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# ON A WEAKLY SYMMETRIC RIEMANNIAN MANIFOLD ADMITTING A SPECIAL TYPE OF SEMI-SYMMETRIC METRIC CONNECTION

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**Abstract**

The object of the present paper is to study a weakly symmetric manifold admitting a special type of semi-symmetric metric connection. It is proved that if a weakly symmetric manifold admits a special type of semi-symmetric metric connection, then it is a particular kind of a weakly Ricci symmetric manifold with non-zero and non-constant scalar curvature and also it is a sub-projective manifold in the sense of Kagan.

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## 1 Introduction

The notion of a weakly symmetric manifold was introduced by L. Tamassy and T. Q. Binh ([1]). Such a manifold has been studied by T. Q. Binh ([2]), M. Prvanović ([3], [4]) and U. C. De and S. Bandyopadhyay [5].

A non-flat Riemannian manifold $(M^n, g)$ $(n > 2)$ is called weakly symmetric if its curvature tensor $R$ satisfies the condition

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(Y)R(X,Z)W \tag{1}$$

$$+C(Z)R(Y,X)W + D(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho,$$

where $\nabla$ denotes the Levi-Civita connection on $(M^n, g)$ and $A, B, C, D$ and $\rho$ are 1-forms and a vector field respectively which are non-zero simultaneously. Such a manifold has been denoted by $(WS)_n$. In [5] De and Bandyopadhyay proved the existence of a weakly symmetric manifold by an example. It was proved in [3] that these 1-forms and the vector field must be related as follows

$$B(X) = C(X) = D(X),\ g(X, \rho) = D(X), \forall X,$$

That is, the weakly symmetric manifold is characterized by the condition

$$(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + D(Y)R(X, Z)W \tag{2}$$

$$+D(Z)R(Y, X)W + D(W)R(Y, Z)X + g(R(Y, Z)W, X)\rho,$$

$$g(X, \rho) - D(X).$$

The 1-form $A$ and $D$ are the first and the second associated 1-forms respectively. Further, in a recent paper [6] Tamassy and Binh introduced the concept of weakly Ricci symmetric manifold. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called weakly symmetric if its Ricci tensor $S$ is not identically zero and if it satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + C(Z)S(Y, X) \tag{3}$$

Where $A$, $B$, $C$ and $\nabla$ have the meaning already stated. Such an n-dimensional manifold is denoted by $(WRS)_n$. From definitions it follows that, in general, a $(WS)_n$ is not necessarily a $(WRS)_n$. It is proved that a $(WS)_n(n > 3)$ admits a special type of semi-symmetric metric connection $\tilde{\nabla}$ whose torsion tensor is given by

$$T(X, Y) = D(Y)X - D(X)Y \tag{4}$$

and whose curvature tensor $\tilde{R}$ and torsion tensor $T$ satisfy the conditions

$$\tilde{R}(X, Y)Z = 0 \tag{5}$$

and

$$(\tilde{\nabla}_X T)(Y, Z) = D(X)T(Y, Z) \tag{6}$$

respectively, then a $(WS)_n$ reduces to a particular kind of a $(WRS)_n$ with non-zero and non-constant scalar curvature. It is also shown that if a $(WS)_n$ admits a type of semi-symmetric metric connection mentioned above, then the manifold is a subprojective manifold in the sense of Kagan.

## 2 Preliminaries

Let $r$ denote the scalar curvature and $L$ denote the symmetric endomorphism of the tangent space at each point of $(WS)_n$ corresponding to the Ricci tensor, i. e.

$$g(LX, Y) = S(X, Y). \tag{7}$$

for every vector fields $X, Y$.

From (2) we get

$$(\nabla_X S)(Y,Z) = A(X)S(Y.Z) + D(Y)S(X,Z) + D(Z)S(Y,X) \qquad (8)$$

$$+D(R(X,Y)Z) + D(R(X,Z)Y).$$

Contracting (8) we have

$$dr(X) = A(X)r + 4S(X,\rho). \qquad (9)$$

We shall use these formulae later.

## 3 $(WS)_n$ $(n > 3)$ admitting a special type of semi-symmetric metric connection

In this section we consider a $(WS)_n$ admitting a special type of semi-symmetric metric connection $\tilde{\nabla}$ whose torsion tensor $T$ and the curvature tensor $\tilde{R}$ satisfy the condition (5) and (6) respectively.

From (4) we have

$$\tilde{\nabla}_X Y = \nabla_X Y + D(Y)X - g(X,Y)\rho, \qquad (10)$$

where $\rho$ is given by

$$g(X,\rho) = D(X).$$

In virtue of (10) from a known result [7] we get

$$\tilde{R}(X,Y)Z = R(X,Y)Z - \alpha(Y,Z)X + \alpha(X,Z)Y \qquad (11)$$

$$-g(Y,Z)QX + g(X,Z)QY,$$

where $\alpha$ is a tensor field of type (0,2) defined by

$$\alpha(X,Y) = (\nabla_X D)(Y) - D(X)D(Y)\frac{1}{2}D(\rho)g(X,Y) \qquad (12)$$

and $Q$ is a tensor field of type (1,1) defined by

$$g(QX,Y) = \alpha(X,Y) \qquad (13)$$

for any vector fields $X, Y$.

Moreover, we have

$$(\tilde{\nabla}_X D)(Y) = (\nabla_X D)(Y) - D(X)D(Y) + D(\rho)g(X,Y). \qquad (14)$$

From (4) we get

$$(C_1^1 T)(Y) = (n-1)D(Y) \tag{15}$$

where $C_1^1$ denotes the operator of contraction.

