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BIHOLOMORPHIC CURVATURE OF AN ALMOST HERMITIAN MANIFOLD

Mileva Prvanović
11000 Beograd, Ljube Stojanovića 14, Yugoslavia

Abstract

In [3], Hsiung, Yang and Friedland obtained the necessary and sufficient condition for an almost Hermitian manifold to be of pointwise constant biholomorphic curvature. In the present paper, in the Section 3, we determine an equivalent condition and derive some consequences. In the Section 4 we investigate the biholomorphic curvature with respect to the tensor (9). In the Section 5, we examine the biholomorphic curvature of an almost Hermitian manifold satisfying the Bianchi-type identity.

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1. Introduction

Let (M, g) be a Riemannian manifold of dimension n $(n \geq 3)$ and R its Riemannian curvature tensor. Let T_p be the tangent vector space of M in $p \in M$, and π a plane in T_p, i. e. a real two-dimensional subspace of T_p. It is well known that the scalar

$$k(X,Y) = \frac{R(X,Y,X,Y)}{(g(X,Y))^2 - g(X,X)g(Y,Y)}, \tag{1}$$

where $X, Y \in \pi$ are two linearly independent vectors, is uniquely determined by the plane π, i. e. is independent of the choice of X and Y in it. The scalar (1) is **the sectional curvature** of M at p with respect to the section π. If at each $p \in M$, the sectional curvature is independent of section π, then it is an absolute constant, i. e. is independent of the point π, too and M is a Riemannian manifold of constant curvature.

The bisectional curvature is determined by two planes in T_p. But now it is important that both planes are oriented. So, we consider two planes,

π and κ of T_p. Let X, Y be a positively oriented basis of π and Z, W a positively oriented basis of κ. The scalar

$$B(\pi,\kappa) = -\frac{R(X,Y,Z,W)}{\begin{vmatrix} g(X,X) & g(X,Y) \\ g(Y,X) & g(Y,Y) \end{vmatrix}^{\frac{1}{2}} \begin{vmatrix} g(Z,Z) & g(Z,W) \\ g(W,Z) & g(W,W) \end{vmatrix}^{\frac{1}{2}}} \tag{2}$$

is uniquely determined by the planes π and κ, i. e. is independent of the choice of X, Y in π and Z, W in κ. This scalar is the **bisectional curvature** of M at p with respect to the sections π and κ ([5]). If $\pi = \kappa$, the bisectional cuvature reduces to the sectional curvature. Thus, the bisectional curvature is a generalization of the sectional curvature.

If a Riemannian manifold is endowed with an almost complex structure, then we can consider the holomorphic sectional and holomorphic bisectional curvatures too and just these curvatures are subjects if the investigation of the present paper. Thus, we consider an almost Hermitian manifold (M, g, J), i. e. a Riemannian manifold (M, g) of dimension $n(= 2m \geq 4)$ endowed withh endomorphism J satisfying

$$J^2 = -I, \; g(JX, JY) = g(X, Y), \tag{3}$$

where I indicates the identity mapping. It is well known that $F(X,Y) = g(JX, Y)$ satisfies

$$F(X,Y) = -F(Y,X). \tag{4}$$

The vectors X and JX are mutually orthogonal and thus determine the plane $\pi(X, JX)$. Such a plane is called a holomorphic plane. It is said to be cannonically oriented if it is positively oriented by X, JX. The sectional curvature with respect to the holomorphic plane at $p \in M$ is **the holomorphic sectonal curvature** (shortly **the holomorphic curvature**). According (1) and in view of (3), the holomorphic curvature with respect to $\pi(X, JX)$ of (M, g, J) at $p \in M$ is given by

$$H(x) = -\frac{R(X, JX, X, JX)}{g(X,X)g(X,X)}. \tag{5}$$

Obviously, it is independent of the choice of the vector X in the holomorphic plane.

