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THE REALIZATION OF THE CONTINUITY PRINCIPLE IN THE RELATIVISTIC PENCILS OF CIRCLES AND SPHERES

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Abstract

The radical axis, as the only straight line in a pencil of circles, brakes the continuity in the infinite succession of the circles. In the same way, in space, the radical plane is an "intruder" in a pencil of spheres which brakes the infinite succession of the spheres.

Therefore, in order to realize the continuity principle in pencils of circles and spheres, we must leave those classical concepts of straight lines and planes as something quite different from circles and spheres, and introduce the new, relativistic, concepts of "straight lines" and "planes" which also are real circles and spheres but with a particular relation to the observer.

Key words and phrases: pencils of circles and spheres, concepts of line and plane, continuity, harmonic equivalence, relativistic geometry.

1 Continuity in Pencils of Circles

A parabolic pencil of circles, for example, consists of uncountably many circles and only one straight line (Fig. 1). Since, in Euclidean geometry, a straight line is something quite different from a circle, this line has to be considered as an intruder into the pencil of circles. Thus, the classical principle of continuity in the pencil is evidently broken by that intruder for nobody can explain intelligible how a circle can be changed into an open-ended Euclidean line, and then, immediately after, the Euclidean line into a circle with opposite curvature.

However, if we observe the forming of the pencil of circles as a continuous process, we will be able to see clearly that the straight line b in fact is not an

