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LOOP GROUP DECOMPOSITIONS IN THE GENERALIZED DPW METHOD *

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Abstract

The DPW method determines classes of surfaces with special features (minimal surfaces, surfaces of constant mean or Gaussian curvature and Willmore surfaces), by means of loop groups. The harmonicity of special maps associated to these surfaces is characterized in terms of meromorphic potentials of a certain shape. The link between these concepts is performed using the Birkhoff and Iwasawa decomposition for loop groups. The paper presents the main features of the method and describes explicitly these decompositions in an extended framework which includes the non-semisimple case.

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1 The proper DPW method

Among the methods which determine harmonic maps from Riemannian surfaces to symmetric spaces or to Lie groups, one of the most recent is the DPW method [8], which proved to be useful in determining classes of CMC-surfaces, minimal surfaces, surfaces of constant Gaussian curvature and Willmore surfaces.

The primary objects of the DPW method are

• a Riemannian compact simply connected surface M of genus $g \geq 1$ and $\mathbf{D} \in \{\mathbf{C}, D^1\}$ its universal cover, where D is the open unit complex disk;

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• a compact connected semisimple Lie group G having the Lie algebra $L(G) = \mathbf{g}$;

• an involution $\sigma \in Aut(G)$ with the fixed point set $G^\sigma \supset K \supset G_0^\sigma$, $\mathbf{g}^\sigma = L(K) \stackrel{n}{=} \mathbf{k}$ (we denoted the induced map of σ on $\mathbf{g}$ by the same symbol);

• a solvable subgroup $B \subset K^{\mathbb{C}}$ which provides an Iwasawa decomposition for the complexified group $K^{\mathbb{C}}$,

$$K^{\mathbb{C}} = K \cdot B, \qquad K \cap B = \{e\}, \tag{1}$$

and the corresponding splitting of the associated Lie algebras

$$\mathbf{k}^{\mathbb{C}} = \mathbf{k} \oplus \mathbf{b}, \qquad \mathbf{b} = L(B). \tag{2}$$

The DPW method determines classes of harmonic maps

$$f : M \to N = G/K = \pi(G),$$

where π is the projection $\pi : G \to G/K$ as described in the following.

The Cartan decomposition

$$\mathbf{g} = \mathbf{k} \oplus \mathbf{p}, \qquad \mathbf{p} = Ker(\sigma + Id), \mathbf{k} = Ker(\sigma - Id), \tag{3}$$

provides the essential relations $[\mathbf{k}, \mathbf{k}] \subset \mathbf{k}, [\mathbf{k}, \mathbf{p}] \subset \mathbf{p}, [\mathbf{p}, \mathbf{p}] \subset \mathbf{k}$.

Assuming M simply connected surface, any map $f : M \to G/K$ provides canonically a lift to $F : M \to G, f = \pi \circ F$, and the $\mathbf{g}$-valued 1-form $\alpha = F^{-1}dF \in \Lambda^1(M, \mathbf{g})$ splits relative to (3), $\alpha = \alpha_0 + \alpha_1$; also, the splitting $TM^{\mathbb{C}} = T'M \oplus T''M$ of $TM^{\mathbb{C}}$ into its (1,0) and (0,1) subspaces, induces the decompositions $d = \partial + \bar{\partial}$, $\alpha_i = {\alpha'}_i + {\alpha''}_i$, $i = \overline{0, 1}$, and hence $\alpha = \alpha'_1 + \alpha_0 + \alpha''_1$.

The harmonicity of f and the Maurer-Cartan equations for α provide the system

$$\begin{cases} d\alpha_0 + \frac{1}{2}[\alpha_0 \wedge \alpha_0] = -[\alpha'_1 \wedge \alpha''_1] \\ \bar{\partial}\alpha'_1 + \frac{1}{2}[\alpha_0 \wedge \alpha'_1] = 0 \end{cases} \tag{4}$$

which are iff conditions for the existence and harmonicity of the function f, constructed as the projection of the lifted frame F provided by the pair of forms $\alpha_0 \in \Lambda^1(M, \mathbf{k})$ and $\alpha_1 \in \Lambda^1(M, \mathbf{p})$ by integration. The system (4) is equivalent to the integrability conditions $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$ of the "loopified" form