If the holomorphic curvature is independent of the holomorphic plane at p, (M, g, J) is said to be of **pointwise constant holomorphic curvature**. Unlike sectional curvature, it need not be an absolute constant. In general, it is dependent of the point, and we express the corresponding condition as follows

$$-\frac{R(X,JX,X,JX)}{g(X,X)g(X,X)}=c(p). \tag{6}$$

The holomorphic bisectional curvature (shortly **the biholomorphic curvature**) $BH(\pi,\kappa)$ at $p\in M$ is the bisectional curvature determined by two cannonically oriented holomorphic planes π and κ at p. If $X\in\pi$, $Y\in\kappa$ we easily obtain, according (2) and (3)

$$BH(\pi,\kappa)=-\frac{R(X,JX,Y,JY)}{g(X,X)g(Y,Y)}. \tag{7}$$

Obviously, $BH(\pi,\kappa)$ is independent of the choice of $X\in\pi$ and $Y\in\kappa$ and $BH(\pi,\pi)=H(\pi)$.

If the biholomorphic curvature at $p\in M$ is independent of the holomorphic planes π and κ, (M,g,J) is said to be of **pointwise constant biholomorphic curvature.** We express the corresponding condition as follows

$$-\frac{R(X,JX,Y,JY)}{g(X,X)g(Y,Y)}=BH(p). \tag{8}$$

The holomorphic and biholomorphic curvatures have been investigated by many authors ([1], [2], [3], [6], [8], [9]). The aim of the present paper is to give some contributions to those investigations. For the latter use, we give in Section 2 some informations about the tensor $A(X,Y,Z,W)$, the curvature-like tensor of Kaehler type. In [3], Hsiung, Yang and Fridland obtained the necessary and sufficient condition for an (M,g,J) to be of pointwise constant biholomorphic curvature. In Section 3, we derive and equivalent necessary and sufficient condition and derive some consequences. In Section 4, we investigate the biholomorphic curvature with respect to the tensor $A(X,Y,Z,W)$ and in Section 5 we examine the bisectional curvature of (M,g,J) satisfying the Bianchi-type identity.

2. Tensor $A(X,Y,Z,W)$

Let us consider the tensor

$$A(X,Y,Z,W)= \tag{9}$$

$$\frac{1}{16}(3[R(X,Y,Z,W)+R(X,Y,JZ,JW)+R(JX,JY,Z,W)+R(JX,JY,JZ,JW)]$$
$$-R(X,Z,JW,JY)-R(JX,JZ,W,Y)-R(X,W,JY,JZ)-R(JX,JW,Y,Z)$$

$+R(JX,Z,JW,Y)+R(X,JZ,W,JY)+R(JX,W,Y,JZ)+R(X,JW,JY,Z))$.

In [4] (**Theorem 1**) it was proved that

$$A(X,Y,Z,W) = -A(X,Y,W,Z), \tag{10}$$

$$A(X,Y,Z,W) = -A(Y,X,Z,W), \tag{11}$$

$$A(X,Y,Z,W) = A(Z,W,X,Y), \tag{12}$$

$$A(X,Y,Z,W) + A(Y,Z,X,W) + A(Z,X,Y,W) = 0, \tag{13}$$

$$A(X,Y,Z,W) = A(X,Y,JZ,JW). \tag{14}$$

Thus, $A(X,Y,Z,W)$ is curvature-like tensor of Kaehler type. If the Riemannian curvature tensor of (M,g,J) is itself of Kaehler type, i. e. if

$$R(X,Y,Z,W) = R(X,Y,JZ,JW),$$

then $A(X,Y,Z,W)$ reduces to $R(X,Y,Z,W)$.

We note the following theorem ([4], Th. 4)

Theorem 1. *An almost Hermitian manifold is a manifold of a pointwise constant holomorphic curvature* $c(p)$ *if and only if*

$$A(X,Y,Z,W) =$$

$$\frac{c(p)}{4}[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \tag{15}$$

$$+F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)].$$

3. Constant biholomorphic curvature

In [3] (Th. 6.2), it was proved the following theorem

Theorem 2. *A necessary and sufficient condition for an almost Hermitian manifold* (M,g,J) *to be of pointwise constant biholomorphic curvature* $BH(p)$ *is that Riemannian curvature tensor satisfies*

$$R(X,Y,Z,W)+R(X,Y,JY,JW)+R(JX,JY,Z,W)+R(JX,JY,JZ,JW) =$$

$$= -4BH(p)F(X,Y)F(Z,W) \tag{16}$$

The following theorem gives an equivalent condition.

