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3–TYPE CURVES IN THE EUCLIDEAN SPACE E^4

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Abstract

In [1] D. Blair gave a complete classification of 3–type curves in the space E^3. In this paper we give a complete classification of 3–type curves in the space E^4.

The notion of curves of finite type was introduced by B. Y. Chen around 1980. A closed curve γ in a Euclidean space E^n has a finite type (type k, $k \in N$) if its Fourier series expansion with respect to an arclength parameter is finite (has exactly k nonzero terms).

It is proved in [3] that a closed curve γ: $[0, 2\pi r] \mapsto E^n$ is of k–type ($k \in N$) if and only if there is a vector $A_0 \in E^n$, natural numbers $p_1 < p_2 < \ldots < p_k$, and vectors $A_1, \ldots, A_k, B_1, \ldots, B_k \in E^n$ such that $\|A_i\|^2 + \|B_i\|^2 \neq 0$ ($i = 1, \ldots, k$) and

$$\gamma(s) = A_0 + \sum_{i=1}^{k}(A_i \cos \frac{p_i s}{r} + B_i \sin \frac{p_i s}{r}).$$

It is shown in [5] that the only interesting case is then $n \leq 2k$.

In particular, 3–type curves in the space E^3, have been investigated several times in the literature (see e.g. [1], [8], [9]). One of the most important papers in that direction is the paper [1] by D. Blair, where a complete classification of such curves in the space E^3 is given.

In this paper we will go a step further, and classify 3–type curves in the space E^4. *In some subsequent papers we shall also consider 3–type curves* *in the spaces E^5 and E^6.* This will obviously complete the investigation of 3–type curves in Euclidean spaces.

Our paper is close to the paper [1], but it is not a simple imitation of this paper. Namely in the space E^4 we often really meet some cases and situations which are contradictory in the space E^3.

We need to mention here that by usual lifting $(x,y,z)\mapsto(x,y,z,0)$ of the space E^3 in E^4, every 3–type curve in the space E^3 becomes a 3–type curve in the space E^4. So, it is interesting to search only for 3–type curve in the space E^4 which are not of such a type.

By the general statement, we have that a curve $\gamma\subseteq E^4$ is of 3–type if there are natural numbers $p_1<p_2<p_3$ (frequency numbers of the curve) such that $\gamma\colon[0,2\pi r]\mapsto E^4$ has the form

$$\gamma(s)=A_0+\sum_{i=1}^{3}(A_i\cos\frac{p_i s}{r}+B_i\sin\frac{p_i s}{r}),$$

where $A_0\in E^4$ and $A_1,A_2,A_3,B_1,B_2,B_3\in E^4$ are such that $\|A_i\|^2+\|B_i\|^2\neq 0$ for each $i=1,2,3$.

It is proved in [3] that the last condition is equivalent to the following system of equations:

$$\sum_{i=1}^{3}p_i^2D_{ii}=2r^2, \tag{O}$$

$$\sum_{\substack{i=1\\2p_i=l}}^{3}p_i^2A_{ii}+2\sum_{\substack{i,j=1\\i>j\\p_i+p_j=l}}^{3}p_ip_jA_{ij}-\sum_{\substack{i,j=1\\i>j\\p_i-p_j=l}}^{3}p_ip_jD_{ij}=0, \tag{$I(l)$}$$

$$\sum_{\substack{i=1\\2p_i=l}}^{3}p_i^2\overline{A}_{ii}+2\sum_{\substack{i,j=1\\i>j\\p_i+p_j=l}}^{3}p_ip_j\overline{A}_{ij}-\sum_{\substack{i,j=1\\i>j\\p_i-p_j=l}}^{3}p_ip_j\overline{D}_{ij}=0, \tag{$\overline{I}(l)$}$$

where

$$A_{ij}=\langle A_i,A_j\rangle-\langle B_i,B_j\rangle,\quad \overline{A}_{ij}=\langle A_i,B_j\rangle+\langle A_j,B_i\rangle,$$
$$D_{ij}=\langle A_i,A_j\rangle+\langle B_i,B_j\rangle,\quad \overline{D}_{ij}=\langle A_i,B_j\rangle-\langle A_j,B_i\rangle,$$

$(i,j=1,2,3)$, and l runs the set

$$\mathcal{A}=\{2p_1,2p_2,2p_3,p_1+p_2,p_1+p_3,p_2+p_3,p_2-p_1,p_3-p_1,p_3-p_2\}.$$

The main theorem of this paper is the following.

Theorem. *If $\gamma(s)$ is a 3–type curve in the Euclidean space E^4, then parameter p_3 equals to some of the numbers $2p_1+p_2$, p_1+2p_2, $2p_2-p_1$ and the curve belongs to a multiparameter family of curves, or the frequency numbers are in the ratio $1:3:9$ and the curve belongs to a 5–parameter family of curves.*

The proof of this theorem follows from a series of propositions which we are going to prove.

In the sequel, the most important thing is to differ the cases when all indices in the set $\mathcal{A}$ are distinct, or some of them coincide.