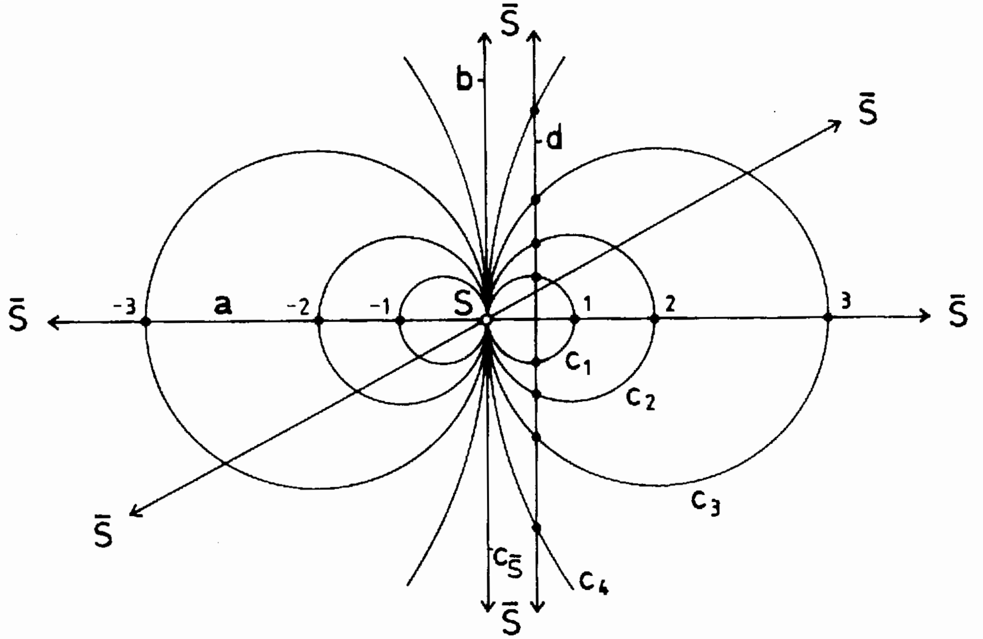

Figure 1: Straight lines intersect twice

intruder because it meets the axis a of the pencil at two points as any other circle does. Namely, every member of the pencil of circles, from the smallest to the greatest, crosses the straight line a at the point S and one more point (1, 2, 3, ...), and so does the straight line b: at S and at the infinitely distant point $\bar{S}$ which lies on the both lines, a and b. It means that two straight lines really intersect at two points. Therefore, the definite conclusion must be that the open-ended Euclidean straight line does not exist, and, if there is no classical straight line, there is no classical plane, either. In other words, the pencil of circles in reality does not lie on an abstract Euclidean plane but on a sphere of unperceivable dimensions, so that all straight lines, in general, secondly intersect at the antipodal point of that spheric "plane". And, really, when the pencil of circles is stereographically projected from the spheric "plane" onto a sphere of perceivable dimensions (Fig. 2), we can see that, for the observer at S, the "straight lines" are all circles which meet secondly at his antipodal point $\bar{S}$. Accordingly, the parallel "lines", a and d, meet at the antipodal point at two consecutive points, that is, they touch each other at the observer's antipodal point.

Since all points of a relativistic "plane" (as a sphere) are equivalent,

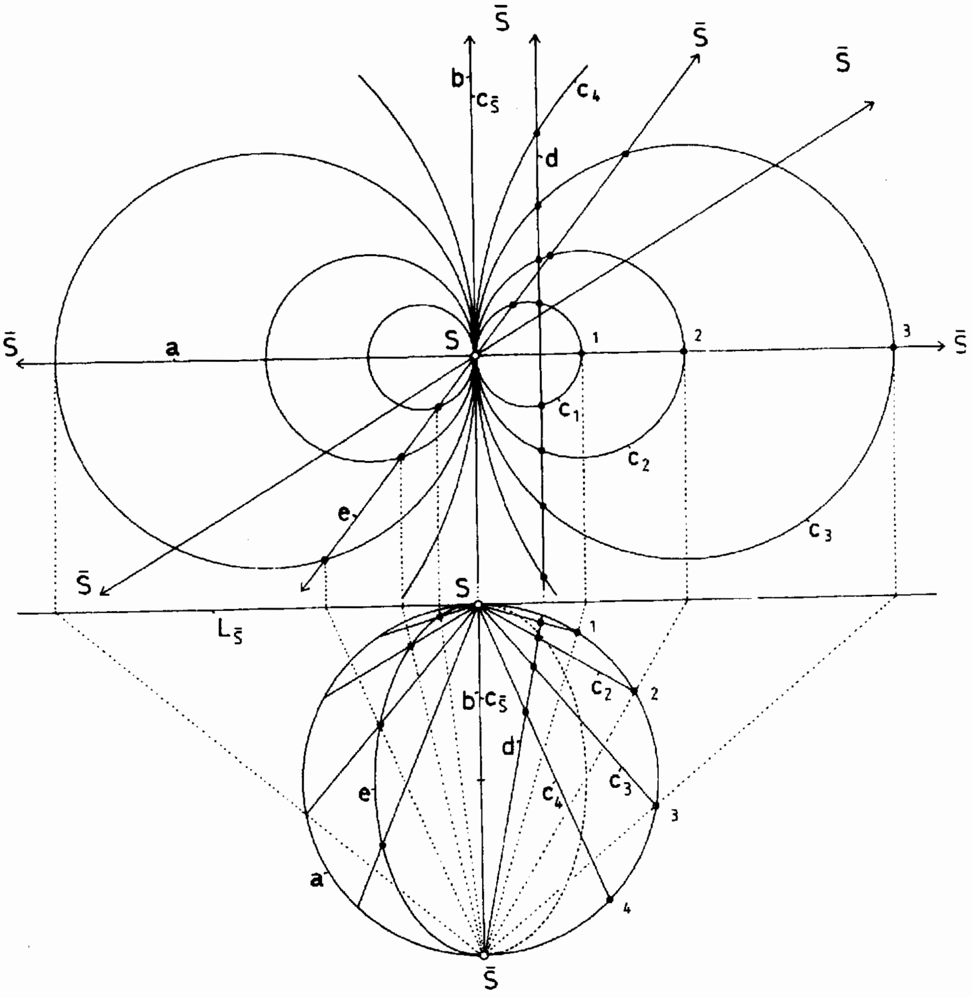

Figure 2: Continuity in a pencil of circles; all "lines" secondly intersect at the observer's antipodal point

any observer has got his own antipodal point and, accordingly, his own system of "straight lines", so that this relativistic concept of "straight lines" leads to the relativistic geometry which enables not only the realization of the continuity principle but also a definite synthesis of Euclidean and non-Euclidean geometries into a unique theory of curves and surfaces with an absolute classification of them in surprisingly wide groups of harmonic equivalents [2].

