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ON VARIATION OF THE VOLUME UNDER INFINITESIMAL BENDING OF A CLOSED ROTATIONAL SURFACE *

Ljubica S. Velimirović
Prirodno-matematički fakultet, Univerzitet u Nišu
Ćirila i Metodija 2, 18000 Niš, Yugoslavia
Email address: vljubica@ziux.grafak.ni.ac.yu

Abstract

I. K. Sabitov [6] made conjecture, that for any infinitesimal C^1 smooth bending of a closed surface the volume, bounded by it, will be stationary, i.e. that the variation of this volume is zero. V. A. Aleksandrov has proved in [1] that the mentioned conjecture is true for the rotational surface of the type 0 or 1 with C^1-smooth meridian not containing a segment perpendicular to the axis of rotation.

At the end of the work [1] the following question is formulated: Is the volume, bounded by a piecewise-smooth surface of rotation, always stationary under its infinitesimal bendings?

In the present work we prove that the answer to this question is positive. Besides, by a direct calculation of the volume variation, we confirm this in the case of Belov's rotational toroid [2].

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1 Introduction

In [1] V. A. Aleksandrov considered I. K. Sabitov's conjecture, given in a note of the editor of the translation ([6], p. 231), that suggested the considering of the Connelly-Sullivan's conjecture at the level of infinitesimal bending for rotational surfaces.

At the starting point V. A. Aleksandrov made a connection between the variation of volume and flow. He has proved the next lemma:

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Lemma (V. A. Aleksandrov [1]) *The variation of volume of domain D_0 under the action of the field $\bar{z}$ is equal to the flux of the field $\bar{z}$ through the outer side of the surface S_0, i.e.,*

$$\delta V(D_0, \bar{z}) = \int\int_{S_0} \bar{z}\bar{n}_1 dS, \tag{1.1}$$

where $\bar{n}_1$ is the outer normal to the surface S_0, and $\bar{z}\bar{n}_1$ is the scalar product of the vectors $\bar{z}$ and $\bar{n}_1$.

Using this lemma, V. A. Aleksandrov has proved the next theorem.

Theorem (V. A. Aleksandrov [1]) *Let the meridian of a connected closed surface of rotation be a C^1-smooth curve and not contain a segment perpendicular to the axis of rotation. Then the flux through the given surface of any C^1-smooth bending field is equal to zero.*

The hypotheses of this theorem satisfy nonrigid surfaces topologically equivalent to a sphere and to a torus.

2 Stationary state of the volume bounded by a closed rotational surface generated by piecewise smooth meridian

V. A. Aleksandrov [1] finished his work with an open question: Is the volume bounded by a piecewise-smooth surface of rotation always stationary under infinitesimal bendings?

In the present paper we are proving the following theorem:

Theorem. *Let the meridian of the closed connected rotational surface be a piecewise smooth curve, not containing a segment perpendicular to the axis of rotation. Then the variation of the volume, bounded by this surface, is equal to zero, under the C^1-smooth field of infinitesimal bending.*

Proof. Let $\rho = \rho(u)$be a meridian, rotating around u-axis at Cartesian coordinate system $uO\rho$. The equation of the rotational surface S_0, at special coordinate system with trihedron $\bar{e}, \bar{a}(v), \bar{a}'(v)$ (see [3]) is

$$S_0 : \bar{r}(u, v) = u\bar{e} + \rho(u)\bar{a}(v). \tag{2.1}$$

Let us suppose that meridian generating surface S_0 is piecewise smooth curve

$$C_0 = \bigcup_{j=1}^{n} C_j, \tag{2.2}$$

where

$$C_j : \rho = \rho_{(j)}(u), \quad j = 1, ..., n; \quad u \in [u_j, u_{j+1}] \subset [\alpha, \beta]. \tag{2.3}$$

At the breaking points $u = \sigma$ of the meridian $\rho = \rho(u)$, i.e. at the points where meridian is continuous, but not smooth curve, we have

$$\rho_{(j)}(\sigma) = \rho_{(j+1)}(\sigma), \quad {\rho'}_{(j)}(\sigma) \neq {\rho'}_{(j+1)}(\sigma). \tag{2.4}$$

Infinitesimal bending field $\bar{z}(u, v)$ can be considered as ([3], p. 91, 92):

$$\bar{z} = \sum_{k=2}^{m} a_k \bar{z}_k, \quad a_k = const., \quad m = 2, 3, ..., \tag{2.5}$$

where

$$\bar{z}_k(u, v) = [\varphi_k(u)e^{ikv} + \tilde{\varphi}_k(u)e^{-ikv}]\bar{e} + [\psi_k(u)e^{ikv} + \tilde{\psi}_k(u)e^{-ikv}]\bar{a}(v)$$

$$+[\chi_k(u)e^{ikv} + \tilde{\chi}_k(u)e^{-ikv}]\bar{a}'(v), \tag{2.6}$$

and $\tilde{\chi}_k(u)$ is conjugated value for $\chi_k(u)$ etc.