The complete classification of all these cases is as follows.

$$p_2 \neq 3p_1,\; p_3 \neq 3p_1, 3p_2, p_2+2p_1, 2p_2 \pm p_1 \qquad (1^0)$$
$$p_2 = 3p_1,\; p_3 \neq 5p_1, 7p_1, 9p_1 \qquad (2^0)$$
$$p_2 \neq 2p_1,\; p_3 = 3p_1 \qquad (3^0)$$
$$p_2 \neq 3p_1,\; p_3 = 3p_2 \qquad (4^0)$$
$$p_2 \neq 3p_1,\; p_3 = p_2+2p_1 \qquad (5^0)$$
$$p_2 = 3p_1,\; p_3 = 5p_1 \qquad (6^0)$$

$$p_2 \neq 3p_1,\; p_3 = p_1+2p_2 \qquad (7^0)$$
$$p_2 = 3p_1,\; p_3 = 7p_1 \qquad (8^0)$$
$$p_2 \neq 2p_1,\; 3p_1, \quad p_3 = 2p_2-p_1 \qquad (9^0)$$
$$p_2 = 2p_1,\; p_3 = 3p_1 \qquad (10^0)$$
$$p_2 = 3p_1,\; p_3 = 9p_1. \qquad (11^0)$$

We shall discuss all these cases separately. We mention that the case $p_3 = 3p_2$ is contradictory in the space E^3 (see [1]), but here in the case (11^0) it gives a family of curves.

Next we introduce the following notations:

$$A_1 = (a_{11}, a_{12}, a_{13}, a_{14}), \quad B_1 = (b_{11}, b_{12}, b_{13}, b_{14}),$$
$$A_2 = (a_{21}, a_{22}, a_{23}, a_{24}), \quad B_2 = (b_{21}, b_{22}, b_{23}, b_{24})$$
$$A_3 = (a_{31}, a_{32}, a_{33}, a_{34}), \quad B_3 = (b_{31}, b_{32}, b_{33}, b_{34}).$$

If some index in the set $\mathcal{A}$ differs of all other indices in this set , we shall call it "single". The set $\mathcal{A}$ obviously has at least two single indices, namely $2p_3$ and p_2+p_3. These indices are evidently the greatest in $\mathcal{A}$.

Lemma 1. *By a suitable change of the coordinate system, we can assume that*

$$A_3 = (\mu, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0) \qquad (\mu \neq 0).$$

In this system we have $b_{21} = -a_{22}$, $b_{22} = a_{21}$, *thus* $B_2 = (-a_{22}, a_{21}, b_{23}, b_{24})$.

Proof. Since the parameters $2p_3$ and $p_2 + p_3 < 2p_3$ are single in $\mathcal{A}$, we have the equalities $I(2p_3), \overline{I}(2p_3), I(p_2+p_3), \overline{I}(p_2+p_3)$, and hence we obtain

$$A_{33} = \overline{A}_{33} = A_{32} = \overline{A}_{32} = 0. \tag{1}$$

From the first two relations we find that $\|A_3\| = \|B_3\|$ and $A_3 \perp B_3$. By changing the coordinate system we can assume that then

$$A_3 = (\mu, 0, 0, 0), \; B_3 = (0, \mu, 0, 0)$$

for some $\mu \neq 0$.

Then the third and the fourth relation give

$$\langle A_3, A_2 \rangle = \langle B_3, B_2 \rangle, \quad \langle A_3, B_2 \rangle = -\langle A_2, B_3 \rangle,$$

hence

$$\mu\, a_{21} = \mu\, b_{22}, \; \mu\, b_{21} = -\mu\, a_{22}.$$

Since $\mu \neq 0$, we obtain $b_{22} = a_{21}$, $b_{21} = -a_{22}$, thus

$$B_2 = (-a_{22}, a_{21}, b_{23}, b_{24}).\square$$

Lemma 2. *If* $2p_2$ *and* $p_3 - p_2$ *are single parameters, then by a suitable change of coordinate system we can assume that*

$$A_2 = (0, 0, \nu, 0), B_2 = (0, 0, 0, \nu),$$

for some $\nu \neq 0$.

Proof. Since $2p_2$ and $p_3 - p_2$ are single, by relations $I(p_3 - p_2), \overline{I}(p_3 - p_2), I(2p_2), \overline{I}(2p_2)$ we obtain

$$D_{32} = \overline{D}_{32} = A_{22} = \overline{A}_{22} = 0.$$

Since

$$A_2 = (a_{21}, a_{22}, a_{23}, a_{24}), \quad B_2 = (-a_{22}, a_{21}, b_{23}, b_{24}),$$

relations $D_{32} = \overline{D}_{32} = 0$ give that $a_{21} = a_{22} = 0$. Next by relation $A_{22} = \overline{A}_{22} = 0$ we find that $\|A_2\| = \|B_2\|$ and $A_2 \perp B_2$, hence we can choose a new coordinate system in which A_2 and B_2 have this form. □

Proposition 1. *Cases* $(1^0), (2^0), (3^0)$ *and* (4^0) *are impossible.*

Proof. Case (1^0). In this case we have that $|\mathcal{A}| = 9$, thus all mentioned parameters are single. By Lemma 2 we can assume that

$$A_2 = (0, 0, \nu, 0), \; B_2 = (0, 0, 0, \nu)$$

for some $\nu \neq 0$.