2 Continuity in Pencils of Spheres

The Euclidean radical plane of a pencil of spheres is also the intruder which apparently brakes the continuity in the succession of the spheres, for nobody can explain how a sphere can be continually changed into an open-ended Euclidean plane, and then, immediately after, that Euclidean plane into a sphere with opposite curvature. Whereas the realization of the continuity principle in pencils of circles has been relatively easy, the continuity in pencils of spheres requires more sophisticated clarification.

While the greatest circle in a relativistic pencil of circles is its radical "line", the greatest sphere in a relativistic pencil of spheres is not its radical "plane", but, seemingly paradoxically, it is a "plane" through the "rectilinear" axis of the pencil. Let us try to elucidate it by comparing the classical concept of symmetry with the relativistic one.

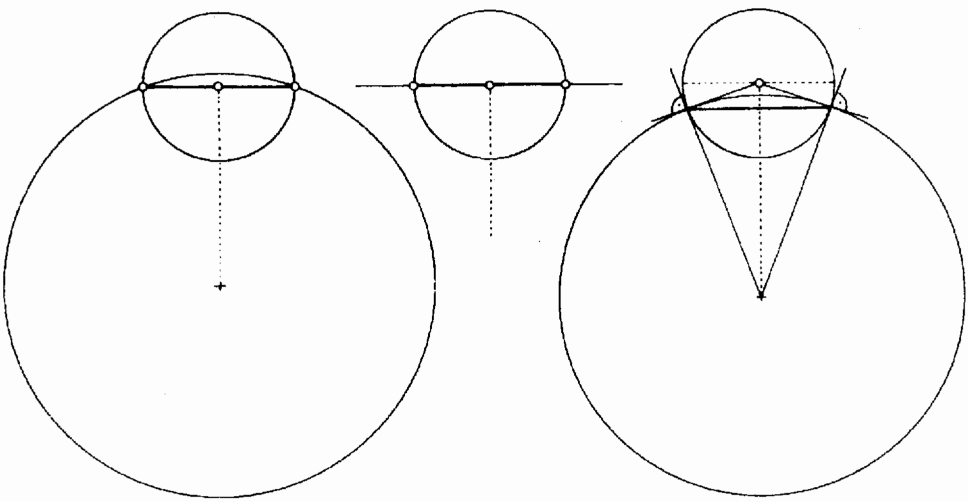

Figure 3: Euclidean symmetry - unattainable ideal of elliptic (left) and hyperbolic symmetry (right)

In classical geometry, the plane of symmetry of a sphere is orthogonal to the sphere, it passes through its centre and intersects it at its great circle (Fig. 3, middle). However, this Euclidean symmetry of a sphere is only unattainable ideal of a relativistic symmetry of a sphere, since whatever great a sphere of hyperbolic symmetry of a given sphere is (Fig. 3, right), it can never become a Euclidean plane, that is to say, the orthogonal sphere can never be at the same time the sphere which intersects the given sphere at its great circle (Fig. 3, left; an elliptic symmetry), and, of course, neither of the two spheres (right and left) can ever pass through the Euclidean centre of the given sphere. But, naturally, when the two spheres become of unperceivable dimensions we cannot tell the difference between them and, consequently, we wrongly equate both of them with an abstract Euclidean plane. In other words, a relativistic "plane", which intersects a sphere at its great circle, can never be orthogonal to the sphere. That also means that two mutually harmonically symmetric parts of a sphere can never be equal, i.e., two equal hemispheres can never be harmonically symmetric.