Infinitesimal bending field $\bar{z}(u, v)$ must be a continuous vector field on the whole surface, including the circles described by the breaking points $u = \sigma$ of the meridian. From (2.5,6) we have

(2.7)

$$\varphi_{k(j)}(\sigma) = \varphi_{k(j+1)}(\sigma), \quad \psi_{k(j)}(\sigma) = \psi_{k(j+1)}(\sigma), \quad \chi_{k(j)}(\sigma) = \chi_{k(j+1)}(\sigma).$$

Surface S_0 consists of smooth parts $S_o^{(j)}$:

$$S_0 = \bigcup_{j=1}^{n} S_0^{(j)}. \tag{2.8}$$

According to Lemma of V. A. Aleksandrov, variation of the volume is flux, calculated as surface integral. Based on the characteristic of the surface integral, according to (1.1), we have:

$$\delta V = \int\int_{S_0} \bar{z}\bar{n}_1 dS = \sum_{j=1}^{n} \int\int_{S_0^{(j)}} \bar{z}\bar{n}_1 dS. \tag{2.9}$$

We transform surface integrals to doubles:

$$\bar{n}_1 = \frac{\bar{r}_u \times \bar{r}_v}{\sqrt{EG - F^2}}, \quad dS = \sqrt{EG - F^2}dudv,$$

(2.10)
$$\delta V = \sum_{j=1}^{n} \iint\limits_{A_0^{(j)}} (\bar{z}, \bar{r}_u, \bar{r}_v)dudv,$$

where $(\bar{z}, \bar{r}_u, \bar{r}_v)$ is the scalar triple product of the vectors $\bar{z}, \bar{r}_u, \bar{r}_v$

(2.11)
$$A_0^{(j)} = [u_j, u_{j+1}] \times [0, 2\pi] \subset [\alpha, \beta] \times [0, 2\pi],$$

$A_0^{(j)}$ is a part of u, v- plane, corresponding to the part $S_o^{(j)}$ of the surface. Using (2.5), we have

(2.12)
$$\delta V = \sum_{j=1}^{n} \sum_{k=2}^{m} a_k \iint\limits_{A_0^{(j)}} (\bar{z}_k, \bar{r}_u, \bar{r}_v)dudv.$$

According to (2.6) and (2.1) we have

$$\begin{aligned}(\bar{z}_k, \bar{r}_u, \bar{r}_v) &= e^{ikv}[\rho(u)\rho'(u)\varphi_k(u) - \rho(u)\psi_k(u)] \\ &+ e^{-ikv}[\rho(u)\rho'(u)\widetilde{\varphi}_k(u) - \rho(u)\widetilde{\psi}_k(u)]\end{aligned}$$

and then

$$\begin{aligned}\iint\limits_{A_0^{(j)}} (\bar{z}_k, \bar{r}_u, \bar{r}_v)dudv &= \int_0^{2\pi} dv \int_{u_j}^{u_{j+1}} \{e^{ikv}\rho(u)[\rho'(u)\varphi_k(u) - \psi_k(u)] \\ &+e^{-ikv}\rho(u)[\rho'(u)\widetilde{\varphi}_k(u) - \widetilde{\psi}_k(u)]\}du \\ &= \int_0^{2\pi} e^{ikv}dv \int_{u_j}^{u_{j+1}} \rho(u)[\rho'(u)\varphi_k(u) - \psi_k(u)]du \\ &+ \int_0^{2\pi} e^{-ikv}dv \int_{u_j}^{u_{j+1}} \rho(u)[\rho'(u)\widetilde{\varphi}_k(u) - \widetilde{\psi}_k(u)]du = 0.\end{aligned}$$

The last equality holds as

$$\int_o^{2\pi} e^{ikv}dv = \int_o^{2\pi} e^{-ikv}dv = 0.$$

Substituting at (2.12) we get

$$\delta V(D_0, \bar{z}) = 0$$

and the theorem is proved.