Next by relations $I(p_1 + p_2), \overline{I}(p_1 + p_2), I(p_2 - p_1)$ and $\overline{I}(p_2 - p_1)$, we easily find that $A_1, B_1 \perp A_2, B_2$.

Similarly, by relations $I(p_1 + p_3), \overline{I}(p_1 + p_3), I(p_3 - p_1)$ and $\overline{I}(p_3 - p_1)$ we find that $A_1, B_1 \perp A_3, B_3$.

But since A_2, B_2, A_3, B_3 span all the space E^4, we obtain a contradiction $A_1 = B_1 = 0$.

Case (2^0). In this case the only equality is $2p_1 = p_2 - p_1$. Hence by Lemma 2 we again find that

$$A_2 = (0, 0, \nu, 0), \; B_2 = (0, 0, 0, \nu).$$

Next by relations $I(p_1 + p_3), \overline{I}(p_1 + p_3), I(p_3 - p_1)$ and $\overline{I}(p_3 - p_1)$ we easily find that $A_1, B_1 \perp A_3, B_3$. Hence we can assume that

$$A_1 = (0, 0, a_{13}, a_{14}), \; B_1 = (0, 0, b_{13}, b_{14}).$$

Next relations $I(p_1 + p_2), \overline{I}(p_1 + p_2)$ give $A_{12} = \overline{A}_{12} = 0$, hence $b_{13} = -a_{14}$, $b_{14} = a_{13}$, that is $B_1 = (0, 0, -a_{14}, a_{13})$, and therefore $\|A_1\| = \|B_1\|, A_1 \perp B_1$.

Now by relations $I(2p_1)$ and $\overline{I}(2p_1)$ we find

$$A_{11} = 3D_{21}, \overline{A}_{11} = 3\overline{D}_{21},$$

and since $A_{11} = 0, \overline{A}_{11} = 0$, we obtain that $D_{21} = 0, \overline{D}_{21} = 0$. Hence $A_1, B_1 \perp A_2, B_2$ which gives a contradiction $A_1 = B_1 = 0$.

Case (3^0). In that case, the only equality is $2p_1 = p_3 - p_1$, and by Lemma 2 we again find that

$$A_2 = (0, 0, \nu, 0), B_2 = (0, 0, 0, \nu) \qquad (\nu \neq 0).$$

By relations $I(p_1+p_2), \overline{I}(p_1+p_2), I(p_2-p_1), \overline{I}(p_2-p_1)$ we get that

$$A_{21} = \overline{A}_{21} = D_{21} = \overline{D}_{21} = 0,$$

hence $A_1, B_1 \perp A_2, B_2$, so we can assume that

$$A_1 = (a_{11}, a_{12}, 0, 0),\ B_1 = (b_{11}, b_{12}, 0, 0).$$

Next by relations $I(p_1+p_3), \overline{I}(p_1+p_3)$ we get that $A_{31} = \overline{A}_{31} = 0$, hence $b_{11} = -a_{12}$, $b_{12} = a_{11}$. So $B_1 = (-a_{12}, a_{11}, 0, 0)$. But then $\|A_1\| = \|B_1\|$ and $A_1 \perp B_1$.

Finaly, by relations $I(2p_1)$ and $\overline{I}(2p_1)$, $\quad (2p_1 = p_3 - p_1)$ we obtain that

$$A_{11} = 3D_{31},\ \overline{A}_{11} = 3\overline{D}_{31}.$$

But since $A_{11} = \overline{A}_{11} = 0$, we obtain that $D_{31} = \overline{D}_{31} = 0$, hence $a_{11} = a_{12} = 0$.

So we again get a contradiction $A_1 = B_1 = 0$.

Case (4^0). $(p_2 \neq 3p_1, p_3 = 3p_2)$. Then the only equality is $2p_2 = p_3 - p_2$.

By relations $I(p_1+p_3), \overline{I}(p_1+p_3), I(p_3-p_1), \overline{I}(p_3-p_1)$ we find that $A_1, B_1 \perp A_3, B_3$. Hence we can put

$$A_1 = (0, 0, a_{13}, a_{14}), \quad B_1 = (0, 0, b_{13}, b_{14}).$$

By relations $I(2p_1), \overline{I}(2p_1)$ we also find that $\|A_1\| = \|B_1\|, A_1 \perp B_1$, so in a suitable chosen coordinate system we can take

$$A_1 = (0, 0, \nu, 0), \quad B_1 = (0, 0, 0, \nu) \qquad (\nu \neq 0).$$

Next by relations $I(p_1+p_2), \overline{I}(p_1+p_2), I(p_2-p_1), \overline{I}(p_2-p_1)$, we get that $A_2, B_2 \perp A_1, B_1$, thus we can take

$$A_2 = (a_{21}, a_{22}, 0, 0), \quad B_2 = (-a_{22}, a_{21}, 0, 0).$$

Then $\|A_2\| = \|B_2\|, A_2 \perp B_2$, and consequently $A_{22} = \overline{A}_{22} = 0$.

