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HYPERBOLIC SPACE GROUPS FOR TWO FAMILIES OF FUNDAMENTAL SIMPLICES

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Abstract

There are investigated six series of hyperbolic fundamental simplices and a non - fundamental one which belong to same family, denoted by F.8. in paper [8] of Molnár and Prok, moreover, one series to F.21.

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1. Introduction

Hyperbolic space groups are isometry groups, acting discontinuously on the hyperbolic 3-space with compact fundamental domain. With the aim of classifying them one may look for their fundamental domains. Face pairing identifications of a given polyhedron give us generators and relations for a space group by Poincaré theorem [1], [2], [5].

The simplest fundamental domains are simplices and truncated simplices by polar planes of vertices when they lie out of the absolute. There are 64 combinatorially different face pairings of fundamental simplices [13], [14], [8], furthermore 35 solid transitive non-fundamental simplex identifications [8]. I. K. Zhuk [13], [14] has classified Euclidean and hyperbolic fundamental simplices of finite volume up to congruence. Some completing cases are discussed in [4], [7], [10], [11], [12]. Algorithmic procedure is given by E. Molnár and I. Prok [7]. In [8] and [9] the authors summarize all these results, arranging identified simplices into 32 families. Each of them is characterized by the so-called maximal series of simplex tilings. Besides spherical, Euclidean, hyperbolic realizations there exist also other metric realizations in 3-dimensional simply connected homogeneous Riemannian spaces, moreover, metrically non-realizable topological simplex tilings occur as well.

Seven simplices investigated in this paper have three equivalence classes of edges: $a\{A_0A_1\}, b\{A_0A_2, A_1A_2\}, c\{A_0A_3, A_1A_3, A_2A_3\}$ and two of vertices $\{A_3\}$ and $\{A_0, A_1, A_2\}$, obtained by their face pairings identifications (one series of them is illustrated in Fig. a)

Sum of dihedral angles around edges in the same equivalence class is always of the form $2\pi/\nu$. That is the reason we shall have three parameters $\bar{u}, \bar{v}$ and $\bar{w}$(for resp. classes a, b, c) for the maximal group $G(\mathcal{S}, 2\bar{u}, 4\bar{v}, 3\bar{w})$ in Sect.3. In this notation e.g. $2\bar{u}$ means that each edge in class a is surrounded with $2\bar{u}$ simplices. For different values of parameters $(\bar{u}, \bar{v}, \bar{w})$ E. Molnár and I. Prok obtained in [8] (in a slightly modified form) various spaces of realization. Simplices are realizable in a space of constant curvature in next cases:

in space $\mathbb{S}^3$ for $(\bar{u}, \bar{v}, \bar{w}) = (\bar{u}, 2, 1), \bar{u} \geq 2, (2, \bar{v}, 1), \bar{v} \geq 1, (3, 3, 1), (3, 4, 1)$;

in space $\mathbb{E}^3$ for (4,3,1) and for (2,1,2) but with ideal vertex A_3;

in space $\mathbb{H}^3$ for (3,5,1), (5,3,1) with all proper vertices, for $(\bar{u}, \bar{v}, 1)$, $1/\bar{u} + 1/\bar{v} = 1/2$, with ideal vertices A_0, A_1, A_2 and proper vertex A_3, for (2,2,2) with all ideal vertices, for $(\bar{u}, \bar{v}, 2), \bar{u}, \bar{v} \geq 2, (\bar{u}, \bar{v}, \bar{w}) \neq (2, 2, 2)$ with outer vertices A_0, A_1, A_2 and ideal vertex A_3 and for $(\bar{u}, \bar{v}, \bar{w}), \bar{u}, \bar{v} \geq 2, \bar{w} \geq 3$ with all outer vertices.

In other cases simplices as orbifolds are realizable in spaces of non constant curvature and it occurs only for $(\bar{u}, 1, 2), \bar{u} \geq 3$ space of realization is $\mathbb{H}^2 \times \mathbb{R}$ with ideal vertex A_3, but it forms an edge in $\mathbb{H}^2 \times \mathbb{R}$. Then we do not get a simplex in usual sense. When vertex A_3 (vertices A_0, A_1, A_2) is (are) out of the absolute, the simplex is not compact and then we truncate it with polar plane(s) of this vertex (of those vertices). The new compact polyhedron obtained in that way is fundamental domain of some larger group, investigated in this paper. It has new triangular faces whose pairing gives new generators. Dihedral angles around new edges are $\pi/2$. That means there are four congruent polyhedra around them in the fundamental space filling.

In section 2. the Poincaré theorem will be given. The results for all seven simplices will be summarized in section 3.

2. Construction of discontinuously acting isometry groups

With any simplex $\mathcal{T}$ of seven ones investigated in this paper, if it is realizable in space of constant curvature, we can fill that space using face pairing identifications which are indicated in the figures and tables.