$$\alpha_\lambda = \lambda^{-1}\alpha'_1 + \alpha_0 + \lambda\alpha''_1,$$

whose associated system $\tilde{F}^{-1}d\tilde{F} = \alpha_\lambda$ integrates to the extended lift $\tilde{F}: M \to \Lambda G_\sigma$, unique up to a gauge transformation $H: M \to K$, where we denoted

$$\Lambda G_\sigma \stackrel{n}{=} \{h \mid h: S^1 \to G, h(e^{2\pi i/m}\lambda) = \sigma h(\lambda), \forall \lambda \in S^1 \equiv U(1)\}.$$

For M simply connected, assuming w.r.g. $M = \mathbf{D}$, the harmonicity of f is equivalent to the existence of a holomorphic map $\tilde{f}: M \to \Lambda G_\sigma/K$ which is provided by a ΛG_σ-translation of a 1-form

$$\theta_\eta = \lambda^{-1}\eta + \lambda\bar{\eta} \in \Lambda \mathbf{p}^{\mathbb{C}};$$

this map is related to f via $\tilde{f}\mid_{\lambda=1} = f$. *The whole family of maps*

$$\tilde{f}_\lambda \stackrel{n}{=} \tilde{f}(\cdot)(\lambda): M \to G/K, \ \forall \lambda \in S^1$$

obtained from such holomorphic forms are harmonic [8].

The procedure which constructs (mod singularities), the harmonic functions f from $\mathbf{p}^{\mathbb{C}}$-valued holomorphic 1-forms is called *the Weierstrass representation of harmonic maps* and is described in [8, 1].

A central question is to obtain the exact form of meromorphic potentials which provide the extended frames $\tilde{F}(z, \bar{z}, \lambda)$. This goal is accomplished by solving "the $\bar{\partial}$-problem" [3] and applying a generalization of the Grauert theorem, to obtain the global holomorphic loop $\tilde{g}$ given by the relation

$$\tilde{F} = \tilde{g}w_+^{-1}.$$

Then the holomorphic potential is $\xi = \tilde{g}^{-1}d\tilde{g}$, and the meromorphic potential is $\xi = \tilde{g}_-^{-1}d\tilde{g}_-$ - obtained from the negative loop $\tilde{g}_-$ given by the further Birkhoff decomposition of $\tilde{g}$

$$\tilde{g} = \tilde{g}_-\tilde{g}_+$$

where $\tilde{g}_\pm \in \Lambda^\pm G_\sigma^{\mathbb{C}}$ and

$$\Lambda^+ G_\sigma{}^{\mathbb{C}} = \{g \in \Lambda G_\sigma{}^{\mathbb{C}} \mid g(0) = e, \ g \text{ extends holomorphically to } D^1\},$$
$$\Lambda^- G_\sigma{}^{\mathbb{C}} = \{g \in \Lambda G_\sigma{}^{\mathbb{C}} \mid g(\infty) = e, \ g \text{ extends holomorphically to } \mathbf{C} \setminus D^1\}.$$

For example, for determining classes of minimal surfaces, the DPW method determines the meromorphic potentials associated to the Gauss map - which is holomorphic (and hence harmonic) and provides via the

Weierstrass representation the concrete surface. Being valued in $S^2 \equiv SO(3)/SO(2) \equiv G/K$, it provides a loopified frame valued in the complex universal cover $G^{\mathbb{C}} = SL(2, \mathbf{C})$; For such loops, the decompositions can be carried on for providing the meromorphic potential from the extended frame, and then recapturing a whole family of frames, after applying the dressing procedure.

The same procedure works for CMC-surfaces, with the difference that generally the Gauss map is harmonic, but not necessarily holomorphic.

For Willmore surfaces, the conformal Gauss map is valued in

$$G/K \equiv SO(4,1)/SO(3,1)$$

and here $G^{\mathbb{C}} = SO(5, \mathbf{C})$; the procedure provides directly the immersion of the surface in $\mathbf{R}^3, S^3 or H^3$ [12].