From (15) it follows

$$(\tilde{\nabla}_X C_1^1 T)(Y) = (n-1)(\tilde{\nabla}_X D)(Y). \tag{16}$$

Again from (6) we get

$$(\tilde{\nabla} X C_1^1 T)(Y) = D(X)(C_1^1 T)(Y). \tag{17}$$

From (15) and (17) we obtain

$$(\tilde{\nabla}_X C_1^1 T)(Y) = (n-1)D(X)D(Y). \tag{18}$$

Hence from (16) and (18) we get

$$(\tilde{\nabla}_X D)(Y) = D(X)D(Y). \tag{19}$$

From (14) and (19) we get

$$(\nabla_X D)(Y) = 2D(X)D(Y) - D(\rho)g(X,Y). \tag{20}$$

Using (20), it follows from (12)

$$\alpha(X,Y) = D(X)D(Y) - \frac{1}{2}D(\rho)g(X,Y). \tag{21}$$

From (13) and (21) we get

$$QX = D(X)\rho - \frac{1}{2}D(\rho)X. \tag{22}$$

By virtue of (21) and (22), the equation (11) takes the form

$$\tilde{R}(X,Y)Z = R(X,Y)Z + D(X)[D(Z)Y - g(Y,Z)\rho] \tag{23}$$

$$+D(Y)[g(X,Z)\rho - D(Z)X] + D(\rho)[g(Y,Z)X - g(X,Z)Y].$$

Now using the condition (5), it follows from (23) that

$$R(X,Y)Z = D(X)[g(Y,Z)\rho - D(Z)Y] \tag{24}$$

$$-D(Y)[g(X,Y)\rho - D(Z)X] + D(\rho)[g(Y,Z)X - g(X,Z)Y].$$

Contracting (24) we have

$$S(Y,Z) = -(n-2)D(\rho)g(Y,Z) + (n-2)D(Y)D(Z). \qquad (25)$$

Now contracting (25) we obtain

$$r = -(n-1)(n-2)D(\rho). \qquad (26)$$

Putting $Z = \rho$ in (25) we get

$$S(Y,\rho) = 0. \qquad (27)$$

Again from (24) we have

$$R(X,Y)Z = 0.$$

Therefore

$$R(X,Y,Z,\rho) = 0,$$

where

$$R(X,Y,Z,U) = g(R(X,Y)Z,U)$$

and hence

$$D(R(X,Y)Z) = 0. \qquad (28)$$

Using (28) from (8) we get

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + D(Y)S(X,Z) + D(Z)S(Y,X). \qquad (29)$$

This shows, in virtue of (3), that the $(WS)_n$ under consideration is a particular kind of $(WRS)_n$.

Since $D(\rho) \neq 0$, it follows from (26) that $r \neq 0$.

Again in virtue of (26) it follows from (9) that $r$ can not be constant, for if $r$ is constant, then $r$ must be zero. Also from (20) we find that the 1-form $D$ is closed. Summing up we can state the following theorem:

**Theorem 1** *If a $(WS)_n$ $(n > 3)$ admits a semi-symmetric metric connection whose torsion tensor is given by (4) and whose curvature tensor $\tilde{R}$ and torsion tensor $T$ satisfy the conditions (5) and (6) respectively, then it is a particular case of $(WRS)_n$ $(n > 3)$ with non-constant and non-zero scalar curvature.*

From (9) and (27) it follows that

$$dr(X) = A(X)r. \tag{30}$$

Further, we get

$$rdA(X,Y) = 0.$$

Since $D(\rho) \neq 0$, it follows from (26) that

$$r \neq 0.$$

Therefore $A$ is closed.
From (7) and (25) it follows that

$$L(X) = -(n-2)D(\rho)X + (n-2)D(X)\rho. \tag{31}$$

Now,

$$(\nabla_X S)(Y,Z) = (\nabla_X S)(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{32}$$

Putting $Z = \rho$ In (32) and using (27) we obtain

$$(\nabla_X S)(Y,\rho) = -S(Y, \nabla_X \rho). \tag{33}$$

By virtue of (7) we get from (33)

$$D(\rho)L(X) = -L(\nabla_X \rho). \tag{34}$$

Now using (31) in (34) it follows that

$$\nabla_X \rho = D(\rho)X + \omega(X)\rho, \tag{35}$$

where

$$\omega(X) = -D(X) + \frac{D(\nabla_X \rho)}{D(\rho)}.$$

Again $D(P) \neq 0$ and $\omega$ is closed, it follows from (35) that $\rho$ is a proper concircular vector field [8]. Hence we can state the following result

**Theorem 2** *If a $(WS)_n$ $(n > 3)$ admits a semi-symmetric metric connection $\tilde{\nabla}$ satisfying (4), (5) and (6), then the vector field $\rho$ is a proper concircular vector field.*

It is known [7] that if a Riemannian manifold $(M^n, g)$ $(n > 3)$ admits a semi-symmetric metric connection whose curvature tensor vanishes, then the manifold is conformally flat. In virtue of (5) it follows that the $(WS)_n$ under consideration is conformally flat. Also it is known [9] that if a conformally flat manifold $(M^n, g)$ $(n > 3)$ admits a proper concircular vector field, then the manifold is a subprojective manifold in the sense of Kagan. Since the $(WS)_n$ under consideration is conformally flat and admits a proper concircular vector field, namely vector field $\rho$, we can state as follows:

**Theorem 3** *If a $(WS)_n$ admits a semi-symmetric metric connection $\tilde{\nabla}$ satisfying (4), (5) and (6), then the manifold is a subprojective manifold in the sense of Kagan.*

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