Finally, by relations $I(2p_2), \overline{I}(2p_2) \quad (2p_2 = p_3 - p_2)$ we obtain

$$A_{22} = 3D_{32}, \quad \overline{A}_{22} = 3\overline{D}_{32},$$

hence $D_{32} = \overline{D}_{32} = 0$. This easily gives $a_{21} = a_{22} = 0$, thus $A_2 = B_2 = 0$, what is a contradiction. □

Proposition 2 (Case (5^0)) ($p_2 \neq 3p_1, p_3 = p_2 + 2p_1$). *In this case a curve $\gamma(s)$ has the type 3 if and only if in a coordinate system we have*

$$A_1 = (a_{11}, a_{12}, a_{13}, a_{14}), \quad B_1 = (-a_{12}, a_{11}, b_{13}, b_{14}),$$
$$A_2 = (a_{21}, a_{22}, \nu, 0), \quad B_2 = (-a_{22}, a_{21}, 0, \nu),$$
$$A_3 = (\mu, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0),$$

where $\mu \neq 0$, $\nu \in R$ and

$$a_{21}\, a_{11} + a_{22}\, a_{12} = -\tfrac{\nu}{2}\,(a_{13} + b_{14}), \qquad (1)$$
$$a_{21}\, a_{12} - a_{22}\, a_{11} = \tfrac{\nu}{2}\,(b_{13} - a_{14}), \qquad (2)$$
$$a_{13}^2 + a_{14}^2 - b_{13}^2 - b_{14}^2 = \tfrac{2p_2p_3\,\mu\, a_{21}}{p_1^2}, \qquad (3)$$
$$a_{13}b_{13} + a_{14}b_{14} = -\tfrac{p_2p_3\,\mu\, a_{22}}{p_1^2}, \qquad (4)$$
$$\nu\,(a_{13} - b_{14}) = \tfrac{p_3\,\mu\, a_{11}}{p_2}, \qquad (5)$$
$$a_{21}\, a_{12} - a_{22}\, a_{11} - \nu\, b_{13} = \tfrac{p_3\,\mu\, a_{12}}{2p_2}. \qquad (6)$$

Remark. The system above has 6 equations and 10 parameters. Hence 4 of these parameters are "free" while the other 6 can be expressed by them. This system always has at least one solution since it corresponds to the case 3.1 by D. Blair.

Proof. In this case the only equalities are

$$2p_1 = p_3 - p_2, \quad p_1 + p_2 = p_3 - p_1.$$

Assuming that $\gamma(s)$ is of type 3 and using relations $I(2p_2)$, $\overline{I}(2p_2)$ we get $A_{22} = \overline{A}_{22} = 0$, hence $||A_2|| = ||B_2||$ and $A_2 \perp B_2$. Since $A_2 = (a_{21}, a_{22}, a_{23}, a_{24}), B_2 = (-a_{22}, a_{21}, b_{23}, b_{24})$, we get that

$$a_{23}^2 + a_{24}^2 = b_{23}^2 + b_{24}^2, \quad a_{23}b_{23} + a_{24}b_{24} = 0.$$

Hence, in a new coordinate system, A_2, B_2 will have the form

$$A_2 = (a_{21}, a_{22}, \nu, 0), \quad B_2 = (-a_{22}, a_{21}, 0, \nu),$$

for some $\nu \in R$.

Next using the equations $I(p_1+p_3), \overline{I}(p_1+p_3)$ we find that $A_{31} = \overline{A}_{31} = 0$, whence

$$\langle A_3, A_1\rangle = \langle B_3, B_1\rangle, \quad \langle A_3, B_1\rangle = -\langle B_3, A_1\rangle.$$

Hence we get $b_{11} = -a_{12}$, $b_{12} = a_{11}$, and consequently $B_1 = (-a_{12}, a_{11}, b_{13}, b_{14})$.

Next using the equations $I(p_2 - p_1), \overline{I}(p_2 - p_1)$ we get $D_{21} = \overline{D}_{21} = 0$, thus

$$\langle A_2, A_1 \rangle = -\langle B_2, B_1 \rangle, \quad \langle A_2, B_1 \rangle = \langle A_1, B_2 \rangle.$$

The last two relations immediately give the equations (1) and (2).