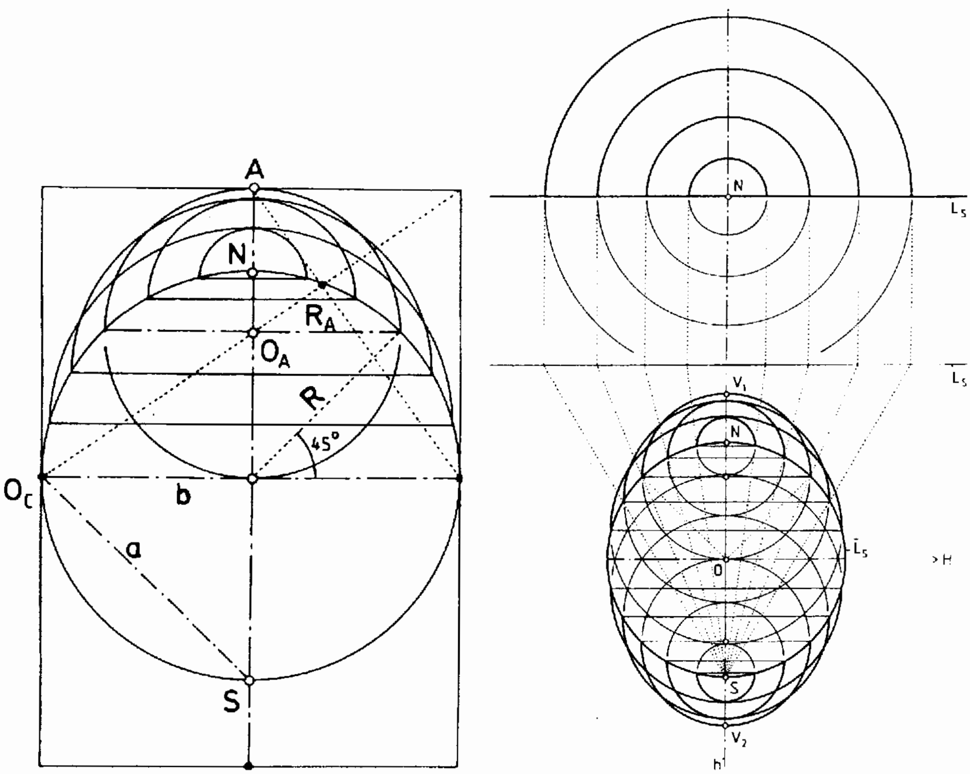

Figure 4: Pencil of "concentric" spheres based on a "plane" and on a sphere

Since the relativistic "plane" is, in fact, a sphere of unperceivable dimensions which intersects all spheres of the pencil at their great circles, in order to see what is really going on here, we will project stereographically these pencils of circles onto a sphere of perceivable dimensions. Now, since geometry is actually a physical science, which can only describe and not prescribe the laws of nature, let us imagine that a multitude of hemispherical cupolas are being built on the globe.

As the relativistic "plane" through the "rectilinear" axis of a pencil of spheres actually is a sphere, the hemispherical cupolas, based with their great circles on that basic sphere, can never have greater diameter than the basic sphere has. Therefore, the heights of the enlarging cupolas range from zero up to a maximal height (the height of the cupola whose sphere passes through the centre of the basic sphere; cf. Figs. 4-8) and back to "zero-height" of the largest "cupola" identical to the basic sphere, which enables continuous transition to the other, symmetric side of the pencil of spheres.

These cupolas, i.e. the spheres of the pencils, form surfaces whose contour curves are the ovals of Descartes: one-part ovals for hyperbolic pencils of spheres (Fig. 5), the cardioid - for a parabolic pencil of spheres (Fig. 6), and two-part ovals for elliptic pencils of spheres (Fig. 7).

