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THE JORDAN DECOMPOSITION OF THE NULL-ADDITIVE SIGNED FUZZY MEASURES

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Abstract. We investigated the signed null-additive fuzzy measure m, the revised monotone set function which vanishes at the empty set and such that $m(B) = 0$ implies $m(A \cup B) = m(A)$ and which is continuous from the above and continuous from below. For such set function m, a Jordan decomposition type theorem was proved and this result enabled the definition of the total variation $|m|$ of m. The absolute continuity of a null-additive signed fuzzy measure with respect to another null-additive signed fuzzy measure was introduced.

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1. Introduction

A wide class of non-additive set functions, the so-called null-additive set functions, include many important set functions as, for example, submeasures [3],[4] , $k-$triangular set functions [9],[10], decomposable measures [5], [6], [8], [11], [12], [13], [25] etc., whose numerous interesting properties are presented in the books of Z. Wang, G. Klir [24] and E. Pap [14].

The signed fuzzy measure was introduced by B. Jiao [7] and investigated also by L. Xuecheng [21]. The notion of signed fuzzy measure introduced in [7] in special case does not reduce to the usual fuzzy measure, but to the modification of this notion [21]. So, we shall use the last version of the notion of signed fuzzy measure.

In this paper we shall use the Hahn decomposition type theorem obtained in [21] for null-additive signed fuzzy measures. This result enables us to introduce the usual notion of total variation for null-additive signed fuzzy measure, and then the notion of absolute continuity of a null-additive signed fuzzy measure with respect to another null-additive signed fuzzy measure.

2. Signed fuzzy measures

Throughout this paper Σ denotes a σ-algebra of subsets of the given set X.

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Definition 1. *A signed fuzzy measure* $m, m : \Sigma \to [-\infty, \infty]$, *is an extended real-valued set function* m *defined on* σ*-algebra* Σ *and with the properties:*

(FM₁) $$m(\emptyset) = 0,$$

(FM₂) *if* $E, F \in \Sigma,\ \ E \cap F = \emptyset$, *then*

(**a**) $m(E) \geq 0,\ \ m(F) \geq 0,\ \ \max(m(E), m(F)) > 0$ *implies*

$$m(E \cup F) \geq \max(m(E), m(F));$$

(**b**) $m(E) \leq 0,\ \ m(F) \leq 0,\ \ \min(m(E), m(F)) < 0$ *implies*

$$m(E \cup F) \leq \min(m(E), m(F));$$

(**c**) $m(E) > 0,\ \ m(F) < 0$ *implies* $m(E) \geq m(E \cup F) \geq m(F)$.

(FM₃) $E_1 \subset E_2 \subset \ldots\ \ , E_n \in \Sigma \quad \Rightarrow \quad m(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n)$,

(FM₄) $E_1 \subset E_2 \subset \ldots\ \ , E_n \in \Sigma$ *and there exists* n_0 *such that* $|m(E_{n_0})| < \infty \Rightarrow \mu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n)$.

The revised monotonicity (**FM₂**) for finite signed measures was introduced in a little bit different form by B. Jiao [7], and in the present form for infinite signed measures by Liu Xuecheng [21].

Example 1. Any non-negative signed fuzzy measure is usually a fuzzy measure. Namely, the condition (FM_2), (a) implies the monotonicity of m, i.e.

$$E \subset F \qquad \Rightarrow \qquad m(E) \leq m(F).$$

But also, each fuzzy measure is a signed fuzzy measure.

Example 2. The classical signed measure is a signed fuzzy measure.

We have by Wang [23]

Definition 2. *A set function* $m,\ m : \Sigma \to [-\infty, \infty]$, *is called null-additive, if we have*

$$m(A \cup B) = m(A)$$

whenever $A, B \in \Sigma,\ \ A \cap B = \emptyset,$ *and* $m(B) = 0$.

3. The Jordan decomposition

Definition 3. *A set A from Σ is called a positive set (resp. negative set) for the signed fuzzy measure m on (Σ, X) if for every subset E of A which belongs to Σ we have $m(E) \geq 0$ (resp. $m(E) \leq 0$).*

We shall need the following Hahn decomposition type theorem proved by Liu Xuecheng [21].

Theorem 1. *Let m be a signed fuzzy measure on (Σ, X). If m takes at most one of the values $-\infty$ or $+\infty$, and if*

$$E \in \Sigma, |m(F)| < +\infty \;\Rightarrow\; |m(F)| < +\infty \;\; (F \subset E, F \in \Sigma),$$

then there exist two disjoint sets A and B from Σ such that $A \cup B = X$ whereby A is a positive set and B is a negative set.

Remark 1. The classical signed measure satisfies the conditions of the previous theorem, so that in this case the previous theorem reduces to the classical Hahn decomposition theorem.

The obtained Hahn decomposition of m in Theorem 1 (which is not unique!) we shall denote by (A, B).

In the rest of the paper we shall suppose, without loss of generality, that the signed fuzzy measure m satisfies the condition

$$-\infty < m(E) \leq +\infty \quad (E \in \Sigma).$$

Now we can prove the following version of the Jordan decomposition theorem.

Theorem 2. *Let m be a null-additive signed fuzzy measure. Then there exist uniquely determined null-additive fuzzy measures m^+ and m^- such that*

$$m^+ \geq m \geq -m^-.$$

If m has a representation in the form

$$m = \lambda_1 - \lambda_2,$$

where λ_1 and λ_2 are null-additive fuzzy measures, then $\lambda_1 \geq m^+$ and $\lambda_2 \geq m^-$.