Theorem 3. *A necessary and sufficient condition for an almost Hermitian manifold (M, g, J) to be of pointwise constant biholomorphic curvature $BH(p)$ is that Riemannian curvature tensor satisfies*

$$R(X,Z,JW,JY)+R(JX,JZ,W,Y)+R(X,W,JY,JZ)+R(JX,JW,Y,Z)$$
$$-R(JX,Z,JW,Y)-R(X,JZ,W,JY)-R(JX,W,Y,JZ)-R(X,JW,JY,Z)=$$
$$=-4BH(p)[g(X,W)g(Y,Z)-g(X,Z)g(Y,W) \tag{17}$$
$$F(X,W)F(Y,Z)-F(X,Z)F(Y,W)+F(X,Y)F(Z,W)].$$

Proof. Let us suppose that (M, g, J) is of pointwise constant biholomorphic curvature. Then, by Theorem 2, (16) holds. Permutting in (16) X, Y, Z cyclically and summing up obtained three equations, we get

$$R(X,Y,JZ,JW)+R(Y,Z,JX,JW)+R(Z,X,JY,JW)$$
$$+R(JX,JY,Z,W)+R(JY,JZ,X,W)+R(JZ,JX,Y,W) \tag{18}$$
$$=-4BH(p)[F(X,Y)F(Z,W)+F(Y,Z)F(X,W)+F(Z,X)F(Y,W)].$$

Putting into (18) JX, JY instead of X, Y respectively, and taking into account (3) and (4), we find

$$R(JX,JY,JZ,JW)-R(JY,Z,X,JW)-R(Z,JX,Y,JW)$$
$$+R(X,Y,Z,W)-R(Y,JZ,JX,W)-R(JZ,X,JY,W) \tag{19}$$
$$=-4BH(p)[F(X,Y)F(Z,W)+g(Y,Z)g(X,W)-g(Z,X)g(Y,W)].$$

Adding (18) to (19), we get

$$R(X,Y,Z,W)+R(X,Y,JZ,JW)+R(JX,JY,Z,W)+R(JX,JY,JZ,JW)$$
$$+R(Y,Z,JX,JW)+R(Z,X,JY,JW)+R(JY,JZ,X,W)+R(JZ,JX,Y,W)$$
$$-R(JY,Z,X,JW)-R(Z,JX,Y,JW)-R(Y,JZ,JX,W)-R(JZ,X,JY,W)$$
$$=-4BH(p)[g(X,W)g(Y,Z)-g(X,Z)g(Y,W)$$
$$+F(X,W)F(Y,Z)-F(X,Z)F(Y,W)+2F(X,Y)F(Z,W)], \tag{20}$$

from which, in view of (16), we get (17).

Now, let us assume that (17) holds. Then, permutting X, Y, Z cyclically and summing up the obtained three equations and using the first Bianchi identity, we obtain (18). As we have seen, we derive (19) fron (18) and adding (18) to (19), we get (20). But (20), in view of (17) reduces to (16). □

Now, putting into (17) $Y = JX$ and using (3) and (4), we get

$$R(X,Z,X,Z)+R(JX,JZ,JX,JZ)+R(X,JZ,X,JZ)+R(JX,Z,JX,Z)$$

$$=2BH(p)[-2g(JX,Z)g(JX,Z)-2g(X,Z)g(X,Z)+g(X,X)g(Z,Z)].$$

This can be rewritten in the form

$$-\frac{R(X,Z,X,Z)}{g(X,X)g(Z;Z)}-\frac{R(JX,JZ,JX,JZ)}{g(X,X)g(Z,Z)}-\frac{R(X,JZ,X,JZ)}{g(X,X)g(Z,Z)}-\frac{R(JX,Z,JX,Z)}{g(X,X)g(Z,Z)}$$

$$=2BH(p)[2\frac{g(JX,Z)g(JX,Z)}{g(X,X)g(Z,Z)}+2\frac{g(X,Z)g(X,Z)}{g(X,X)g(Z,Z)}-1]. \tag{21}$$