The boundary cases of the ovals are: the ellipse ($a = b\sqrt{2}$, foci N and S) for the pencil of "concentric" spheres (Fig. 4; the parallel planes of the basic circles of the cupolas intersect at the conjugate polar through the pole H at infinity), and, the contour circle of the basic sphere $\bar{L}_S$ (cf. Fig. 7) when the pole E is at the centre O of the sphere (thus all the basic circles of these "cupolas" are the great circles of the basic sphere so that all the "cupolas" are identical to basic sphere).

In all the cases, the spheres of a pencil, even if considered as surfaces without thickness, actually form filled up solids (through any point inside the solid two spheres pass). In the boundary case of the hollow solids formed by elliptical pencils of spheres (cf. Fig. 7), when the pole E reaches the centre O of the basic sphere, the "solid" becomes identical to the hollow basic sphere as a surface without thickness.

The vertices V_1 and V_2 and of two highest cupolas always lie on the polar of a chosen pole with respect to the contour circle of the basic sphere $\bar{L}_S$.

By moving the pole from H at infinity to E at the centre of the basic sphere (Fig. 8) all forms of the solids, i.e. all forms of their contour Descartes' ovals, between the starting ellipse and the finishing circle, can be obtained: for the poles between H and H_3 - one-part convex ovals; for the pole H_3 (60°) - one-part oval with a flattened vertex; for H_2 (45°) - one-part oval with a bitangent and two inflectional points; for H_1 (30°) - the bitangent passes through H_1 (cf. Fig. 5); for P - the cardioid (cf. Fig. 6); for the poles between P and E_1 - two-part ovals with a bitangent and two inflectional points on the outer part of the curve; for E_1 (30°) - two-part oval with a flattened vertex (cf. Fig. 7); for the poles between E_1 and E - the outer oval is also convex (for E_2 - the moving vertex of the inner oval reaches the centre E of the sphere).