Identifications are face pairings on the simplex $\mathcal{T}$ by isometries, satisfying the following conditions:

a) For each face $f_{g^{-1}}$ of $\mathcal{T}$ there is another face f_g and an identifying isometry g of the space $S^3(H^3)$, which maps $f_{g^{-1}}$ onto f_g and $\mathcal{T}$ onto $\mathcal{T}^g \not\cong \mathcal{T}$, the neighbour of $\mathcal{T}$ along f_g.

b) The isometry g^{-1} maps the face f_g onto $f_{g^{-1}}$ and $\mathcal{T}$ onto $\mathcal{T}^{g^{-1}}$, joining the simplex $\mathcal{T}$ along $f_{g^{-1}}$.

The face pairing identifications of $\mathcal{T}$ generate an isometry group G.

These generators induce subdivision of the edges into oriented segments such that a segment does not contain two equivalent points in its interior. An equivalence class consisting of edge segments $e_1, e_2, \dots, e_r$ with dihedral angles $\varepsilon(e_1), \varepsilon(e_2), \dots, \varepsilon(e_r)$, respectively, is defined as follows.

We consider an edge segment, say e_1, and choose one of the faces denoted by $f_{g_1^{-1}}$ whose boundary contains e_1. The isometry g_1 maps e_1 and $f_{g_1^{-1}}$ onto e_2 and $f_{g_1^{-1}}$. There exists exactly one other face $f_{g_2^{-1}}$ with e_2 on the boundary, furthermore the isometry g_2 maps e_2 and $f_{g_2^{-1}}$ onto e_3 and $f_{g_2^{-1}}$, and so on. We obtain a cycle of isometries $g_1, g_2, \dots, g_r$ according to the scheme

$$(e_1, f_{g_1^{-1}}) \xrightarrow{g_1} (e_2, f_{g_1}); \quad (e_2, f_{g_2^{-1}}) \xrightarrow{g_2} (e_3, f_{g_2}); \quad \dots \tag{1}$$

$$(e_r, f_{g_r^{-1}}) \xrightarrow{g_r} (e_1, f_{g_r})$$

where the symbols are not necessarily distinct. More precisely, we have two essentially different cases for the scheme (1)

1) if a plane reflection $m_i = g_i$ occurs then $e_{i+1} = e_i$, and we turn back to e_1, then, say, e_{-1} comes. Furthermore, another plane reflection $m_{-j} = g_{-j}$ shall appear in the cycle. Then each edge segment comes two times in the scheme (1), and the cycle transformation is of the form

$$\begin{aligned} c &= g_1 g_2 \cdots g_r = \\ &= (g_1 \cdots g_{i-1}\, m_i\, g_{i-1}^{-1} \cdots g_1^{-1})(g_{-1}^{-1} \cdots g_{-j+1}^{-1}\, m_{-j}\, g_{-j+1} \cdots g_{-1}) \end{aligned}$$

2)there is no plane reflection in the cycle, this will be the simpler (general) case. (In dimension 3 we have 5 subcases for full edges at all [4].)

In other words the segment e_1 is successively surrounded by simplices

$$\mathcal{T}, \mathcal{T}^{g_1^{-1}}, \mathcal{T}^{g_2^{-1}g_1^{-1}}, \dots, \mathcal{T}^{g_r^{-1}\dots g_2^{-1}g_1^{-1}}$$

which fill an angular region of measure $2\pi/\nu$. In above case 1) holds

$$\varepsilon(e_1) + \dots + \varepsilon(e_i) + \varepsilon(e_{-1}) + \dots + \varepsilon(e_{-j+1}) = \pi/\nu. \qquad (2)$$

In case 2) we have

$$\varepsilon(e_1) + \dots + \varepsilon(e_r) = 2\pi/\nu. \qquad (3)$$

Finally, the cycle transformation $c = g_1 g_2 \dots g_r$ belonging to the edge segment class $\{e_i\}$ is a rotation, say, of order ν. Thus we have the cycle relation in both cases

$$(g_1 g_2 \dots g_r)^\nu = 1 \qquad (4)$$

c) Assume that (2) or (3) holds for the face angles at $\{e_1\}$ in each segment equivalence class.

We need the specified Poincaré theorem:

Theorem 1. *Let $\mathcal{T}$ be a simplex, or a truncated simplex in a space $\mathcal{S}^3$ of constant curvature and G be the group generated by the face identifications, satisfying conditions a) − c). Then G is discontinuously acting group on $\mathcal{S}^3$, $\mathcal{T}$ is a fundamental domain for G and the cycle relations of type (4) for every equivalence class of edge segments form a complete set of relations for G, if we also add the relations $g_i^2 = 1$ to the occasional involutive generators $g_i = g_i^{-1}$.*