On the other hand, according to (1), the sectional curvature, determined by linearly independent vectors X, Z is given by

$$k(X,Z)=\frac{R(X,Z,X,Z)}{(g(X,Z))^2-g(X,X)g(Z,Z)}. \tag{22}$$

Taking into account that

$$\cos(X,Z)=\frac{g(X,Z)}{[g(X,X)g(Z,Z)]^{\frac{1}{2}}},$$

we have

$$(g(X,Z))^2=\cos^2(X,Z)g(X,X)g(Z,Z), \tag{23}$$

because of which (22) can be rewritten in the form

$$-\frac{R(X,Z,X,Z)}{g(X,X)g(Z,Z)}=k(X,Z)\sin^2(X,Z). \tag{24}$$

Similarly, using (3), we have

$$-\frac{R(JX,JZ,JX,JZ)}{g(X,X)g(Z,Z)}=k(JX,JZ)\sin^2(X,Z), \tag{25}$$

$$(g(X,JZ))^2=\cos^2(X,JZ)g(X,X)g(Z,Z), \tag{26}$$

$$-\frac{R(X,JZ,X,JZ)}{g(X,X)g(Z,Z)}=k(X,JZ)\sin^2(X,JZ) \tag{27}$$

and

$$-\frac{R(JX,Z,JX,Z)}{g(X,X)g(Z,Z)}=k(JX,Z)\sin^2(X,JZ). \tag{28}$$

Substituting (24), (25), (27) and (28) into (21), we find

$$\begin{aligned}&[k(X,Z)+k(JX,JZ)]\sin^2(X,Z)\\&+[k(X,JZ)+k(JX,Z)]\sin^2(X,JZ)=\\&=-2BH(p)[2\cos^2(X,Z)+2\cos^2(X,JZ)-1].\end{aligned} \tag{29}$$

Thus, we state

Theorem 4. *If (M,g,J) is a manifold of pointwise constant biholomorphic curvature $BH(p)$, then for any linearly independent vectors $X,Z\in T_p$, the relation (29) holds.*

4. Biholomorphic curvature with respect to the tensor (9)

It is easy to see that

$$A(X,JX,X,JX)=R(X,JX,X,JX).$$

This means that the holomorphic curvature of the plane $\pi(X,JX)\subset T_p$ with respect to the tensor A is the same with respect to the tensor R. In general,

$$A(X,JX,Y,JY)\neq R(X,JX,Y,JY)$$

and thus it makes the sense to examine the biholomorphic curvature with respect to the tensor A.

Let π and κ be two canonically oriented holomorphic planes at $p\in M$. We define $BH_A(\pi,\kappa)$, the biholomorphic curvature with respect to the tensor $A(X,Y,Z,W)$ determined by π and κ as follows

$$BH_A(\pi,\kappa)=-\frac{A(X,JX,Y,JY)}{g(X,X)g(Y,Y)}, \tag{30}$$

where $X\in\pi$, $Y\in\kappa$. To make difference between the biholomorphic curvature with respect to $A(X,Y,Z,W)$ and with respect to $R(X,Y,Z,W)$, we will denote the last one by $BH_R(\pi,\kappa)$.

Like $BH_R(\pi,\kappa)$, $BH_A(\pi,\kappa)$ is uniquelly determined by the planes π and κ i. e. independent of the choice of $X\in\pi$, $Y\in\kappa$.

Putting into (9) $Y = JX$, $W = JZ$ and using (3), we find

$$8A(X,JX,Z,JZ) = 6R(X,JX,Z,JZ) + R(X,Z,X,Z) + R(JX,JZ,JX,JZ)$$
$$+R(X,JZ,X,JZ) + R(JX,Z,JX,Z).$$

This can be rewritten in the form

$$8\frac{A(X,JX,Z,JZ)}{g(X,X)g(Z,Z)} = 6\frac{R(X,JX,Z,JZ)}{g(X,X)g(Z,Z)}$$
$$+\frac{R(X,Z,X,Z)}{g(X,X)g(Z,Z)} + \frac{R(JX,JZ,JX,JZ)}{g(X,X)g(Z,Z)} \tag{31}$$
$$+\frac{R(X,JZ,X,JZ)}{g(X,X)g(Z,Z)} + \frac{R(JX,Z,JX,Z)}{g(X,X)g(Z,Z)},$$

or, in view of (7), (30), (24), (25) and (28), in the form

$$8HB_A(\pi,\kappa) = 6BH_R(\pi,\kappa)$$
$$+[k(X,Z) + k(JX,JZ)]\sin^2(JX,JZ) \tag{32}$$
$$+[k(X,JZ) + k(JX,Z)]\sin^2(X,JZ),$$

where we put $\pi = \pi(X,JX)$, $\kappa = \kappa(Z,JZ)$. Thus, we can state

Theorem 5. *Let π and κ be two holomorphic planes at $p \in M$. Then, for any $X \in \pi$, $Z \in \kappa$ the relation (32) holds*