In the following we present briefly a series of results of the extended DPW framework, in which the same techniques are used to study the harmonicity of maps valued to general Lie groups. Namely, we characterize the harmonicity of these maps in terms of conditions on the associated Maurer-Cartan forms, and provide natural extensions of the Birkhoff and Iwasawa decompositions for loop groups, in the non-twisted case (similar to the twisted case of the proper DPW method).

2 Harmonic maps

Let a Lie group G endowed with a left invariant pseudoriemannian metric g, and let $\varphi : \mathbf{D} \to G$ a weakly conformal immersion [23]. Then the immersion φ is harmonic, if it is a critical point for the *the energy* functional is [10, 11, 23]

$$E(\varphi) = \int_{\mathbf{D}} \frac{1}{2} g_{ij} \frac{\mathrm{d}\varphi^i}{\partial x^a} \frac{\partial \varphi^j}{\partial x^b} \gamma^{ab} \mathrm{d\ vol},$$

where (γ^{ab}) is the inverse of $\gamma = \varphi^* g$. The energy $E(\varphi)$ is conformally invariant (e.g., [23]), and can be rewritten

$$E(\varphi) = \frac{1}{2} \int_{\mathbf{D}} (\mid A_{(x)} \mid^2 + \mid A_{(y)} \mid^2) dx dy, \tag{5}$$

where

$$A_{(x)} = \varphi^{-1} \varphi_x,\ A_{(y)} = \varphi^{-1} \varphi_y \in \Lambda^1(\mathbf{D}, \mathbf{g}). \tag{6}$$

Here we use $\mathbf{g} = Lie(G)$ and the subscripts x, y, xx, etc denote the partial differentiation with respect to the corresponding variable(s).

The harmonicity (Euler-Lagrange) equations for φ, are provided by

Theorem. *Let G be a Lie group admitting a left-invariant pseudoriemannian metric g. Then the map $\varphi : \mathbf{D} \to G$ is harmonic iff the associated vector-valued 1-forms $A_{(x)}$ and $A_{(y)}$ (6) satisfy the equation*

$$(adA_{(x)})^* A_{(x)} + (adA_{(y)})^* A_{(y)} - (\partial_x A_{(x)} + \partial_y A_{(y)}) = 0, \tag{7}$$

where the star superscript indicates the adjoint w.r.t. the nondegenerate bilinear form induced by g on the Lie algebra $\mathbf{g}$.

Corollary. *If the metric g is bi-invariant, then the harmonicity condition becomes:*

$$\partial_x A_{(x)} + \partial_y A_{(y)} = 0, \tag{8}$$

with $A_{(x)}$, $A_{(y)}$ defined in (6).

For complex coordinates $z = x + iy, \bar{z} = x - iy$ on $\mathbf{D}$, the equations (8) rewrite

$$\partial_{\bar{z}} A_{(z)} + \partial_z A_{(\bar{z})} = 0, \tag{9}$$

where $\partial_z = (\partial_x - i\partial_y)/2$, $\partial_{\bar{z}} = (\partial_x + i\partial_y)/2$, and $A_{(z)} = \varphi^{-1}\varphi_z$, $A_{(\bar{z})} = \varphi^{-1}\varphi_{\bar{z}}$.

We can characterize the harmonicity of the map $\varphi : \mathbf{D} \to G$, in terms of its associated Maurer-Cartan form

$$\alpha \equiv \varphi^{-1}\mathrm{d}\varphi : \mathbf{D} \to \mathbf{g} \equiv Lie(G), \alpha \in \Lambda^1(\mathbf{D}, \mathbf{g})$$

as follows:

Proposition. *The following statements are equivalent:*

a) The map φ is harmonic.

b) The form α satisfies the integrability and harmonicity equations:

$$\begin{cases} \mathrm{d}\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0 \\ \partial_{\bar{z}}\alpha' + \partial_z \alpha'' = 0. \end{cases} \tag{10}$$

c) The "loopified form"

$$\alpha_\lambda = \frac{1 + \lambda^{-1}}{2}\alpha' + \frac{1 + \lambda}{2}\alpha'' \in \Lambda^1(\mathbf{D}, \mathbf{g}^{\mathbb{C}}), \forall \lambda \in S^1$$

is integrable for $\forall \lambda \in S^1$, i.e., it satisfies the integrability condition:

$$\mathrm{d}\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0,$$

where $\mathrm{d} = \partial_z + \partial_{\bar{z}}$ and the form α splits canonically

$$\alpha \equiv \alpha' + \alpha'' \equiv A_{(z)}\mathrm{d}z + A_{(\bar{z})}\mathrm{d}\bar{z} : T\mathbf{D}^{1,0} \oplus T\mathbf{D}^{0,1} \to \mathbf{g}^{\mathbb{C}}, \forall \lambda \in S^1.$$