Proof. Using Theorem 1 we obtain a positive set A and a negative set B for m, both from Σ. We define

$$m^+(E) = m(E \cap A) \quad \text{and} \quad m^-(E) = -m(E \cap B).$$

We shall prove that the previous definition is correct. If we take two different Hahn decompositions (A_1, B_1) and (A_2, B_2) of m, i.e., $X = A_i \cup B_i$ with $A_i \cap$

$B_i = \emptyset \quad (i = 1, 2)$, and A_i are positive sets and B_i are negative sets with respect to m, then we have

$$m(E \cap A_1) = m(E \cap A_2) \text{ and } m(E \cap B_1) = m(E \cap B_2).$$

We shall prove only the first equality since the second can be proved in a quite similar way.
Since $E \cap (A_2 \setminus A_1)$ is a subset of $E \cap A_2$ we have

$$m(E \cap (A_2 \setminus A_1)) \geq 0.$$

Since $E \cap (A_2 \setminus A_1)$ is also a subset of the set $E \cap B_1$ we have

$$m(E \cap (A_2 \setminus A_1)) \leq 0.$$

The last two inequalities imply

$$m(E \cap (A_2 \setminus A_1)) = 0.$$

Exchanging A_1 and A_2 in the last equality we obtain

$$m(E \cap (A_2 \setminus A_1)) = 0.$$

By the null-additivity of m we have

$$\begin{aligned} m(E \cap A_1) &= m(E \cap (A_1 \cap A_2) \cup (E \cap (A_1 \setminus A_2)) \\ &= m(E \cap (A_1 \cap A_2) \cup (E \cap (A_1 \setminus A_2) \cup (E \cap (A_2 \setminus A_1))) \\ &= m(E \cap (A_1 \cap A_2) \cup (E \cap (A_2 \setminus A_1))) \\ &= m(E \cap A_2). \end{aligned}$$

It is easy to see that m^+ and m^- are fuzzy measures. We shall prove that m^+ is null-additive. Namely, if $C \in \Sigma$ is such that $m^+(C) = 0$ we have $m(C \cap A) = 0$ and so we obtain

$$m^+(E \cup C) = m((E \cup C) \cap A) = m((E \cap A) \cup (C \cap A)) = m(E \cap A) = m^+(E).$$

In a quite analogous way we can prove that m^- is also null-additive.
By the revised monotonicity of m (condition (**c**) in (**FM$_2$**)) and the definitions of m^+ and m^- we obtain

$$m^+ \geq m \geq -m^-.$$

To prove the second part of the theorem suppose that m is represented in the form $\lambda_1 - \lambda_2$, where λ_1 and λ_2 are null-additive fuzzy measures. Then, by the inequality $m \leq \lambda_1$ we have

$$m^+(E) \leq m(E \cap A) \leq \lambda_1(E \cap A) \leq \lambda_1(E) \quad (E \in \Sigma).$$

In an analogous way we obtain by the inequality $-m \leq \lambda_2$ that

$$m^-(E) = -m(E \cap B) \leq \lambda_2(E \cap B) \leq \lambda_2(E) \quad (E \in \Sigma).$$

Open problem: Does there exist for any null-additive signed fuzzy measure m a representation $m = \lambda_1 - \lambda_2$, where λ_1 and λ_2 are null-additive fuzzy measures?

4. The variation

By the previous result for a null-additive signed fuzzy measure m the set functions m^+ and m^- are uniquely determined. So we can introduce the notion of a total variation of m in an analogous way as for the classical measure.

Definition 4. *The total variation $|m|$ of a null-additive signed fuzzy measure m is given by*

$$|m| = m^+ + m^-.$$

Now we have the following analogous property as for classical signed measure.

Theorem 3. *The total variation $|m|$ of a null-additive signed fuzzy measure m is a null-additive fuzzy measure which satisfies the inequality*

$$|m(E)| \leq |m|(E) \quad (E \in \Sigma).$$

Proof. It is obvious that $|m|$ is a fuzzy measure. We shall use the inequality

$$m^+ \geq m \geq -m^-.$$

Suppose first that $m(E) > 0$. Then by the last inequality $m^+(E) \geq m(E)$. Therefore,

$$|m|(E) = m^+(E) + m^-(E) \geq m(E) = |m(E)|.$$

Suppose now that $m(E) < 0$. Then we have $-m(E) \leq m^-(E)$. Therefore

$$|m(E)| = -m(E) \leq m^-(E) \leq |m|(E).$$

If $m(E) = 0$, then since $|m|$ is always non-negative, we have also in this case the desired inequality.

Definition 5. *Let m and λ be two null-additive signed fuzzy measures on Σ. m is absolutely continuous with respect to λ, $m \ll \lambda$, if $m(E) = 0$ for every $E \in \Sigma$ such that $|\lambda|(E) = 0$.*

We have the following characterization of the notion of absolute continuity.

Theorem 4. *If m and λ are null-additive signed fuzzy measures, then the following conditions are equivalent to each other*

(i) $m \ll \lambda$,

(ii) $m^+ \ll \lambda$ *and* $m^- \ll \lambda$,

(iii) $|m| \ll |\lambda|$.

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