Now, let us assume that (15) holds. Then, in view of (5) and (4) we have

$$A(X,JX,Z,JZ) =$$
$$-\frac{c(p)}{2}[g(JX,Z)g(JX,Z) + g(X,Z)g(X,Z) + g(X,X)g(Z,Z)].$$

Thus,

$$-\frac{A(X,JX,Z,JZ)}{g(X,X)g(Z,Z)} = \frac{c(p)}{2}\left[\frac{(g(JX,Z))^2}{g(X,X)g(Z,Z)} + \frac{(g(X,Z))^2}{g(X,X)g(Z,Z)} + 1\right],$$

or, in view of (30), (23) and (26),

$$BH_A(\pi,\kappa) = \frac{c(p)}{2}[1 + \cos^2(X,Z) + \cos^2(X,JZ)].$$

Sustituting this into (32), we find

$$6BH_R(\pi,\kappa) = 4c(p)[1+\cos^2(X,Z)+\cos^2(X,JZ)]$$
$$-[k(X,Z)+k(JX,JZ)]\sin^2(X,Z) \tag{33}$$
$$-[k(X,JZ)+k(JX,Z)]\sin^2(X,JZ).$$

Specially, if $c(p)=0$, we have

$$6BH_R(\pi,\kappa) = -[k(X,Z)+k(JX,JZ)]\sin^2(X,Z)$$
$$-[k(X,JZ)+k(JX,Z)]\sin^2(X,JZ). \tag{34}$$

Thus, and in view of Theorem 1, we can state

Theorem 6. *If (M,g,J) is a manifold of pointwise constant holomorphic curvature $c(p)$, then, for any two holomorphic planes $\pi,\kappa \subset T_p$, the biholomorphic curvature $BH_R(\pi,\kappa)$ can be expressed in the form (33), where $X\in\pi$, $Z\in\kappa$ are any non-null vectors. Specialy, if $c(p)=0$, then (34) holds.*

An other special case of (33) is the case when (M,g,J) is a manifold of constant sectional curvature k. Then

$$c(p)=k(X,Z)=k(JX,JZ)=k(X,JZ)=k(JX,Z)=k,$$

and if $k\neq 0$, (33) reduces to

$$BH_R(\pi,\kappa)=k[\cos^2(X,Z)+\cos^2(X,JZ)]. \tag{35}$$

Thus, we have the following corollary (cf. [4], Th. 7.5):

Corollary 1. *If (M,g,J) is a manifold of constant sectional curvature k and $k\neq 0$, the biholomorphic curvature $BH_R(\pi,\kappa)$ has the form (35), where $X\in\pi$ and $Z\in\kappa$ are any non-null vectors.*

For a manifold of pointwise constant curvature $BH_A(\pi,\kappa)$, we have the following theorem:

Theorem 7. *If (M,g,J) is a manifold of pointwise constant curvature $BH_A(\pi,\kappa)$, then this constant is zero.*

Proof. Let us suppose that $BH_A(\pi,\kappa)$ is independent of the choice of holomorphic planes π and κ at p on M. Then $BH_A(\pi,\kappa) = BH_A(p)$ and according to (30), the relation

$$A(X,JX,Z,JZ) = -BH_A(p)g(X,X)g(Z,Z) \tag{36}$$

is satisfied for any vectors $X, Z \in T_p$. Thus we have

$$A(X+Y,J(X+Y),Z,JZ) = -BH_A(p)g(X+Y,X+Y)g(Z,Z)$$

i. e.

$$\begin{gathered}A(X,JX,Z,JZ) + A(X,JY,Z,JZ) + A(Y,JX,Z,JZ)\\ +A(Y,JX,Z,JZ) + A(Y,JY,Z,JZ) =\\ = -BH_A(p)[g(X,X)g(Z,Z) + g(Y,Y)g(Z,Z) + 2g(X,Y)g(Z,Z)].\end{gathered}$$

