponding unperturbed equation.

The method presented here could be appropriately extended to stochastic integral and integrodifferential equations of the Itô type, as well as to stochastic differential equations including martingales and martingale measures instead of the Brownian motion process.

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