The general conclusion about pencils of circles and pencils of spheres is

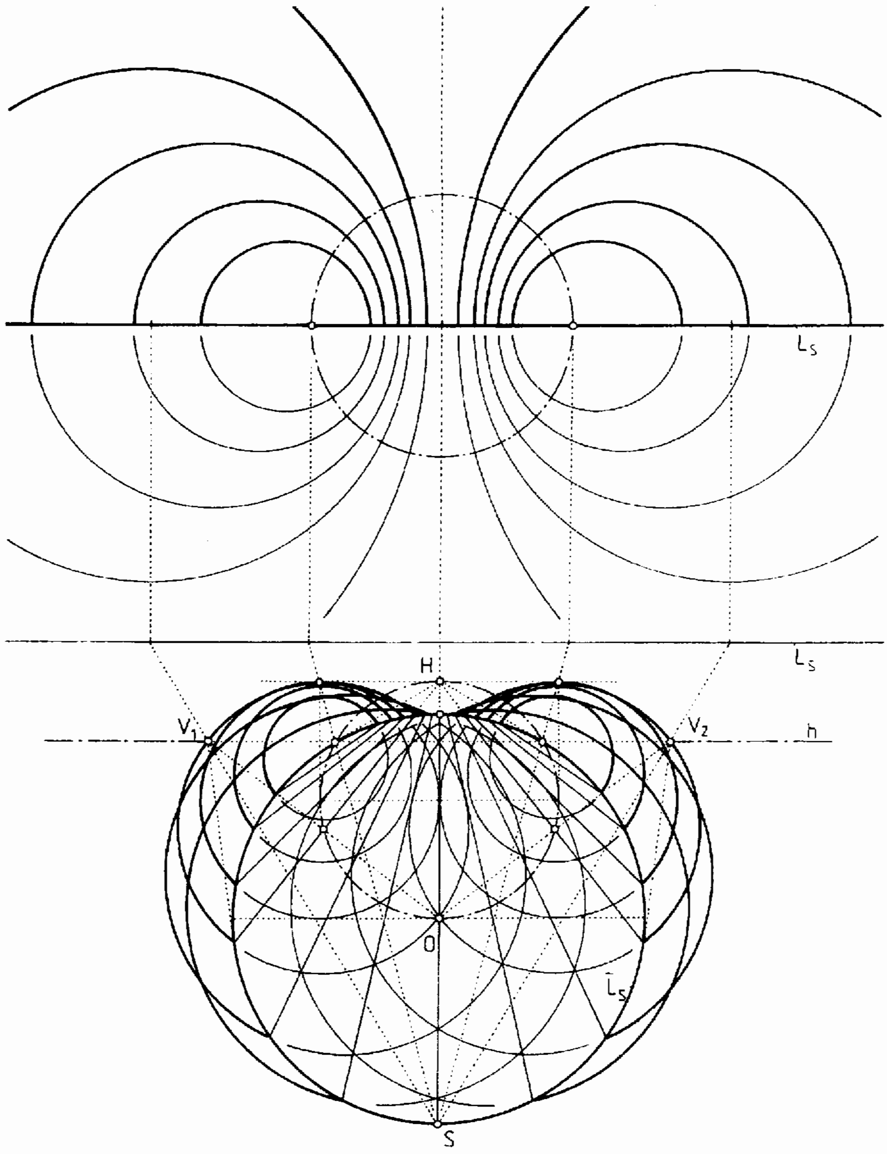

Figure 5: Hyperbolic pencil of spheres based on a "plane" and on a sphere

that their classical Euclidean interpretations are as illusive as the abstract Euclidean lines and planes are. Since an observer on a real sphere of unperceivable dimensions sees that large sphere as an abstract Euclidean plane, he cannot tell the difference between, for instance, a real hyperbolic pencil of