Taking into account (36), we find that

$$A(X,JY,Z,JZ) + A(Y,JX,Z,JZ) = -2BH_A(p)g(X,X)g(Z,Z) \tag{37}$$

is valid for all $X, Y, Z \in T_p$. From (37), we obtain

$$\begin{gathered}A(X,JY,Z+W,J(Z+W)) + A(Y,JX,Z+W,J(Z+W))\\ = -2BH_A(p)g(X,Y)g(Z+W,Z+W),\end{gathered}$$

or

$$\begin{gathered}A(X,JY,Z,JZ) + A(X,JY,Z,JW) + A(X,JY,W,JZ) + A(X,JY,W,JW)\\ +A(Y,JX,Z,JZ)+A(Z,JX,Z,JW)+A(Y,JX,W,JZ)+A(Y,JX,W,JW) =\\ = -2BH_A(p)g(X,Y)[g(Z,Z) + g(W,W) + 2g(Z,W)].\end{gathered}$$

Thus, and in view of (37), we find

$$\begin{gathered}A(X,JY,Z,JW)+A(X,JY,W,JZ)+A(Y,JX,Z,JW)+A(Y,JX,W,JZ) =\\ = -4BH_A(p)g(X,Y)g(Z,W).\end{gathered}$$

Putting JY and JW instead of Y and W respectively and taking into account (3), we get

$$\begin{gathered}A(X,Y,Z,W)+A(X,Y,JZ,JW)+A(JX,JY,Z,W)+A(JX,JY,JZ,JW) =\\ = -4BH_A(p)F(X,Y)F(Z,W).\end{gathered}$$

But $A(X,Y,Z,W)$ satisfies conditions (10), (11), (12) and (14) and by this reason the preceeding relation reduces to

$$A(X,Y,Z,W) = -BH_A(p)F(X,Y)F(Z,W). \tag{38}$$

In view of (13), we obtain

$$BH_A(p)[F(X,Y)F(Z,W) + F(Y,Z)F(X,W) + F(Z,X)F(Y,W)] = 0.$$

This relation, taking into account the assumption $n \geq 4$ implies $BH_A(p) = 0$. □

Because of $BH_A(p) = 0$, (38) gives $A(X,Y,Z,W) = 0$. Specially,

$$\frac{A(X,JX,X,JX)}{(g(X,X))^2} = \frac{R(X,JX,X,JX)}{(g(X,X))^2} = 0.$$

This means that the holomorphic sectional curvature of the manifold, $c(p)$, is zero, too. Thus, as a consequence of the Theorems 6. and 7, we have

Corollary 2 *If (M,g,J) is a manifold of pointwise constant curvature $BH_A(\pi,\kappa)$, then the biholomorphic curvature $BH_R(\pi,\kappa)$ can be expressed in the form (34).*

Finally, let us consider the manifold (M,g,J) satisfying the condition

$$BH_A(\pi,\kappa) = BH_R(\pi,\kappa). \tag{39}$$

We have

Theorem 8. *An almost Hermitian manifold (M,g,J) satisfies the condition (39) if and only if the Riemannian curvature tensor satisfies*

$$\begin{aligned} &R(X,Y,Z,W) + R(X,Y,JZ,JW) + R(JX,JY,Z,W) \\ &+R(JX,JY,JZ,JW) + R(X,Y,JW,JZ) + R(JX,JZ,W,Y) \\ &+R(X,W,JY,JZ) + R(JX,JW,Y,Z) - R(JX,Z,JW,Y) \\ &-R(X,JZ,W,JY) - R(JX,W,Y,JZ) - R(X,JW,JY,Z) = 0. \end{aligned} \tag{40}$$