3. The isometry groups for simplices

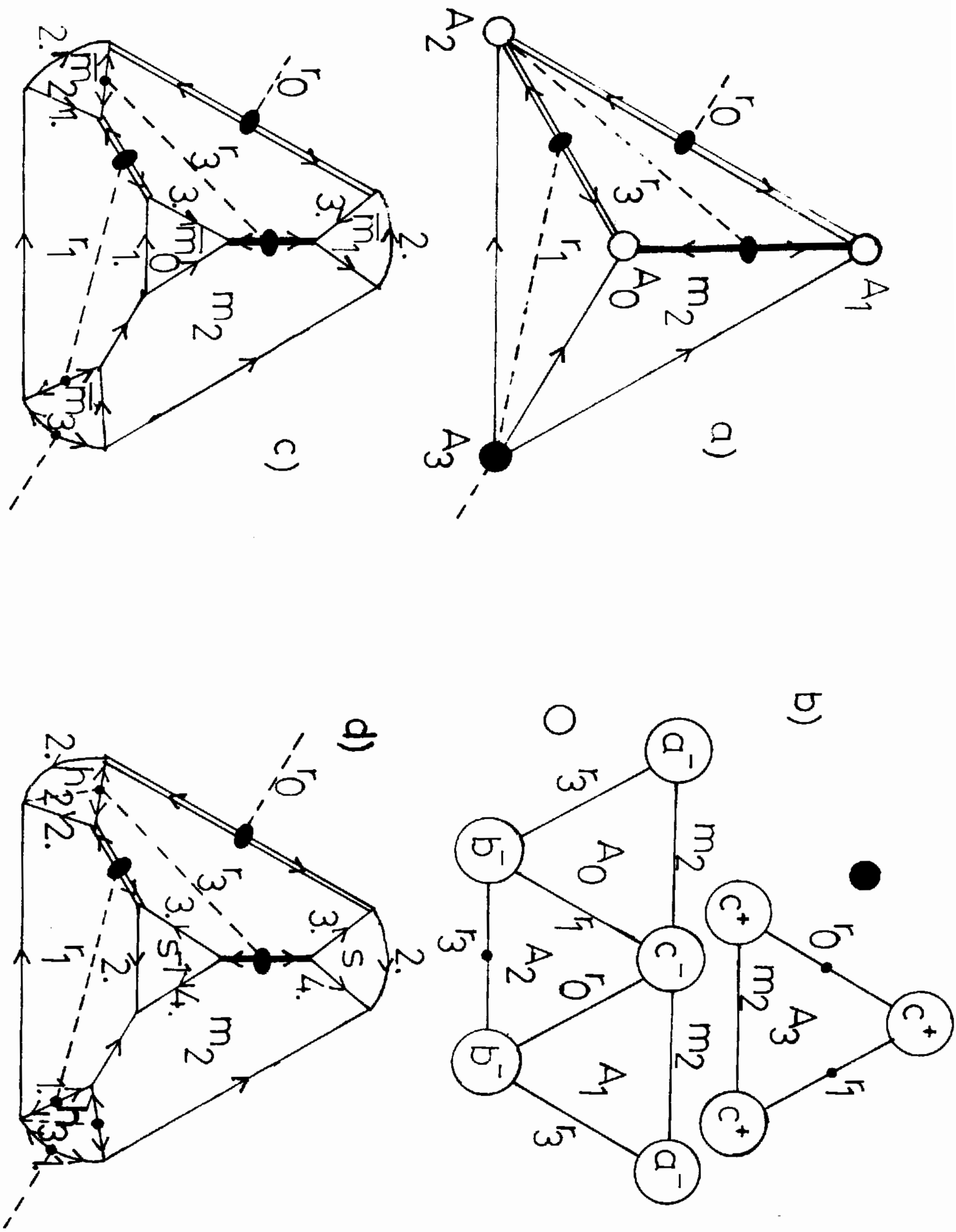
Isometries which identify faces of simplex $\mathcal{T}_1$(Fig. *a*) are

$$r_0: \begin{pmatrix} A_1\, A_2\, A_3 \\ A_2\, A_1\, A_3 \end{pmatrix};\ r_1: \begin{pmatrix} A_0\, A_2\, A_3 \\ A_2\, A_0\, A_3 \end{pmatrix};\ r_3: \begin{pmatrix} A_0\, A_1\, A_2 \\ A_1\, A_0\, A_2 \end{pmatrix};\ m_2: \begin{pmatrix} A_0\, A_1\, A_3 \\ A_0\, A_1\, A_3 \end{pmatrix}.$$

By notations in [8], [9] it is representing simplices from Family 8 with groups of simplex tilings $\Gamma_9(4u; 4v; 6w)$. The face pairing isometries divide edges into two equivalent oriented segments in classes a and b . Relations for the isometry group are obtained by Theorem 1. and the presentation is

$$G(\mathcal{T}_1, 4u, 4v, 6w) = (r_0, r_1, r_3, m_2, -(r_3 m_2 r_3 m_2)^u = (r_1 r_3 r_0 r_3)^v =$$

$$= (r_1 r_0 m_2 r_0 r_1 m_2)^w = r_0^2 = r_1^2 = r_3^2 = m_2^2 = 1;\ u \geq 1, v \geq 1, w \geq 1).$$



