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ON SYMMETRIC RIEMANNIAN MANIFOLDS

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Abstract

Symmetric spaces were studied by a large number of authors such as Chaki [1], Roter [4], Glodek [7], Miyazawa [5] and K. Anur [6] etc. In this paper we have discussed the nature of scalar curvature r for conformally symmetric Riemannian manifold. Some results for $(1,3)$ type tensors have also been obtained in an Einstein manifold.

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1 Introduction

Let M be a smooth Riemannian manifold of dimension n and let $\chi(M)$ denote the set of differentiable vector fields on M. Let $X, Z \in \chi(M)$; $D_X Y$ denotes the covariant derivative of Y with respect to X and $K(X,Y,Z)$ be the Riemannian curvature tensor of type (3). A Riemannian manifold is said to be symmetric if ([2], [6])

$$(D_U K)(X,Y,Z) = 0. \tag{1}$$

Contracting (1) with respect to X, we get

$$(D_U Ric)(Y,Z) = 0, \tag{2}$$

where $Ric = (C_1^1 K) = \Sigma K(E_i, Y, Z, E_i); E_i$ are orthonormal basic vector fields of the tangent space $T_x M$ at $x \in M$.

From (2), we have

$$(D_U R)(Y) = 0, \tag{3}$$

where R is the Ricci tensor of type (1), defined as

$$Ric(Y,Z) = g(R(Y),Z). \tag{4}$$

A symmetric Riemannian manifold is Ricci-symmetric, but the converse is not true.

In this paper we have considered a non-flat n-dimensional Riemannian manifold in which the conformal curvature tensor $C(X,Y,Z)$ satisfies the condition

$$(D_U C)(X,Y,Z) = 0, \tag{5}$$

where $C(X,Y,Z)$ is defined by ([3])

$$\begin{aligned} C(X,Y,Z) = K(X,Y,Z) - \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)R(X) \\ -g(X,Z)R(Y)] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{6}$$

Such an n-dimensional Riemannian manifold has been called a conformally symmetric manifold ([1], [4]).

Weyl concircular curvature tensor $W(X,Y,Z)$, projective curvature tensor $P(X,Y,Z)$ and conharmonic curvature tensor $N(X,Y,Z)$ are given by ([3])

$$W(X,Y,Z) = K(X,Y,Z) - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{7}$$

$$P(X,Y,Z) = K(X,Y,Z) - \frac{1}{n-1}[Ric(Y,Z)X - Ric(X,Z)Y] \tag{8}$$

$$\begin{aligned} N(X,Y,Z) = K(X,Y,Z) - \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y \\ +g(Y,Z)R(X) - g(X,Z)R(Y)] \end{aligned} \tag{9}$$

respectively.

A Riemannian manifold is said to be concircularly symmetric ([5]), projectively symmetric ([7]) and conharmonically symmetric ([5]) if $W(X,Y,Z)$, $P(X,Y,Z)$ and $N(X,Y,Z)$ satisfy the conditions:

$$(D_U W)(X,Y,Z) = 0 \tag{10}$$

$$(D_U O)(X,Y,Z) = 0 \tag{11}$$

and

$$(D_U N)(X,Y,Z) = 0 \tag{12}$$

respectively.

A manifold is said to be an Einstein manifold if ([2])

$$Ric(Y, Z) = kg(Y, Z) \tag{13}$$

where k is constant. From (13), we have

$$R(Y) = kY. \tag{14}$$

Contracting (14) with respect to Y, we get

$$r = nk \tag{15}$$

where r is scalar curvature.