Finally, using the equations $I(2p_1), \overline{I}(2p_1)$ $(2p_1 = p_3 - p_2)$, $I(p_1+p_2), \overline{I}(p_1+p_2)$ $(p_1 + p_2 = p_3 - p_1)$, we get

$$A_{11} = \frac{p_2 p_3}{p_1^2} D_{32}, \quad \overline{A}_{11} = \frac{p_2 p_3}{p_1^2} \overline{D}_{32}, \quad A_{21} = \frac{p_3}{2p_2} D_{31}, \quad \overline{A}_{21} = \frac{p_3}{2p_2} \overline{D}_{31}.$$

Since

$$\begin{aligned}
A_{11} &= ||A_1||^2 - ||B_1||^2 = a_{13}^2 + a_{14}^2 - b_{13}^2 - b_{14}^2, \\
\overline{A}_{11} &= 2\langle A_1, B_1 \rangle = 2(a_{13}b_{13} + a_{14}b_{14}), \\
A_{21} &= \langle A_2, A_1 \rangle - \langle B_2, B_1 \rangle = \nu\,(a_{13} - b_{14}), \\
\overline{A}_{21} &= \langle A_2, B_1 \rangle + \langle A_1, B_2 \rangle = 2\langle A_2, B_1 \rangle = 2(-a_{21}\,a_{12} + a_{22}\,a_{11} + \nu\, b_{13}), \\
D_{32} &= \langle A_3, A_2 \rangle + \langle B_3, B_2 \rangle = 2\mu\, a_{21}, \\
\overline{D}_{32} &= \langle A_3, B_2 \rangle - \langle A_2, B_3 \rangle = -2\mu\, a_{22}, \\
D_{31} &= \langle A_3, A_1 \rangle + \langle B_3, B_1 \rangle = 2\mu\, a_{11}, \\
\overline{D}_{31} &= \langle A_3, B_1 \rangle - \langle A_1, B_3 \rangle = -2\mu\, a_{12},
\end{aligned}$$

a substitution immediately gives the equations (3)–(6).

Since the converse is immediate, the proof is complete. □

Proposition 3 (Case (6^0)) $(p_1 : p_2 : p_3 = 1 : 3 : 5)$. *In this case a curve $\gamma(s)$ is of type 3 if and only if, in a coordinate system we have*

$$\begin{aligned}
A_1 &= (a_{11}, a_{12}, a_{13}, a_{14}), \quad B_1 = (b_{11}, b_{12}, b_{13}, b_{14}), \\
A_2 &= (a_{21}, a_{22}, a_{23}, a_{24}), \quad B_2 = (-a_{22}, a_{21}, b_{23}, b_{24}), \\
A_3 &= (\mu, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0),
\end{aligned}$$

where $\mu \neq 0$ and

$$\begin{aligned}
a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{14}^2 - b_{11}^2 - b_{12}^2 - b_{13}^2 - b_{14}^2 &= \\
= 3\,(a_{21}\,a_{11} + a_{22}\,a_{12} + a_{23}\,a_{13} + a_{24}\,a_{14}) &+ \\
+3\,(-a_{22}\,b_{11} + a_{21}\,b_{12} + b_{23}b_{13} + b_{24}b_{14}) &+ 30\mu\, a_{21},
\end{aligned} \tag{1}$$

$$\begin{aligned} a_{11}b_{11} + &a_{12}b_{12} + a_{13}b_{13} + a_{14}b_{14} = \\ &= 1.5\,(a_{21}\,b_{11} + a_{22}\,b_{12} + a_{23}\,b_{13} + a_{24}\,b_{14}) - \\ &-1.5\,(-a_{22}\,a_{11} + a_{21}\,a_{12} + b_{23}\,a_{13} + b_{24}\,a_{14}) - 15\mu\,a_{22}, \end{aligned} \tag{2}$$

$$\begin{aligned} (a_{21}\,a_{11} + &a_{22}\,a_{12} + a_{23}\,a_{13} + a_{24}\,a_{14}) - \\ &-(-a_{22}\,b_{11} + a_{21}\,b_{12} + b_{23}\,b_{13} + b_{24}\,b_{14}) = \tfrac{5}{6}\,\mu\,(a_{11} + b_{12}), \end{aligned} \tag{3}$$

$$\begin{aligned} (a_{21}\,b_{11} + &a_{22}\,b_{12} + a_{23}\,b_{13} + a_{24}\,b_{14}) + \\ &+(-a_{22}\,a_{11} + a_{21}\,a_{12} + b_{23}\,a_{13} + b_{24}\,a_{14}) = \tfrac{5}{6}\,\mu\,(b_{11} - a_{12}), \end{aligned} \tag{4}$$

$$a_{23}^2 + a_{24}^2 - b_{23}^2 - b_{24}^2 = -\tfrac{10}{9}\,\mu\,(a_{11} - b_{12}), \tag{5}$$

$$a_{23}b_{23} + a_{24}b_{24} = -\tfrac{5}{9}\,\mu\,(b_{11} + a_{12}). \tag{6}$$

Remark. Since the system above has 6 equations with 15 parameters, 9 of these parameters are "free" while the other 6 can be expressed by free ones. This system has at least one solution since it corresponds to the case 2.1 by D. Blair.