Figure

Considering vertex figures on a 2-dimensional surface around the vertices, we can obtain a fundamental domain for the stabilizer group G_{A_3} of vertex

A_3 and e.g. G_{A_2} of vertex A_2. Transformation r_0 is mapping vertex figure $\mathcal{T}_{A_2}$ onto $\mathcal{T}_{A_1}^{r_0}$ and r_1 is mapping $\mathcal{T}_{A_2}$ onto $\mathcal{T}_{A_0}^{r_1}$. That means that $\mathcal{T}_{A_2}$ and $\mathcal{T}_{A_1}^{r_0}$ have joint edge corresponding to the joint face f_{r_0} of the simplices $\mathcal{T}_1$ and $\mathcal{T}_1^{r_0}$ and similarly, $\mathcal{T}_{A_2}$ and $\mathcal{T}_{A_0}^{r_1}$ have joint edge corresponding to f_{r_1}. One of the domains for G_{A_2}(Fig. b) is

$$\mathcal{P}_{A_2} := \mathcal{T}_{A_0}^{r_1} \bigcup \mathcal{T}_{A_2} \bigcup \mathcal{T}_{A_1}^{r_0}.$$

In the diagram for $\mathcal{P}_{A_2}$ the minus sign in notations a^-, b^- and c^- means that edges in these classes are directed to vertex according to vertex figure, the plus sign in diagram for $\mathcal{P}_{A_3}$ means opposite direction.

When parameters u, v, w are such that simplex $\mathcal{T}_1$ is hyperbolic and that the vertices either in the first or in the second equivalence class are out of the absolute, it is possible to truncate the simplex by polar planes of the vertices. Then we get a compact polyhedron denoted by $\mathcal{O}_1$. If we equip $\mathcal{O}_1$ with additional face pairing isometries, it will be a fundamental domain for the group $G(\mathcal{O}_1, 4u, 4v, 6w)$ which will be a supergroup for $G(\mathcal{T}_1, 4u, 4v, 6w)$. Trivial group extension with plane reflections in polar planes of the outer vertices is always possible. In the case of vertices A_0, A_1, A_2 the new relations, which are necessary to add to group $G(\mathcal{T}_1, 4u, 4v, 6w)$ to obtain supergroup in this way, are (Fig. c)

$$r_3\bar{m}_2r_3\bar{m}_2 = \bar{m}_2r_1\bar{m}_0r_1 = \bar{m}_2r_0\bar{m}_1r_0 = \bar{m}_0r_3\bar{m}_1r_3 =$$

$$= (\bar{m}_0m_2)^2 = (\bar{m}_1m_2)^2 = \bar{m}_0^2 = \bar{m}_1^2 = \bar{m}_2^2 = 1$$

where $\bar{m}_i$ is plane reflection corresponding to the vertex A_i.

For vertex A_3 these relations with trivial extension are $r_0\bar{m}_3r_0\bar{m}_3 = r_1\bar{m}_3r_1\bar{m}_3 = (m_2\bar{m}_3)^2 = \bar{m}_3^2 = 1$.

There is one more possibility to equip new triangular faces corresponded to outer vertices A_0, A_1, A_2(i.e. vertex A_3) with pairing isometries. New additional face pairings of $\mathcal{O}_1$ for vertices A_0, A_1, A_2 have to satisfy the next criteria. Polar plane of A_2 and so group G_{A_2} will be invariant under these new transformations, fixing A_2, and exchanging half spaces obtained by the polar plane. Thus, fundamental domain $\mathcal{P}_{A_2}$ is divided into two parts, and the new stabilizer of the polar plane will be supergroup for G_{A_2}, namely of index two. Symmetries of the $\mathcal{P}_{A_2}$-tiling give us the idea how to introduce the new generators. If h_2 is the new half-turn mapping vertex figure $\mathcal{T}_{A_2}$ to itselves and $\mathcal{T}_{A_0}^{r_1}$ to $\mathcal{T}_{A_1}^{r_0}$, but exchanging the half-spaces, then the new generators for $G(\mathcal{O}_1, 4u, 4v, 6w)$ will be h_2 and $s = r_0h_2r_1$ and new relators (Fig. d)

$$(h_2r_3)^2 = r_1sr_0h_2 = (r_3s)^2 = sm_2s^{-1}m_2 = h_2^2 = 1.$$

Similarly for outer vertex A_3, it is possible to equip new triangular face with half-turn h_3, so that new relators are

$$h_3 m_2 h_3 m_2 = r_1 h_3 r_0 h_3 = h_3^2 = 1.$$

REMARK: Truncation of vertices in different equivalence classes and variants of their equipping with face pairing identifications are independent, so there are more possibilities to create supergroup of $G(\mathcal{T}_1, 4u, 4v, 6w)$ in this way. For all cases of such supergroups is used same (common) notation $G(\mathcal{O}_1, 4u, 4v, 6w)$.