2 Conformally symmetric space

From (5) and (6) it follows that

$$\begin{aligned}(D_UK)(X, Y, Z) = \frac{1}{n-2}[(D_URic)(Y, Z)X - (D_URic)(X, Z)Y \\ +g(Y, Z)(D_UR)(X) - g(X, Z)(D_UR)(Y)] \\ -\frac{Ur}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y].\end{aligned} \tag{16}$$

Permuting equation (16) twice with respect to U, X, Y, adding the three equations and using the second Bianchi's identity, we have

$$\begin{aligned}(D_URic)(Y, Z)X - (D_URic)(X, Z)Y + g(Y, Z)(D_UR)(X)- \\ g(X, Z)(D_UR)(Y)+(D_XRic)(U, Z)Y-(D_XRic)(Y, Z)U+g(U, Z)(D_UR)(Y) \\ -g(Y, Z)(D_XR)(U) + (D_YRic)(X, Z)U - (D_YRic)(U, Z)X \\ +g(X, Z)(D_YR)(U) - g(U, Z)(D_YR)(X) \\ = \frac{r}{n-2}[U(g(Y, Z)X - g(X, Z)Y) + X(g(U, Z)Y \\ -g(y, Z)U) + Y(g(X, Z)U - g(U, Z)X)].\end{aligned} \tag{17}$$

Contracting (17) with respect to X, we get

$$9n - 1(D_URic)(Y, Z) + g(Y, Z)(Ur) - g((D_UR)(Y), Z) \tag{18}$$

$$+(D_Y Ric)(U,Z) - (D_U Ric)(Y,Z) + \frac{1}{2}g(U,Z)(Yr) - \frac{1}{2}g(Y,Z)(Ur)$$
$$+(1-n)(D_Y Ric)(U,Z) + g((D_Y R)(U),Z) - g(U,Z)(Yr)$$
$$= \frac{r}{(n-1)}[(n-1)g(Y,Z)U + ng(U,Z)Y - ng(Y,Z)U - (n-1)g(U,Z)Y].$$

By virtue of (4), equation (18) reduces to the form

$$(n-1)(D_U R)(Y) + Y(Ur) - (D_U R)(Y) + (D_Y R)(U) - (D_U R)(Y) + \frac{1}{2}U(Yr)$$
$$-\frac{1}{2}Y(Ur) + (1-n)(D_Y R)(U) + (D_Y R)(U) - U(Yr)$$
$$= \frac{r}{n-1}[(n-1)YU + nUY - nYU - (n-1)UY],$$

which yield

$$(D_U R)(Y) - (D_Y R)(U) = 0. \tag{19}$$

Contracting (19) with respect to Y, we get

$$Ur - \frac{1}{2}Ur = 0,$$

or

$$Ur = 0.$$

This leads to the following:

Theorem 1 *The scalar curvature r of a conformally symmetric Riemannian manifold is constant.*

We now prove the following:

Theorem 2 *In an Einstein manifold, the conformal curvature tensor satisfies the following identity:*

$$(D_U C)(X,Y,Z) + (D_X C)(Y,U,Z) + (D_Y C)(U,X,Z) = 0.$$

Proof. Let M be an Einstein manifold. Then from (13), (14), (15) and (6) it follows that

$$C(X,Y,Z) = K(X,Y,Z) - \frac{k}{n-1}[g(Y,Z)X - g(X,Z)Y]. \tag{20}$$

Taking covariant derivative of (20) with respect to U, we get

$$(D_U C)(X,Y,Z) = (D_U K)(X,Y,Z). \tag{21}$$

Taking cyclic permutation of (21) twice with respect to U, X, Y, adding the three equations and using Bianchi's second identity, we get the required result.

From (21) we deduce the following

Corollary 1 *An Einstein space is symmetric if and only if it is conformally symmetric.*

3 Some results on symmetric spaces

From (6) and (9), we have

$$C(X,Y,Z) = N(X,Y,Z) + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \tag{22}$$

Let M be an Einstein manifold. Then by virtue of (15), equation (22) reduces to

$$C(X,Y,Z) = N(X,Y,Z) + \frac{nk}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \tag{23}$$

Taking covariant derivative of (23), we get

$$(D_U C)(X,Y,Z) = (D_U N)(X,Y,Z). \tag{24}$$

From (24) it is evident that if any of equations (5) and (12) holds, then the other one also holds.