3 Loop group decompositions

Let G be a simply connected real analytic Lie group which admits a faithful finite dimensional continuous representation. Then, the Levi theorem [15, 24] provides the following subgroups: • H - a reductive, analytic subgroup of G, and

• B - a simply connected solvable normal subgroup of G,
such that G can be written as a semidirect product

$$G = H \circledS B.$$

The complexified groups, $B^{\mathbb{C}}$ and $H^{\mathbb{C}}$ inherit the properties of B and H , hence $H^{\mathbb{C}}$ is a reductive complex group, and $B^{\mathbb{C}}$ is also simply connected, solvable normal subgroup of $G^{\mathbb{C}}$. Then $G^{\mathbb{C}}$ admits a faithful finite-dimensional complex representation and satisfies

$$G^{\mathbb{C}} = H^{\mathbb{C}} \circledS B^{\mathbb{C}}. \tag{11}$$

Being complex and reductive, $H^{\mathbb{C}}$ is the complexification of a (maximal) real compact subgroup K of $H^{\mathbb{C}}$ ($H^{\mathbb{C}} = K^{\mathbb{C}}$). On the other hand, being solvable and simply connected, the group $B^{\mathbb{C}}$ decomposes canonically

$$B^{\mathbb{C}} = A^{\mathbb{C}} N^{\mathbb{C}} \tag{12}$$

where $A^{\mathbb{C}}$ is abelian and $N^{\mathbb{C}} = [B^{\mathbb{C}}, B^{\mathbb{C}}]$ is simply connected and is the nilradical of $B^{\mathbb{C}}$ [2]. $N^{\mathbb{C}}$ is closed in $B^{\mathbb{C}}$ and has the Lie algebra $Lie(N^{\mathbb{C}}) = [Lie(B^{\mathbb{C}}), Lie(B^{\mathbb{C}})]$.

We remark that

$$B^{\mathbb{C}}/N^{\mathbb{C}} \cong A^{\mathbb{C}} \cong V \times W, \tag{13}$$

where $V \cong \mathbf{C}^k = (\mathbf{R}^k)^{\mathbb{C}}$ has an abelian additive Lie algebra, and W is a complex torus which has a multiplicative abelian Lie algebra.

Let ρ be a finite-dimensional faithful representation of $G^{\mathbb{C}}$ and for $g : S^1 \to G^{\mathbb{C}}$ smooth, let $\tilde{g} = \sum_{k \in \mathbf{Z}} a_k \lambda^k, \quad a_k \in \rho(G^{\mathbb{C}}), \lambda \in S^1$ be the associated Fourier series. Then the loop group $\Lambda G^{\mathbb{C}}$ defined by

$$\Lambda G^{\mathbb{C}} = \{ g : S^1 \to G^{\mathbb{C}} \mid \tilde{g} \text{ is absolutely convergent} \}$$

can be organized as a complex Banach Lie group with the Wiener topology [17].

We shall use also its subgroups

$\Lambda G = \{g \in \Lambda G^{\mathbb{C}} \mid g(\lambda) \in G, \forall \lambda \in S^1\}$
$\Lambda^+ G^{\mathbb{C}} = \{g \in \Lambda G^{\mathbb{C}} \mid g \text{ extends holomorphically to} \mathbf{D}\}$,
$\Lambda^- G^{\mathbb{C}} = \{g \in \Lambda G^{\mathbb{C}} \mid g \text{ extends holomorphically to } \mathbf{C} \setminus \mathbf{D}\}$,
where $\mathbf{D}$ is the open unit complex disk and $e \in G$ is the unity of the group G.