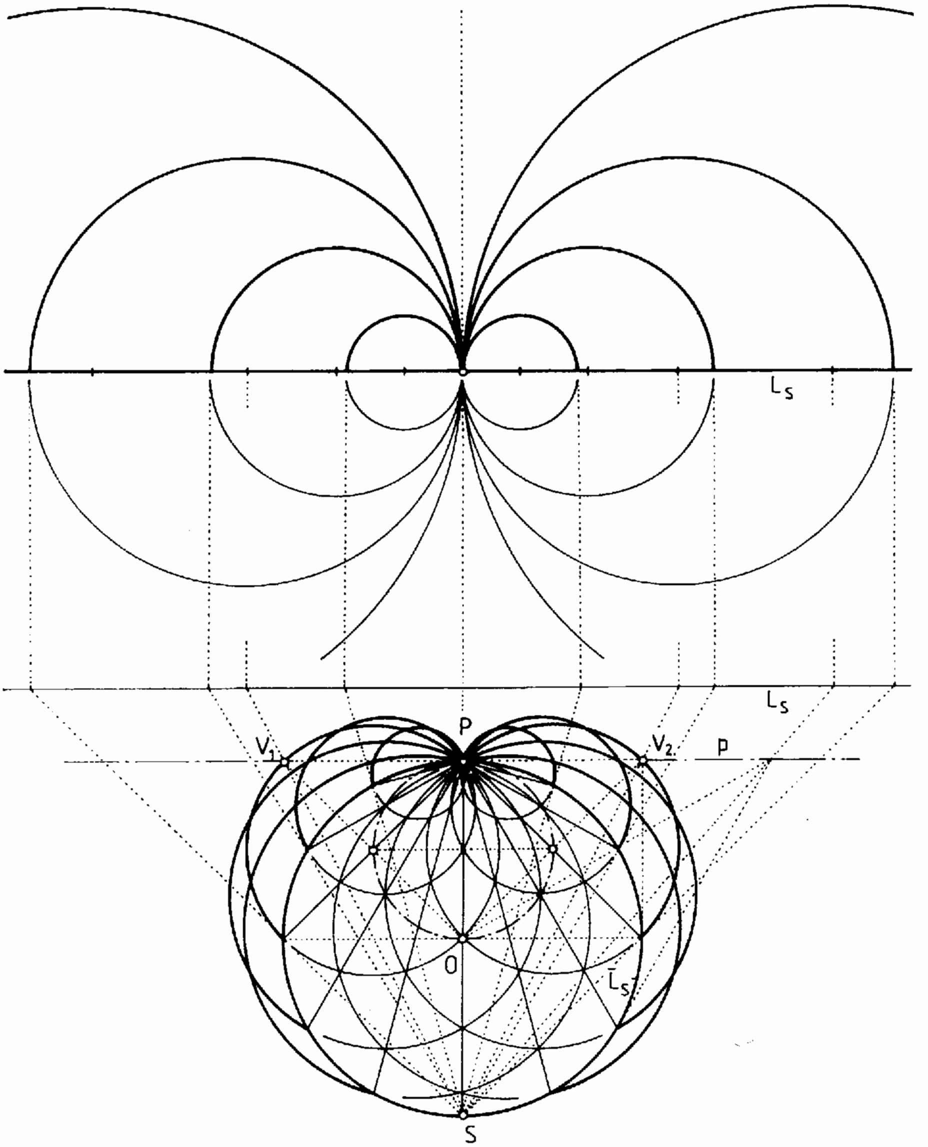

Figure 6: Parabolic pencil of spheres based on a "plane" and on a sphere

hemispherical cupolas based on the large sphere and an abstract Euclidean pencil of cupolas based on an abstract Euclidean plane. He cannot tell any difference because of the fact that in the starting fase of the cupolas constructing the difference is imperceptible, and later on, when the difference