Proof. The condition (39), in view of (7) and (30) means that

$$A(X,JX,Z,JZ) = R(X,JX,Z,JZ) \tag{41}$$

is valid for any two vectors $X, Z \in T_p$. From (41), imitating the way described in the proof of Theorem 7, we find

$$A(X,JY,Z,JW)+A(X,JY,W,JZ)+A(Y,JX,Z,JW)+A(Y,JX,W,JZ) =$$

$$= R(X,JY,Z,JW)+R(X,JY,W,JZ)+R(Y,JX,Z,JW)+R(Y,JX,W,JZ).$$

Putting into this relation JY and JW instead of Y and W respectively and taking into account (3), we get

$$A(X,Y,Z,W)+A(X,Y,JZ,JW)+A(JX,JY,Z,W)+A(JX,JY,JZ,JW) =$$

$$= R(X,Y,Z,W)+R(X,Y,JZ,JW)+R(JX,JY,Z,W)+R(JX,JY,JZ,JW).$$

This relation, in view of (10), (11), (12) and (14) reduces to

$$4A(X,Y,Z,W) = \tag{42}$$

$$= R(X,Y,Z,W)+R(X,Y,JZ,JW)+R(JX,JY,Z,W)+R(JX,JY,JZ,JW).$$

Substituting $A(X,Y,Z,W)$ from (9), we find (40).

Conversely, let us suppose that (40) holds. Then (42) holds too, and from this fact we get (41), that is, (39).□

5. Manifolds satisfying the Bianchi-type identity

For an almost Hermitian manifold (M, g, J), G. B. Rizza introduced in [7] the following condition

$$R(X,Y,JZ,JW) + R(X,Z,JW,JY) + R(X,W,JY,JZ) \tag{43}$$

$$+R(JX,JY,Z,W) + R(JX,JZ,W,Y) + R(JX,JW,Y,Z) = 0.$$

If $R(X,Y,JZ,JW) = R(X,Y,Z,W)$ (in particular if (M,g,J) is a Kaehler manifold), (43) reduces to the Bianchi identity. For this reason, Rizza called (43) the Bianchi-type identity.

Putting into (43) $Y = JX, W = JZ$ and using (3), we get

$$2R(X,JX,Z,JZ) = R(X,Z,X,Z) + R(JX,JZ,JX,JZ)$$

$$+R(X,JZ,X,JZ) + R(JX,Z,JX,Z).$$

Thus, for non-null vectors $X, Z \in T_p$, we have

$$2\frac{R(X,JX,Z,JZ)}{g(X,X)g(Z,Z)} = \frac{R(X,Z,X,Z)}{g(X,X)g(Z,Z)} + \frac{R(JX,JZ,JX,JZ)}{g(X,X)g(Z,Z)}$$

$$+\frac{R(X,JZ,X,JZ)}{g(X,X)g(Z,Z)}+\frac{R(JX,Z,JX,Z)}{g(X,X)g(Z,Z)},$$

from which, in view of (7), (24), (25), (27) and (28), we get

$$2BR_R(\pi,\kappa)=[k(X,Z)+k(JX,JZ)]\sin^2(X,Z) \tag{44}$$

$$+[k(JX,Z)+k(X,JZ)]\sin^2(X,JZ).$$

Thus, we have

Theorem 9. *If (M,g,J) satisfies the Bianchi-type identity (43), then the biholomorphic curvature $BH_R(\pi,\kappa)$ is given by (44) for any non-null vectors $X\in\pi, Z\in\kappa$.*

If (M,g,J) is a manifold with pointwise constant holomorphic curvature, then (33) holds. From (33) and (44) we immediately derive (cf. [7], corollary 1):

Corollary 3 *If (M,g,J) has constant holomorphic curvature $c(p)$ and satisfies the Bianchi-type identity (43), then the biholomorphic curvature has the form*

$$BH_R(\pi,\kappa)=\frac{c(p)}{2}[1+\cos^2(X,Z)+\cos^2(X,JZ)],$$

for any non-null vectors $X\in\pi, Z\in\kappa$.

Putting into (43) $Z=JZ, W=JW$, we find

$$R(X,Y,Z,W)-R(X,JZ,W,JY)-R(X,JW,JY,Z)$$

$$+R(JX,JY,JZ,JW)-R(JX,Z,JW,Y)-R(JX,W,Y,JZ)=0. \tag{45}$$

Summing up (43) and (45), we get (40). Conversely, permutting in (40) X,Y,Z cyclically, we can obtain (43). Thus, from the Theorem 8 we derive

Theorem 10. *The Bianchi-type identity (43) is necessary and sufficient condition for an (M,g,J) to satisfy the condition $BH_A(\pi,\kappa)=BH_R(\pi,\kappa)$ for any holomorphic planes $\pi,\kappa\in T_p$.*

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