Proof. The only equalities in this case are

$$2p_1 = p_2 - p_1 = p_3 - p_2, \quad p_1 + p_2 = p_3 - p_1, \quad p_1 + p_3 = 2p_2.$$

Assuming that $\gamma(s)$ is of type 3 we firstly find that vectors A_2, B_2, A_3, B_3, in a suitable choosen coordinate system, have the mentioned form. Next by relations $I(2p_1), \overline{I}(2p_1), I(p_1 + p_2), \overline{I}(p_1 + p_2), I(p_1 + p_3), \overline{I}(p_1 + p_3)$ we find the equations

$$\begin{aligned} &A_{11} - 3D_{21} - 15D_{32} = 0, \\ &\overline{A}_{11} - 3\overline{D}_{21} - 15\overline{D}_{32} = 0, \\ &A_{21} = \tfrac{5}{6}\,D_{31}, \quad \overline{A}_{21} = \tfrac{5}{6}\,\overline{D}_{31}, \\ &A_{22} = -\tfrac{10}{9}\,A_{31}, \quad \overline{A}_{22} = -\tfrac{10}{9}\,\overline{A}_{31}. \end{aligned}$$

Since

$$\begin{aligned} D_{32} &= \langle A_3, A_2\rangle + \langle B_3, B_2\rangle = 2\mu\,a_{21}, \\ \overline{D}_{32} &= \langle A_3, B_2\rangle - \langle A_2, B_3\rangle = -2\mu\,a_{22}, \\ D_{31} &= \langle A_3, A_1\rangle + \langle B_3, B_1\rangle = \mu\,(a_{11} + b_{12}), \\ \overline{D}_{31} &= \langle A_3, B_1\rangle - \langle A_1, B_3\rangle = \mu\,(b_{11} - a_{12}), \\ A_{31} &= \langle A_3, A_1\rangle - \langle B_3, B_1\rangle = \mu\,(a_{11} - b_{12}), \\ \overline{A}_{31} &= \langle A_3, B_1\rangle + \langle A_1, B_3\rangle = \mu\,(b_{11} + a_{12}), \end{aligned}$$

a substitution immediately gives the equations (1)–(6).

Since the converse statement is immediate, the proof is completed. □

Proposition 4 (Case (7^0)) ($p_2 \neq 3p_1, p_3 = p_1 + 2p_2$). *In this case a curve $\gamma(s)$ is of type 3 if and only if, in a coordinate system we have*

$$A_1 = (a_{11}, a_{12}, \nu, 0), \quad B_1 = (-a_{12}, a_{11}, 0, \nu),$$
$$A_2 = (a_{21}, a_{22}, a_{23}, a_{24}), \quad B_2 = (-a_{22}, a_{21}, b_{23}, b_{24}),$$
$$A_3 = (\mu, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0),$$

where $\mu \neq 0$, $\nu \in R$ and

$$a_{21}\, a_{11} + a_{22}\, a_{12} = -\tfrac{\nu}{2}\, (a_{23} + b_{24}), \qquad (1)$$
$$a_{21}\, a_{12} - a_{22}\, a_{11} = \tfrac{\nu}{2}\, (a_{24} - b_{23}), \qquad (2)$$
$$a_{23}^2 + a_{24}^2 - b_{23}^2 - b_{24}^2 = \tfrac{2p_1 p_3 \mu}{p_2^2}\, a_{11}, \qquad (3)$$
$$a_{23} b_{23} + a_{24} b_{24} = -\tfrac{p_1 p_3 \mu}{p_2^2}\, a_{12}, \qquad (4)$$
$$\nu\, (a_{23} - b_{24}) = \tfrac{p_3 a_{21}\, \mu}{p_1}, \qquad (5)$$
$$\nu\, (a_{24} + b_{23}) = -\tfrac{p_3 a_{22} \mu}{p_1}. \qquad (6)$$

Remark. The system above has 6 equations and 10 parameters. Hence 4 of these parameters are "free" while the other 6 can be expressed by them. This system always has at least one solution since it corresponds to the case 1.2 by D. Blair.

Proof. In this case the only equalities are

$$2p_2 = p_3 - p_1, \quad p_1 + p_2 = p_3 - p_2.$$

If $\gamma(s)$ has the type 3, then A_i, B_i ($i = 2, 3$) have the form as was described earlier. Next by relations $I(2p_1), \overline{I}(2p_1)$ we have $A_{11} = \overline{A}_{11} = 0$, hence $\|A_1\| = \|B_1\|$ and $A_1 \perp B_1$.

Next by relations $I(p_1 + p_3), \overline{I}(p_1 + p_3), I(p_2 - p_1), \overline{I}(p_2 - p_1)$, we get

$$A_{31} = \overline{A}_{31} = D_{21} = \overline{D}_{21} = 0,$$

therefore

$$\langle A_3, A_1\rangle = \langle B_3, B_1\rangle, \quad \langle A_3, B_1\rangle = -\langle B_3, A_1\rangle,$$
$$\langle A_2, A_1\rangle = -\langle B_2, B_1\rangle, \quad \langle A_2, B_1\rangle = \langle A_1, B_2\rangle.$$

First two of these equations give

$$\mu a_{11} = \mu b_{12}, \quad \mu b_{11} = -\mu a_{12},$$

hence $b_{12} = a_{11}$, $b_{11} = -a_{12}$ because $\mu \neq 0$. Hence $B_1 = (-a_{12}, a_{11}, b_{13}, b_{14})$.