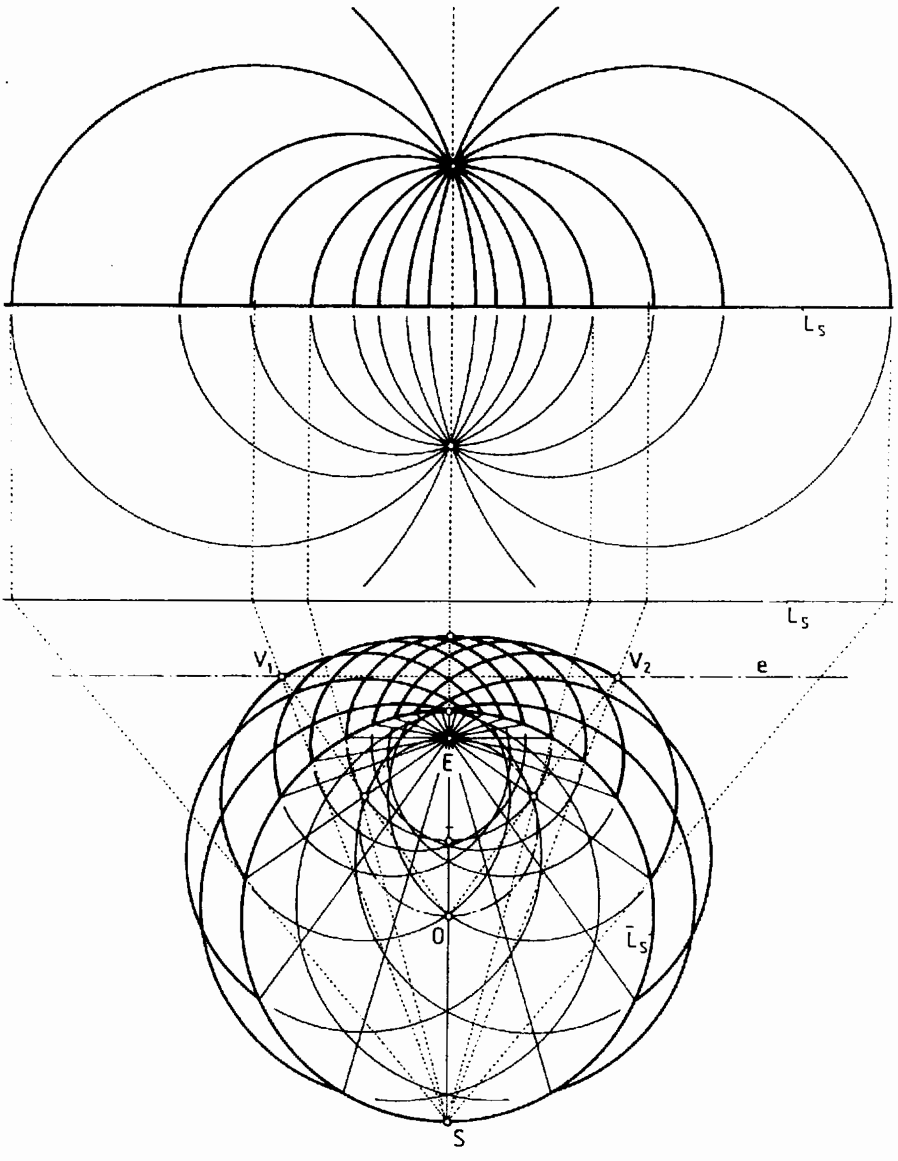

Figure 7: Elliptic pencil of spheres based on a "plane" and on a sphere

could be seen, the respective cupolas are out of his sight.

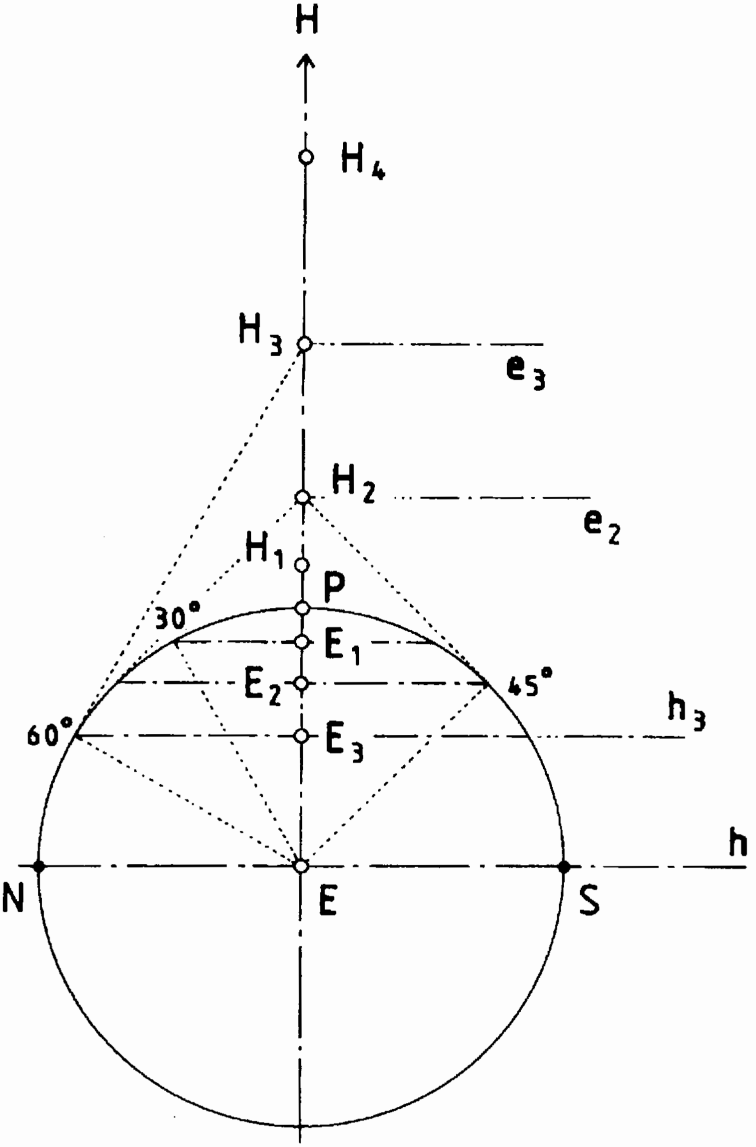

Figure 8: Poles and polars for different forms of Descartes' ovals

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