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AREA SWEPT BY LINE SEGMENT UNDER A PLANAR MOTION

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Abstract

Formula for the area swept by line segment under a planar motion in the Euclidean plane is presented.

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Let $\mathcal{M}$ be any motion of Euclidean plane. Then the motion $\mathcal{M}$ can be obtained as a rolling of some curve c along a fixed curve l. For the curve l we can take the polode of trajectory of any point.

A roulette $\mathcal{R}(c, l, P)$ is a curve traced out by a point P which is in fixed position with respect to a rolling curve c, and which rolls without slipping along a fixed base curve l. Area bounded by the roulette and base curve, area $\mathcal{R}(c, l, P)$ in some cases is minimal if the tracing point is barycenter B of curve c (which is) pondered by a difference of curvature measures $\kappa_c - \kappa_l$.

Let $c = \widehat{C_1C_2}$ and $l = \widehat{L_1L_2}$ be oriented piecewise regular curves with parametric equations $r(t) = (x(t), y(t))$, where $a \leq t \leq b$, and $\rho(t) = (u(t), v(t))$, where $c \leq t \leq d$, respectively. Tangent vector $\dot{r}(t) = (\dot{x}(t), \dot{y}(t))$ at regular point $X(t)$ is non-zero. A parametric representation is regular if vector function r is of class C^1 and $\dot{r}(t) \neq 0$ for all $a \leq t \leq b$. Tangent vectors can be translated so that they have same origin. The total curvature κ_c^T is difference $\theta_1 - \theta_0$ in the values of inclination θ of the tangent to the curve at the end points of the curve. For closed curve c ($C_1 = C_2$) it holds $\kappa_c^T = 2z\pi$, z is an integer. Natural parameter is length of arc $s = \int |\dot{r}(t)|\, dt$; then tangent vectors have length 1. If c is smooth in some neighborhood of point X, then curvature of curve c at point X is

$$\kappa(X) = \lim \frac{\Delta\alpha}{\Delta s} = \frac{d\alpha}{ds}, \qquad \Delta s \to 0, \qquad Y \to X,$$

$\Delta\alpha$ is angle between tangent vectors at X and Y, Δs is length of arc $\widehat{XY}$. Classical formula for the curvature is

$$\kappa(X) = \frac{\dot{x}(t)\ddot{y}(t) - \ddot{x}(t)\dot{y}(t)}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{3/2}}.$$

At singular point S the point curvature is $\kappa(S) = \angle(\dot{r}_-(S), \dot{r}_+(S))$, so that $-\pi \leq \kappa(S) \leq \pi$. We take $\kappa(S) = \pm\pi$ if $\lim \angle(\dot{r}_-(S), \widehat{SX}) = \pm\pi$, $X \to S$, $x \in \widehat{SC_2}$. If the whole arc $\widehat{SC_2}$ is straight segment opposite to $\dot{r}_-(s)$, we can choose either π or $-\pi$. Mention that curvature κ as function of length of arc gives natural equation of the curve. If two curves have equal curvatures as functions of length, then they can coincidence by motion.

Definition 1 *Curvature measure κ on curve c is defined on regular segments by the curvature functional $\kappa(t)$, and at singular points the point-measure is equal to point-curvature.*

Definition 2 *Steiner point B of curve c is the barycenter of curve c which is pondered by the curvature measure κ.*

Theorem 1 ([1]) *Let B be a barycenter of curve c by difference of curvature measures $\kappa_c - \kappa_l$ pondered, and let κ^T be total curvature. Then, area bounded by a roulette and base curve l is*

$$\begin{aligned} area\,\mathcal{R}(c,l,P) &= area\ cone\,Pc + \frac{1}{2}\int_c PX^2\, d(\kappa_c - \kappa_l), \qquad X \in c \\ &= area\,\mathcal{R}(c,l,B) + \frac{\kappa_c^T - \kappa_l^T}{2} BP^2 + area\,\Box PC_1BC_2. \end{aligned}$$

For closed rolling curve area cone Pc = area c and area $\Box PC_1BC_2 = 0$.

Theorem 2 ([1]) *Let $area\,\mathcal{R}(c,l,PQ)$ be the area traced by a segment PQ in fixed position with respect to the rolling curve c, i.e., the area between two roulettes traced by points P and Q which ends are connected by straight segments. Then*

$$area\,\mathcal{R}(c,l,PQ) = \frac{1}{2}(\kappa_c^T - \kappa_l^T)(BP^2 - BQ^2).$$

Specially,

$$area\,\mathcal{R}(c,l,BP) = \frac{1}{2}(\kappa_c^T - \kappa_l^T)BP^2.$$

Theorem 3 *Let $\mathcal{M}$ be a motion of Euclidean plane which polode contains only finite points, and let ρ be the total angle of rotation of $\mathcal{M}$. Then there exists a point J such that the area swept by any segment PQ under the motion $\mathcal{M}$ is*

$$\mathit{area}\,\mathcal{M}(PO) = \frac{\rho}{2}(JP^2 - JQ^2).$$

Proof. Let the motion $\mathcal{M}$ can be presented as a rolling of some curve c along the polode l. Also, take that J is Steiner point of the system c and l. Then Theorem 3 follows from Theorem 2. □

References

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