Groups $G(\mathcal{T}_1, 4u, 4v, 6w), G(\mathcal{O}_1, 4u, 4v, 6w)$ are not maximal, i.e. they are subgroups of certain groups $G(\mathcal{S}, 2\bar{u}, 4\bar{v}, 3\bar{w})$ and $G(\mathcal{Q}, 2\bar{u}, 4\bar{v}, 3\bar{w})$, respectively, which leave invariant the tilings with simplices $\mathcal{T}_1$, resp. truncated simplices $\mathcal{O}_1$. Each of domains $\mathcal{T}_1$, $\mathcal{O}_1$ can be divided into smaller parts, each with a face pairing to be fundamental domain of a larger group, i.e. supergroup of starting one. In our cases it is possible to halve $\mathcal{T}_1$ or $\mathcal{O}_1$ with plane through edge A_2A_3 and midpoint M of A_0A_1. Equipping the new simplex $\mathcal{S}$, resp. the truncated simplex $\mathcal{Q}$ with the face pairing, we obtain the groups $G(\mathcal{S}, 2\bar{u}, 4\bar{v}, 3\bar{w})$ and $G(\mathcal{Q}, 2\bar{u}, 4\bar{v}, 3\bar{w})$.

New face pairings for $\mathcal{S}$ are

$$r_0: \begin{pmatrix} A_1\ A_2\ A_3 \\ A_2\ A_1\ A_3 \end{pmatrix};\ m: \begin{pmatrix} M\ A_2\ A_3 \\ M\ A_2\ A_3 \end{pmatrix};\ m_2: \begin{pmatrix} M\ A_1\ A_3 \\ M\ A_1\ A_3 \end{pmatrix};\ m_3: \begin{pmatrix} M\ A_1\ A_2 \\ M\ A_1\ A_2 \end{pmatrix}.$$

and triangular faces of $\mathcal{Q}$ corresponded to outer vertices A_1, A_2 or A_3 of $\mathcal{S}$ is possible to equip only with plane reflections.

Group for $\mathcal{S}$

$$\begin{aligned} G(\mathcal{S}, 2\bar{u}, 4\bar{v}, 3\bar{w}) = (r_0, m, m_2, m_3 - (m_2 m_3)^{\bar{u}} = (r_0 m_3 r_0 m_3)^{\bar{v}} = \\ (r_0 m_2 r_0 m)^{\bar{w}} = (m m_2)^2 = (m m_3)^2 = r_0^2 = \\ m^2 = m_2^2 = m_3^2 = 1; \bar{u} \geq 2, \bar{v} \geq 1, \bar{w} \geq 1) \end{aligned}$$

where $u = \bar{u}/2, v = \bar{v}, w = \bar{w}/2$, if integers. Then the tiling $G(\mathcal{T}_1, 4u, 4v, 6w)$ has just the automorphism group equivariant to $G(\mathcal{S}, 2\bar{u}, 4\bar{v}, 3\bar{w})$. This restricts the realizations of $G(\mathcal{T}_1, 4u, 4v, 6w)$, of course. Additional relators for $\mathcal{Q}$ and vertices A_1, A_2 are

$$\bar{m}_2 r_0 \bar{m}_1 r_0 = (\bar{m}_2 m)^2 = (\bar{m}_2 m_3)^2 = (\bar{m}_1 m_2)^2 = (\bar{m}_1 m_3)^2 = \bar{m}_1^2 = \bar{m}_2^2 = 1$$

and for vertex A_3

$$\bar{m}_3 r_0 \bar{m}_3 r_0 = (\bar{m}_3 m_2)^2 = (\bar{m}_3 m)^2 = \bar{m}_3^2 = 1.$$

Generators of $\mathcal{T}_1$ and $\mathcal{O}_1$ expressed by these of $\mathcal{S}$ and $\mathcal{Q}$ are

$$r_0 \equiv r_0, r_1 = m r_0 m, r_3 = m_3 m, m_2 \equiv m_2$$

$$h_2 = m\bar{m}_2, h_3 = m\bar{m}_3.$$

If $\bar{u} \neq 2\bar{v}$ groups $G(\mathcal{S}, 2\bar{u}, 4\bar{v}, 3\bar{w})$ and $G(\mathcal{Q}, 2\bar{u}, 4\bar{v}, 3\bar{w})$ are maximal i.e. without further (combinatorial) symmetries.