3 An example of direct calculation of variation of the volume

K. M. Belov [2] considered toroid rotational surface generated by quadrangular meridian $\mathcal{B}$ at $uO\rho$-plane:

$$\mathcal{B}: \quad A(-1,b), \quad B(0,b+c_1), \quad C(1,b), \quad D(0,b-c_2). \tag{3.1}$$

Necessary and sufficient condition for this surface to be nonrigid is:

$$1/c_2 - 1/c_1 = k^2/b, \quad (k \in N, k \geq 2). \tag{3.2}$$

According to the previous Theorem variation of the volume bounded by the Belov's surface is zero. We shall here prove this fact in an another way, by direct calculation of the variation of the volume. For that purpose we shall find the equation of deformed surface under the infinitesimal bendings. Let us consider rotational surface S at coordinate system with base $\bar{e}$, $\bar{a}\,(v)$, $\bar{a}'\,(v)$. The equation of the surface S is given by (2.1), and deformed surface S_ε, with infinitesimal bending field $\bar{z}(u,v)$

$$S_\varepsilon : \bar{r}(u,v,\varepsilon) = \bar{r}(u,v) + \varepsilon \bar{z}(u,v). \tag{3.3}$$

The field $\bar{z}_k(u,v)$ (2.6), with every $k \in \{2,3,...\}$, can be considered as a bending field (see [3]). The equation (2.6) can be written as

$$\begin{aligned} \bar{z}(u,v) = {} & \{\Re e[\varphi(u)]\cos kv - \Im m[\varphi(u)]\sin kv\}\bar{e} \\ & +\{\Re e[\psi(u)]\cos kv - \Im m[\psi(u)]\sin kv\}\bar{a}(v) \\ & +\{\Re e[\chi(u)]\cos kv - \Im m[\chi(u)]\sin kv\}\bar{a}'(v). \end{aligned} \tag{3.4}$$

It is known [8] that a circular torus is rigid. Among surfaces topologically equivalent to the torus, Belov [2] pointed out a class of nonrigid toroids with quadrangular meridian of a special form, which can be convex or nonconvex, but he didn't determine infinitesimal bending field. We have done this at [7]. A class of nonrigid surfaces was enlarged at [4] and [5]. According to [7], for $M_1 = 1$, we have

$$\psi_{(1)}(u) = (u-P), \quad \psi_{(2)}(u) = -(u+P),$$

$$\psi_{(3)}(u) = \left[\frac{c_1}{c_2}(u-1) - (P+1)\right], \tag{3.5}$$

$$\psi_{(4)}(u) = -\left[\frac{c_1}{c_2}(u+1) + P + 1\right],$$

$$\chi_{(1)}(u) = \frac{i}{k}(u-P), \quad \chi_{(2)}(u) = -\frac{i}{k}(u+P),$$

$$\chi_{(3)}(u) = \frac{i}{k}\left[\frac{c_1}{c_2}(u-1) - (P+1)\right], \tag{3.6}$$

$$\chi_{(4)}(u) = -\frac{i}{k}\left[\frac{c_1}{c_2}(u+1) + P + 1\right],$$

$$\varphi_{(1)}(u) = \varphi_{(2)}(u) = \varphi_{(3)}(u) = \varphi_{(4)}(u) = -c_1 u \tag{3.7}$$

$$P = \frac{b + c_1}{c_1(k^2-1)} = \frac{bc_2 + c_1c_2}{bc_1 - bc_2 - c_1c_2},$$

$$Q = \frac{bc_1 - c_1c_2}{bc_1 - bc_2 - c_1c_2}.$$

where $\varphi_{(i)}(u), \psi_{(i)}(u), \chi_{(i)}(u)$ are the functions $\varphi(u)$, $\psi(u)$, $\chi(u)$, $i = 1, \ldots, 4$, on the parts generated by the sides AB, BC, CD, DA respectively.