This leads to the following

Theorem 3 *An Einstein manifold is conformally symmetric if and only if it is conharmonically symmetric.*

From (8) and (9), we have

$$N(X,Y,Z) = P(X,Y,Z) - \frac{1}{(n-2)}[\frac{1}{n-1}(Ric(Y,Z)X - Ric(X,Z)Y) \tag{25}$$

$$+g(Y,Z)R(X) - g(X,Z)R(Y)].$$

Taking covariant derivative of (25) and using (11), we have

$$(D_U N)(X,Y,Z) = -\frac{1}{(n-2)}[\frac{1}{n-1}((D_U Ric)(Y,Z)X - (D_U Ric)(X,Z)Y \\ +g(Y,Z)(D_U R)(X) - g(X,Z)(D_U R)(Y)]. \tag{26}$$

Let us suppose a projectively symmetric space to be conharmonically symmetric space. Then by virtue of (12) equation (26) reduces to

$$[(D_U Ric)(Y,Z)X - (D_U Ric)(X,Z)Y] \\ +(n-1)[g(Y,Z)(D_U R)(X) - g(X,Z)(D_U R)(Y)] = 0. \tag{27}$$

Conversely, if a projectively symmetric space satisfies the condition (27), then the equation (26) reduces to $(D_U N)(X,Y,Z) = 0$, which shows that the space is conharmonically symmetric.

Thus we have the following

Theorem 4 *The necessary and sufficient condition for a projectively symmetric space to be a conharmonically symmetric is that*

$$[(D_U Ric)(Y,Z)X - (D_U Ric)(X,Z)Y] \\ +(n-1)[g(Y,Z)(D_U R)(X) - g(X,Z)(D_U R)(Y)] = 0.$$

We now prove

Theorem 5 *Suppose that M is a smooth Riemannian manifold of dimension $n(n>2)$, which is Ricci symmetric. Then we have the following identity*

$$(D_U C)(X,Y,Z) + (D_X C)(Y,U,Z) + (D_Y C)(U,X,Z) \\ +\frac{1}{n-2}[(D_U W)(X,Y,Z) + (D_X W)(Y,U,Z) + (D_Y W)(U,X,Z)] = 0.$$

Proof. From (6) and (7), it follows that

$$C(X,Y,Z) = \frac{1}{(n-2)}[2(n-1)K(X,Y,Z) - nW(X,Y,Z) - Ric(Y,Z)X \\ +Ric(X,Z)Y - g(Y,Z)R(X) + g(X,Z)R(Y)]. \tag{28}$$

Taking the covariant derivative of (28) with respect to U, we get

$$(D_U C)(X,Y,Z) = \frac{1}{(n-2)}[2(n-1)(D_U K)(X,Y,Z) - n(D_U W)(X-Y,Z) \tag{29}$$

$$-(D_U Ric)(Y,Z)X + (D_U Ric)(X,Z)Y$$
$$-g(Y,Z)(D_U R)(X) + g(X,Z)(D_U R)(Y)].$$

Using (2) and (3) in (29), we get

$$(D_U C)(X,Y,Z) = \frac{1}{(n-2)}[2(n-1)(D_U K)(X,Y,Z) - n(D_U W)(X,Y,Z)] \tag{30}$$

Permuting equation (30) twice with respect to U, X, Y, adding the three equations and using Bianchi's second identity, we get the required result.

From (30) it is evident that if any two of the equations (1), (5) and (10) hold, then the third one also holds. This leads to the following:

Theorem 6 *For a Ricci-symmetric Riemannian manifold of dimension* $n(n > 2)$, *if any two of the following properties hold, then the third one also holds:*

(a) It is a conformally symmetric space,

(b) It is concircularly symmetric space,

(c) It is a symmetric space.

From Theorem 5, we deduce the following:

Corollary 2 *Let* M *be a smooth Riemannian manifold of dimension* $n(n > 2)$, *which is Ricci-symmetric. Then it is conformally symmetric if and only if it is concircularly symmetric.*

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