Since $\|A_1\| = \|B_1\|$ and $A_1 \perp B_1$, we can again change the coordinate system so that A_1, B_1 have the form

$$A_1 = (a_{11}, a_{12}, \nu, 0), \quad B_1 = (-a_{12}, a_{11}, 0, \nu).$$

Another two of the above equations now easily give equations (1) and (2).

Finally, by relations $I(2p_2), \overline{I}(2p_2)$ $(2p_2 = p_3 - p_1), I(p_1 + p_2), \overline{I}(p_1 + p_2)$ $(p_1 + p_2 = p_3 - p_2)$ we get

$$A_{22} = \frac{p_1 p_3}{p_2^2} D_{31}, \quad \overline{A}_{22} = \frac{p_1 p_3}{p_2^2} \overline{D}_{31}, \quad A_{21} = \frac{p_3}{2p_1} D_{32}, \quad \overline{A}_{21} = \frac{p_3}{2p_1} \overline{D}_{32}.$$

Hence we easily find the equations (3)–(6).

Since the converse is immediate, we completed the proof. □

Proposition 5 (Case (8^0)) $(p_1 : p_2 : p_3 = 1 : 3 : 7)$. *In this case a curve $\gamma(s)$ is of type 3 if and only if, in a coordinate system, we have*

$$A_1 = (a_{11}, a_{12}, a_{13}, a_{14}), \quad B_1 = (-a_{12}, a_{11}, b_{13}, b_{14}),$$
$$A_2 = (a_{21}, a_{22}, a_{23}, a_{24}), \quad B_2 = (-a_{22}, a_{21}, b_{23}, b_{24}),$$
$$A_3 = (\mu, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0),$$

where $\mu \neq 0$ and

$$a_{23}^2 + a_{24}^2 - b_{23}^2 - b_{24}^2 = \frac{14}{9} \mu\, a_{11}, \tag{1}$$

$$a_{23} b_{23} + a_{24} b_{24} = -\frac{7}{9} \mu\, a_{12}, \tag{2}$$

$$a_{13}\, a_{23} + a_{14}\, a_{24} - b_{13} b_{23} - b_{14} b_{24} = 7 \mu\, a_{21}, \tag{3}$$

$$b_{13}\, a_{23} + b_{14}\, a_{24} + a_{13}\, b_{23} + a_{14}\, b_{24} = -7 \mu\, a_{22}, \tag{4}$$

$$a_{13}^2 + a_{14}^2 - b_{13}^2 - b_{14}^2 = 3\,(2a_{21}\, a_{11} + 2a_{22}\, a_{12} + a_{23}\, a_{13} + a_{24}\, a_{14} + b_{13} b_{23} + b_{14} b_{24}), \tag{5}$$

$$a_{13} b_{13} + a_{14} b_{14} = 1.5\,(-2a_{21}\, a_{12} + 2a_{22}\, a_{11} + a_{23}\, b_{13} + a_{24}\, b_{14} - b_{23}\, a_{13} - b_{24}\, a_{14}). \tag{6}$$

Remark. Since the system above (1)–(6) has 6 equations and 13 parameters, then 7 of these parameters are "free" while the remaining 6 can be expressed by them.

This system has at least one solution since this case corresponds to the case 1.1 by D. Blair. Thus it is enough to take $a_{14} = b_{14} = a_{24} = b_{24} = 0$ and the corresponding example of D. Blair in the space E^3.

Proof. The only equalities in this case are

$$2p_1 = p_2 - p_1, \quad 2p_2 = p_3 - p_1, \quad p_1 + p_2 = p_3 - p_2.$$

Assuming that $\gamma(s)$ is a 3–type curve and using equalities $I(p_1+p_3)$ and $\overline{I}(p_1 + p_3)$ we have $A_{31} = \overline{A}_{31} = 0$. Hence if $A_1 = (a_{11}, a_{12}, a_{13}, a_{14})$, $B_1 = (b_{11}, b_{12}, b_{13}, b_{14})$ we easily get $b_{11} = -a_{12}$, $b_{12} = a_{11}$, thus $B_1 = (-a_{12}, a_{11}, b_{13}, b_{14})$.