Table 1: Simplex $\mathcal{T}_2$

1. $F_8, \Gamma_{16}(2u; 4v; 3w), u \geq 2, v \geq 1, w \geq 1$

2. $r_0: \begin{pmatrix} A_1 & A_2 & A_3 \\ A_2 & A_1 & A_3 \end{pmatrix}; \; r_1: \begin{pmatrix} A_0 & A_2 & A_3 \\ A_2 & A_0 & A_3 \end{pmatrix}; \; r_2: \begin{pmatrix} A_0 & A_1 & A_3 \\ A_1 & A_0 & A_3 \end{pmatrix}; \; r_3: \begin{pmatrix} A_0 & A_1 & A_2 \\ A_1 & A_0 & A_2 \end{pmatrix}$

3. $G(\mathcal{T}_2, 2u, 4v, 3w) = (r_0, r_1, r_2, r_3 - (r_2 r_3)^u = (r_1 r_3 r_0 r_3)^v = (r_1 r_2 r_0)^w = r_0^2 = r_1^2 = r_2^2 = r_3^2 = 1, u \geq 2, v \geq 1, w \geq 1)$

4.a) $\bar{\mathbf{m}}_2 r_3 \bar{m}_2 r_3 = r_0 \bar{m}_1 r_0 \bar{m}_1 = r_1 \bar{m}_0 r_1 \bar{m}_2 = r_2 \bar{m}_1 r_2 \bar{m}_0 = r_3 \bar{m}_1 r_3 \bar{m}_0 = \bar{m}_0^2 = \bar{m}_1^2 = \bar{m}_2^2 = 1$

b) $\mathbf{r}_1 \bar{m}_3 r_1 \bar{m}_3 = r_2 \bar{m}_3 r_2 \bar{m}_3 = r_0 \bar{m}_3 r_0 \bar{m}_3 = \bar{m}_3^2 = 1$

5.a) $(r_3 h_2)^2 = r_1 s r_0 h_2 = (r_3 s)^2 = (s r_2)^2 = h_2^2 = 1$

b) $(r_2 h_3)^2 = r_1 h_3 r_0 h_3 = h_3^2 = 1$

6. $\mathbf{u} = \bar{u}, v = \bar{v}, w = \bar{w} r_0 \equiv r_0, r_1 = m r_0 m, r_2 = m_2 m, r_3 = m_3 m h_2 = m\bar{m}_2, h_3 = m\bar{m}_3$

Results for other simplices $\mathcal{T}_i (i = 2, \ldots, 7)$ are arranged in tables where is indicated:

1. notation used in [8] for family and group of simplex tiling;

2. face pairing identifications;

3. isometry group for simplex $\mathcal{T}_i (i = 2, \ldots, 7)$;

4. additional relators for trivial equipping of triangular faces of $\mathcal{Q}_i (i = 2, \ldots, 7)$

a) corresponding to outer vertices A_0, A_1, A_2

b) corresponding to outer vertex A_3;

5. another possibility of isometries identifying triangular faces of $\mathcal{Q}_i$(not existing for $\mathcal{T}_7$) and relators

a) for A_0, A_1, A_2

Table 2: Simplex $\mathcal{T}_3$

1. $F_8, \Gamma_{22}(2u;4v;6w), u \geq 2, v \geq 1, w \geq 1$

$$\textbf{2. } m_2\colon \begin{pmatrix} A_0\, A_1\, A_3 \\ A_0\, A_1\, A_3 \end{pmatrix};\ m_3\colon \begin{pmatrix} A_0\, A_1\, A_2 \\ A_0\, A_1\, A_2 \end{pmatrix};\ z\colon \begin{pmatrix} A_1\, A_2\, A_3 \\ A_2\, A_0\, A_3 \end{pmatrix}$$

$$\textbf{3. } G(\mathcal{T}_3, 2u, 4v, 6w) = (m_2, m_3, z - (m_2 m_3)^u = (z m_3 z^{-1} m_3)^v = (z^2 m_2 z^{-2} m_2)^w = m_2^2 = m_3^2 = 1, u \geq 2, v \geq 1, w \geq 1)$$

$$\textbf{4.a) } (\bar{m}_2 m_3)^2 = (\bar{m}_1 m_2)^2 = z\bar{m}_0 z^{-1}\bar{m}_2 = z\bar{m}_2 z^{-1}\bar{m}_1 = (\bar{m}_1 m_3)^2 = (\bar{m}_0 m_3)^2 = (\bar{m}_0 m_2)^2 = \bar{m}_0^2 = \bar{m}_1^2 = \bar{m}_2^2 = 1$$

$$\textbf{b) } (m_2 \bar{m}_3)^2 = z\bar{m}_3 z^{-1}\bar{m}_3 = \bar{m}_3^2 = 1$$

$$\textbf{5.a) } \mathbf{m}_3 h_2 m_3 h_2 = z h_2 z s = s m_2 s^{-1} m_2 = s m_3 s^{-1} m_3 = h_2^2 = 1$$

$$\textbf{b) } \mathbf{m}_2 h_3 m_2 h_3 = z h_3 z h_3 = h_3^2 = 1$$

$$\textbf{6. } \mathbf{u} = \bar{u}, v = \bar{v}, w = \bar{w}/2$$

$$z = r_0 m, m_2 \equiv m_2, m_3 \equiv m_3$$

$h_2 = m\bar{m}_2, h_3 = m\bar{m}_3$

b) for A_3;

6. expresion of parameters u, v, w by $\bar{u}, \bar{v}, \bar{w}$ and generators of $\mathcal{T}_i$ and $\mathcal{O}_i (i = 2, \dots, 6)$ expressed by these of $\mathcal{S}$ and $\mathcal{Q}$(groups $G(\mathcal{T}_7, 2u, 8v, 6w)$ and $G(\mathcal{O}_7, 2u, 8v, 6w)$ are maximal).