Let $N^{\mathbb{C}}$ be the nilradical of $B^{\mathbb{C}}$; then $N^{\mathbb{C}} \triangleleft G^{\mathbb{C}}$. We assume w.r.g. that in the given representation, $\Lambda N^{\mathbb{C}}$ is provided by upper triangular matrices with ones on the diagonal.

Under the assumptions on G stated above, the Birkhoff decomposition for the loop group $\Lambda G^{\mathbb{C}}$ is provided by [5]

Theorem. *a) Any element $g \in \Lambda G^{\mathbb{C}}$ can be written as*

$$g = g_- D g_+ \tag{14}$$

where $g_\pm \in \Lambda^\pm G^{\mathbb{C}}$, $D = s b_-^+ d \in \Lambda^d G^{\mathbb{C}} \times (\Lambda^- B^{\mathbb{C}})_s^+ \times \Lambda^d B^{\mathbb{C}}$ and

$$\Lambda^d G^{\mathbb{C}} = \{ \textit{ upper-diagonal polynomial loops in } \Lambda G^{\mathbb{C}}\},$$

$$(\Lambda^- B^{\mathbb{C}})_s^\pm = \{b \in \Lambda^- B^{\mathbb{C}} \mid sbs^{-1} \in \Lambda^\pm B^{\mathbb{C}}\}.$$

The expression (14) will be called canonic Birkhoff decomposition, *with $g_\pm$ having a specific form [5].*

b) The "big cell" $P_{G^{\mathbb{C}}}$ is open and dense in $\Lambda G^{\mathbb{C}}$, and the mapping

$$\Lambda^- G^{\mathbb{C}} \cdot \Lambda^+ G^{\mathbb{C}} \to \Lambda G^{\mathbb{C}}$$

provides a surjective submersion onto the big cell.

Let G be a connected, simply connected real analytic Lie group which admits a faithful finite dimensional continuous representation.

We have the decomposition of its reductive subgroup H as the product of a semisimple Lie group S and a compact abelian Lie group (complex torus) K, $H = S \cdot K$, which leads to the decomposition $H^{\mathbb{C}} = S^{\mathbb{C}} \cdot K^{\mathbb{C}}$. Then, regarding the loop group $\Lambda H^{\mathbb{C}}$, the Iwasawa decomposition is provided by the following generalisation of a result of P.Kellersch [18],

Theorem. *If H is connected, then*

$$\Lambda H^{\mathbb{C}} = \Lambda H \cdot \Lambda^m H \cdot \Lambda^+ H^{\mathbb{C}}, \tag{15}$$

where: $\Lambda H = \Lambda S \cdot \Lambda K$, $\Lambda^m H = \Lambda^m S$, $\Lambda^+ H^{\mathbb{C}} = \Lambda^+ S^{\mathbb{C}} \cdot \Lambda^+ K^{\mathbb{C}}$, and

$$\Lambda^m S \subset \{g \in \Lambda S \mid g \textit{ is a finite-series diagonal loop}\}$$

consisting of loops valued in the Cartan subgroup of S.
Using this result, we get the following theorem which describes the Iwasawa decomposition for an arbitrary loop group $\Lambda G^{\mathbb{C}}$:

Theorem. *a) Let G be a connected, simply connected Lie group, which admits a finite-dimensional faithful representation. Then*

$$\Lambda G^{\mathbb{C}} = \Lambda G \cdot \Lambda^m G^{\mathbb{C}} \cdot \Lambda^+ G^{\mathbb{C}}, \quad (16)$$

where the double cosets are indexed by the middle terms $\Lambda^m G^{\mathbb{C}} = \bigcup_{s \in \Lambda^m S} (\Lambda^+ S^{\mathbb{C}})_s^-$.

b) The "big cell" $\tilde{P} = \Lambda\ G \cdot \Lambda^+ G^{\mathbb{C}}$ *of the decomposition is open in* $\Lambda G^{\mathbb{C}}$. *Moreover, for S the compact real form of* $S^{\mathbb{C}}$, *we have*

$$\tilde{P} \equiv \Lambda G \cdot \Lambda^+ G^{\mathbb{C}} = \Lambda G^{\mathbb{C}}. \quad (17)$$

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