According to (3.3) and (1.1) for the parts of deformed surface S_ε, corresponding to the sides of the meridian we have

$$\bar{r}_{(i)}(u,v,\varepsilon) = u\bar{e} + \rho_{(i)}(u)\bar{a}(v) + \varepsilon\bar{z}_{(i)}(u,v), \quad i = 1,2,3,4, \tag{3.8}$$

where $\rho_{(i)}(u)$ are the values of $\rho = \rho(u)$ on the corresponding sides of meridian. Using (3.4-3.7) we have

$$\bar{r}_{(1)}(u,v,\varepsilon) = u(1 - \varepsilon c_1 \cos kv)\bar{e}$$

$$+[c_1u + b + c_1 + \varepsilon(u-P)\cos kv]\bar{a}(v) + \frac{\varepsilon}{k}(P-u)\sin kv\bar{a}'(v), \tag{3.9a}$$

$$\bar{r}_{(2)}(u,v,\varepsilon) = u(1 - \varepsilon c_1 \cos kv)\bar{e}$$

$$+[-c_1u + b + c_1 - \varepsilon(u+P)\cos kv]\bar{a}(v) + \frac{\varepsilon}{k}(P+u)\sin kv\bar{a}'(v), \tag{3.9b}$$

$$\bar{r}_{(3)}(u,v,\varepsilon) = u(1-\varepsilon c_1 \cos kv)\bar{e}$$

$$(3.9c) \quad +[c_2 u + b - c_2 + \varepsilon \frac{c_1}{c_2}(u-Q)\cos kv]\bar{a}(v) + \varepsilon \frac{c_1}{kc_2}(Q-u)\sin kv \bar{a}'(v),$$

$$\bar{r}_{(4)}(u,v,\varepsilon) = u(1-\varepsilon c_1 \cos kv)\bar{e}$$

$$(3.9d) \quad +[c_1 u + b + c_1 + \varepsilon(u-P)\cos kv]\bar{a}(v) + \frac{1}{k}\varepsilon(P-u)\sin kv \bar{a}'(v).$$

At the plane $(\bar{a}, \bar{a}')$ we introduce xOy coordinate system, so that for $v=0$, we obtain the positive part of x-axis, and u-axis is identical to z-axis. Then

$$(3.10) \quad \bar{a}(v) = \cos v\bar{i} + \sin v\bar{j}, \quad \bar{a}'(v) = -\sin v\bar{i} + \cos v\bar{j}, \quad \bar{e} = \bar{k},$$

and from (3.9) we have

$$x_{(1)}(u,v,\varepsilon) = (c_1 u + b + c_1)\cos v$$

$$(3.9'a) \quad +\varepsilon(u-P)(\cos kv \cos v + \frac{1}{k}\sin kv \sin v),$$

$$y_{(1)}(u,v,\varepsilon) = (c_1 u + b + c_1)\sin v$$

$$+\varepsilon(u-P)(\cos kv \sin v + \frac{1}{k}\sin kv \cos v),$$

$$z_{(1)}(u,v,\varepsilon) = u(1-\varepsilon c_1 \cos kv),$$

$$(u \in [-1,0], v \in [0,2\pi]),$$

$$x_{(2)}(u,v,\varepsilon) = (-c_1 u + b + c_1)\cos v$$

$$-\varepsilon(u+P)(\cos kv \cos v + \frac{1}{k}\sin kv \sin v),$$

$$y_{(2)}(u,v,\varepsilon) = (-c_1 u + b + c_1)\sin v$$

$$(3.9'b) \quad -\varepsilon(u+P)(\cos kv \sin v - \frac{1}{k}\sin kv \cos v),$$

$$z_{(2)}(u,v,\varepsilon) = u(1-\varepsilon c_1 \cos kv), \quad (u \in [0,1], v \in [0,2\pi]),$$

and, analogously, for the remaining two parts. From these equations we can see that deformed surface is symmetrical compared to the plane xOy (i.e. $(\bar{a}, \bar{a}')$).

From the equation (3.9′) we have

1. For $u = 0$

$$\begin{aligned} x_{(1)}(0,v,\varepsilon) &= x_{(2)}(0,v,\varepsilon) = (b+c_1)\cos v \\ &-\varepsilon P(\cos kv\cos v - \frac{1}{k}\sin kv\sin v), \\ y_{(1)}(0,v,\varepsilon) &= y_{(2)}(0,v,\varepsilon) = (b+c_1)\sin v \\ &-\varepsilon P(\cos kv\sin v - \frac{1}{k}\sin kv\cos v), \\ z_{(1)}(0,v,\varepsilon) &= z_{(2)}(0,v,\varepsilon) = 0, \end{aligned} \tag{3.11}$$

i.e. the circle circumscribed by the apex B, is deformed at the plane curve $(z = 0)$ that is not a circle.