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Table 3: Simplex $\mathcal{T}_4$

$$\mathbf{1}.\ F_8, \Gamma_{26}(4u;4v;3w), u \geq 1, v \geq 1, w \geq 1$$

$$\mathbf{2}.\ r_2: \begin{pmatrix} A_0\, A_1\, A_3 \\ A_1\, A_0\, A_3 \end{pmatrix};\ m_3: \begin{pmatrix} A_0\, A_1\, A_2 \\ A_0\, A_1\, A_2 \end{pmatrix};\ z: \begin{pmatrix} A_1\, A_2\, A_3 \\ A_2\, A_0\, A_3 \end{pmatrix}$$

$$\mathbf{3}.\ G(\mathcal{T}_4, 4u, 4v, 3w) = (r_2, m_3, z - (r_2 m_3 r_2 m_3)^u = (z m_3 z^{-1} m_3)^v = (z r_2 z)^w =$$
$$= r_2^2 = m_3^2 = 1, u \geq 1, v \geq 1, w \geq 1)$$

$$\mathbf{4.a})\ (\bar{m}_2 m_3)^2 = (\bar{m}_1 m_3)^2 = (\bar{m}_0 m_3)^2 = r_2 \bar{m}_1 r_2 \bar{m}_0 = z \bar{m}_0 z^{-1} \bar{m}_2 =$$
$$= z \bar{m}_2 z^{-1} \bar{m}_1 = \bar{m}_0^2 = \bar{m}_1^2 = \bar{m}_2^2 = 1$$

$$\mathbf{b})\ \mathbf{r}_2 \bar{m}_3 r_2 \bar{m}_3 = z \bar{m}_3 z^{-1} \bar{m}_3 = \bar{m}_3^2 = 1$$

$$\mathbf{5.a})\ \mathbf{m}_3 h_2 m_3 h_2 = z h_2 z s = s m_3 s^{-1} m_3 = (s r_2)^2 = h_2^2 = 1$$

$$\mathbf{b})\ (r_2 h_3)^2 = z h_3 z h_3 = h_3^2 = 1$$

$$\mathbf{6}.\ \mathbf{u} = \bar{u}/2, v = \bar{v}, w = \bar{w}$$

$$z = r_0 m, r_2 = m_2 m, m_3 \equiv m_3$$

$$h_2 = m \bar{m}_2, h_3 = m \bar{m}_3$$

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Table 4: Simplex $\mathcal{T}_5$

$$\mathbf{1}.\ F_8, \Gamma_{27}(4u;4v;6w), u \geq 1, v \geq 1, w \geq 1$$

$$\mathbf{2}.\ m_2 : \begin{pmatrix} A_0\, A_1\, A_3 \\ A_0\, A_1\, A_3 \end{pmatrix};\ r_3 : \begin{pmatrix} A_0\, A_1\, A_2 \\ A_1\, A_0\, A_2 \end{pmatrix};\ z : \begin{pmatrix} A_1\, A_2\, A_3 \\ A_2\, A_0\, A_3 \end{pmatrix}$$

$$\mathbf{3}.\ G(\mathcal{T}_5, 4u, 4v, 6w) = (m_2, r_3, z - (r_3 m_2 r_3 m_2)^u = (z r_3 z r_3)^v =$$

$$= (z^2 m_2 z^{-2} m_2)^w = m_2^2 = r_3^2 = 1, u \geq 1, v \geq 1, w \geq 1)$$

$$\mathbf{4.a})\ (m_2 \bar{m}_0)^2 = (m_2 \bar{m}_1)^2 = z \bar{m}_0 z^{-1} \bar{m}_2 = z \bar{m}_2 z^{-1} \bar{m}_1 = r_3 \bar{m}_2 r_3 \bar{m}_2 =$$

$$= r_3 \bar{m}_1 r_3 \bar{m}_0 = \bar{m}_0^2 = \bar{m}_1^2 = \bar{m}_2^2 = 1$$

$$\mathbf{b})\ (m_2 \bar{m}_3)^2 = z \bar{m}_3 z^{-1} \bar{m}_3 = \bar{m}_3^2 = 1$$

$$\mathbf{5.a})\ (h_2 r_3)^2 = z h_2 z s = s m_2 s^{-1} m_2 = (s r_3)^2 = h_2^2 = 1$$