2. For $u = -1$ and $u = 1$:

$$\begin{aligned} x_{(1)}(-1,v,\varepsilon) &= x_{(2)}(1,v,\varepsilon) = b\cos v \\ &-\varepsilon(1+P)(\cos kv\cos v + \frac{1}{k}\sin kv\sin v), \\ y_{(1)}(-1,v,\varepsilon) &= y_{(2)}(1,v,\varepsilon) = b\sin v \\ &-\varepsilon(1+P)(\cos kv\sin v - \frac{1}{k}\sin kv\cos v), \\ -z_{(1)}(-1,v,\varepsilon) &= z_{(2)}(1,v,\varepsilon) = 1 - \varepsilon c_1\cos kv. \end{aligned} \tag{3.12}$$

Hence, we conclude that the circles circumscribed by the apeces A, C are deformed to the plane curves that are symmetrical compared to the $x0y$-plane and have the same projection.

The first two equations at (3.11) determine closed curve L, and at (3.12) closed curve L_1 at xOy-plane. Let us prove that L_1 is inside L. It is sufficient to prove that $d^2 - d_1^2 > 0$, where d is the distance from $M \in L$ to O, and d_1 is the distance from $M_1 \in L_1$ to O. From (3.11, 12) we have:

$$d^2 - d_1^2 = (x_{(2)}(0,v,\varepsilon))^2 + (y_{(2)}(0,v,\varepsilon))^2 - (x_{(2)}(1,v,\varepsilon))^2 - (y_{(2)}(1,v,\varepsilon))^2$$

$$= 2bc_1 + c_1^2 - (1+2P)\varepsilon^2(\cos kv^2 + \frac{1}{k^2}\sin^2 kv) + 2\varepsilon(b - Pc_1)\cos kv > 0$$

for sufficiently small ε. Let us observe the parts of the volume of toroid of Belov with $V_{(1)}, V_{(2)}, V_{(3)}, V_{(4)}$, for the parts that are between xOy-plane and the parts that are generated by AB, BC, CD, DA respectively, and with $V_{(j)}(\varepsilon)$, $j = 1, ..., 4$ the volume of corresponding deformed part. If $V(\varepsilon)$ denotes the volume of deformed toroid, we will have

$$V(\varepsilon) = 2V_{(2)}(\varepsilon) + 2V_{(3)}(\varepsilon). \tag{3.13}$$

The part that has volume $V_{(2)}(\varepsilon)$ is projected at the ring between L and L_1, i.e. on the area denoted as $R(\varepsilon)$. Then

$$V_{(2)}(\varepsilon) = \iint_{R(\varepsilon)} z_{(2)}(u,v,\varepsilon)|J|dudv = \int_0^{2\pi} dv \int_0^1 z_{(2)}(u,v,\varepsilon)|J|du, \tag{3.14}$$

where $z_{(2)}(u,v,\varepsilon)$ is given at (3.9′b). From

$$J = \frac{\partial(x_{(2)}, y_{(2)})}{\partial(u,v)} = (c_1u - b - c_1)(c_1 + \varepsilon\cos kv) - \frac{k^2-1}{k^2}\varepsilon^2(u+P)\sin^2 kv,$$

for sufficiently small ε, $J \approx c_1(c_1u - b - c_1)$. As $u \in [0,1]$, we have that $J < 0$. From (3.14, 9′b) we get

$$V_{(2)}(\varepsilon) = \int_0^{2\pi} dv \int_0^1 u(1 - \varepsilon c_1\cos kv)$$

$$[(b + c_1 - c_1u)(c_1 + \varepsilon\cos kv) + \frac{k^2-1}{k^2}\varepsilon^2(u+P)\sin^2 kv]du$$

$$= \pi(\frac{1}{3}c_1^2 + bc_1) + \pi\varepsilon^2\lambda(b, c_1, c_2).$$

In the same manner

$$V_{(3)}(\varepsilon) = \pi(-\frac{1}{3}c_1^2 + bc_2) + \pi\varepsilon^2\mu(b, c_1, c_2),$$

where λ, μ are the functions of their arguments. Substituting at (3.13) we have

$$V(\varepsilon) = \frac{2\pi}{3}(c_1 + c_2)(3b + c_1 - c_2) + 2\pi\varepsilon^2(\lambda + \mu),$$

and

$$\delta V(\varepsilon) = (\frac{\partial V}{\partial \varepsilon})_{\varepsilon=0} = 0.$$

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