$$\mathbf{b})\ \mathbf{m}_2 h_3 m_2 h_3 = z h_3 z h_3 = h_3^2 = 1$$

$$\mathbf{6}.\ \mathbf{u} = \bar{u}/2, v = \bar{v}, w = \bar{w}/2$$

$$z = r_0 m, m_2 \equiv m_2, r_3 = m_3 m$$

$$h_2 = m \bar{m}_2, h_3 = m \bar{m}_3$$

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Table 5: Simplex $\mathcal{T}_6$

$$\mathbf{1.}\ F_8, \Gamma_{37}(2u; 4v; 3w), u \geq 2, v \geq 1, w \geq 1$$

$$\mathbf{2.}\ r_2: \begin{pmatrix} A_0\, A_1\, A_3 \\ A_1\, A_0\, A_3 \end{pmatrix};\ r_3: \begin{pmatrix} A_0\, A_1\, A_2 \\ A_1\, A_0\, A_2 \end{pmatrix};\ z: \begin{pmatrix} A_1\, A_2\, A_3 \\ A_2\, A_0\, A_3 \end{pmatrix}$$

$$\mathbf{3.}\ G(\mathcal{T}_6, 2u, 4v, 3w) = (r_2, r_3, z - (r_2 r_3)^u = (z r_3 z r_3)^v = (z r_2 z)^w =$$
$$= r_2^2 = r_3^2 = 1, u \geq 2, v \geq 1, w \geq 1)$$

$$\mathbf{4.a)}\ \mathbf{z\bar{m}}_0 z^{-1} \bar{m}_2 = z \bar{m}_2 z^{-1} \bar{m}_1 = r_3 \bar{m}_2 r_3 \bar{m}_2 = r_3 \bar{m}_1 r_3 \bar{m}_0 = r_2 \bar{m}_1 r_2 \bar{m}_0 =$$
$$= \bar{m}_0^2 = \bar{m}_1^2 = \bar{m}_2^2 = 1$$

$$\mathbf{b)}\ \mathbf{r}_2 \bar{m}_3 r_2 \bar{m}_3 = z \bar{m}_3 z^{-1} \bar{m}_3 = \bar{m}_3^2 = 1$$

$$\mathbf{5.a)}\ (h_2 r_3)^2 = z h_2 z s = (s r_2)^2 = (s r_3)^2 = h_2^2 = 1$$

$$\mathbf{b)}\ (h_3 r_2)^2 = z h_3 z h_3 = h_3^2 = 1$$

$$\mathbf{6.}\ \mathbf{u} = \bar{u}, v = \bar{v}, w = \bar{w}$$

$$z = r_0 m, r_2 = m_2 m, r_3 = m_3 m$$

$$h_2 = m \bar{m}_2, h_3 = m \bar{m}_3$$

Table 6: Simplex $\mathcal{T}_7$

$$\mathbf{1.}\ F_{21}, \Gamma_{12}(2a; 6b; 8c), a \geq 2, b \geq 1, c \geq 1$$

$$\mathbf{2.}\ m_0: \begin{pmatrix} A_1\, A_2\, A_3 \\ A_1\, A_2\, A_3 \end{pmatrix};\ r_1: \begin{pmatrix} A_0\, A_2\, A_3 \\ A_2\, A_0\, A_3 \end{pmatrix};\ r_2: \begin{pmatrix} A_0\, A_1\, A_3 \\ A_1\, A_0\, A_3 \end{pmatrix};\ r_3: \begin{pmatrix} A_0\, A_1\, A_2 \\ A_1\, A_0\, A_2 \end{pmatrix}$$

$$\mathbf{3.}\ G(\mathcal{T}_7, 2u, 8v, 6w) = (m_0, r_1, r_2, r_3 - (r_2 r_3)^u = (r_3 r_1 r_3 m_0 r_3 r_1 r_3 m_0)^v =$$
$$= (r_1 r_2 m_0 r_2 r_1 m_0)^w = m_0^2 = r_1^2 = r_2^2 = r_3^2 = 1, u \geq 2, v \geq 1, w \geq 1)$$

$$\mathbf{4.a)}\ \bar{\mathbf{m}}_2 r_3 \bar{m}_2 r_3 = r_1 \bar{m}_0 r_1 \bar{m}_2 = r_2 \bar{m}_1 r_2 \bar{m}_0 = r_3 \bar{m}_1 r_3 \bar{m}_0 = (m_0 \bar{m}_2)^2 =$$
$$= (m_0 \bar{m}_1)^2 = \bar{m}_0^2 = \bar{m}_1^2 = \bar{m}_2^2 = 1$$

$$\mathbf{b)}\ \mathbf{r}_1 \bar{m}_3 r_1 \bar{m}_3 = r_2 \bar{m}_3 r_2 \bar{m}_3 = (m_0 \bar{m}_3)^2 = \bar{m}_3^2 = 1$$