Next relations $I(2p_2), \overline{I}(2p_2), I(p_1 + p_2), \overline{I}(p_1 + p_2)$ give:

$$A_{22} = \frac{7}{9}\, D_{31}, \quad \overline{A}_{22} = \frac{7}{9}\,\overline{D}_{31}, \quad A_{21} = \frac{7}{2}\, D_{32}, \quad \overline{A}_{21} = \frac{7}{2}\,\overline{D}_{32}.$$

Since

$$\begin{aligned} D_{31} &= \langle A_3, A_1\rangle + \langle B_3, B_1\rangle = 2\mu\, a_{11}, \\ \overline{D}_{31} &= \langle A_3, B_1\rangle - \langle A_1, B_3\rangle = -2\mu\, a_{12}, \\ D_{32} &= \langle A_3, A_2\rangle + \langle B_3, B_2\rangle = 2\mu\, a_{21}, \\ \overline{D}_{32} &= \langle A_3, B_2\rangle - \langle A_2, B_3\rangle = -2\mu\, a_{22}, \end{aligned}$$

the last four equations give equations (1)–(4).

Finally, equations $I(2p_1), \overline{I}(2p_1)$ give $A_{11} = 3D_{21}$, $\overline{A}_{11} = 3\overline{D}_{21}$, that is

$$\begin{aligned} &||A_1||^2 - ||B_1||^2 = 3\,(\langle A_2, A_1\rangle + \langle B_2, B_1\rangle), \\ &\langle A_1, B_1\rangle = 1.5\,(\langle A_2, B_1\rangle - \langle B_2, A_1\rangle), \end{aligned}$$

thus we have the equations (5) and (6).

The converse is immediate. □

Proposition 6 (Case (9^0)) $(p_2 \neq 2p_1, 3p_1, p_3 = 2p_2 - p_1)$. *In this case a curve $\gamma(s)$ is of type 3 if and only if in a coordinate system we have*

$$\begin{aligned} &A_1 = (a_{11}, a_{12}, \nu, 0), \quad B_1 = (a_{12}, -a_{11}, 0, \nu), \\ &A_2 = (a_{21}, a_{22}, a_{23}, a_{24}), \quad B_2 = (-a_{22}, a_{21}, b_{23}, b_{24}), \\ &A_3 = (\mu, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0), \end{aligned}$$

where $\mu \neq 0$, $\nu \in R$ and

$$a_{21}\, a_{11} + a_{22}\, a_{12} = \tfrac{\nu}{2}\,(b_{24} - a_{23}), \qquad (1)$$
$$a_{22}\, a_{11} - a_{21}\, a_{12} = \tfrac{\nu}{2}\,(b_{23} + a_{24}), \qquad (2)$$
$$a_{23}^2 + a_{24}^2 - b_{23}^2 - b_{24}^2 = -\tfrac{6p_1p_3}{p_2^2}\,\mu\, a_{11}, \qquad (3)$$
$$a_{23}\, b_{23} + a_{24} b_{24} = -2\tfrac{p_1p_3}{p_2^2}\,\mu\, a_{12}, \qquad (4)$$
$$\nu\,(a_{23} + b_{24}) = -\tfrac{2p_3}{p_1}\,\mu\, a_{21}, \qquad (5)$$
$$\nu\,(a_{24} - b_{23}) = \tfrac{2p_3}{p_1}\,\mu\, a_{22}. \qquad (6)$$

Remark. The system above has 6 equations and 10 parameters, so that 4 of these parameters are "free" while the other can be expressed by them. This system always has at least one solution because it corresponds to the case 2.2 by D. Blair.

Proof. The only equalities in this case are

$$2p_2 = p_1 + p_3, \quad p_2 - p_1 = p_3 - p_2.$$

Assuming that $\gamma(s)$ has the type 3, by relations $I(2p_1), \overline{I}(2p_1)$ we get $A_{11} = \overline{A}_{11} = 0$, hence $\|A_1\| = \|B_1\|$ and $A_1 \perp B_1$.

Next by relations $I(p_3 - p_1), \overline{I}(p_3 - p_1)$ we get $D_{31} = \overline{D}_{31} = 0$, which means that

$$\langle A_3, A_1\rangle = -\langle B_3, B_1\rangle, \quad \langle A_3, B_1\rangle = \langle A_1, B_3\rangle.$$

This easily gives $b_{11} = a_{12}$, $b_{12} = -a_{11}$, thus $B_1 = (a_{12}, -a_{11}, b_{13}, b_{14})$.

Since $\|A_1\| = \|B_1\|$, $A_1 \perp B_1$, we can again change the coordinate system so that

$$A_1 = (a_{11}, a_{12}, \nu, 0), \quad B_1 = (a_{12}, -a_{11}, 0, \nu) \quad (\nu \in R).$$

Next by relations $I(p_1 + p_2), \overline{I}(p_1 + p_2)$, $I(2p_2), \overline{I}(2p_2)$ $(2p_2 = p_1 + p_3), I(p_2 - p_1), \overline{I}(p_2 - p_1)$ $(p_2 - p_1 = p_3 - p_2)$, we get

$$\langle A_2, A_1\rangle = \langle B_2, B_1\rangle, \quad \langle A_2, B_1\rangle = -\langle B_2, A_1\rangle,$$
$$A_{22} = -\tfrac{2p_1p_3}{p_2^2}\, A_{31}, \quad \overline{A}_{22} = -\tfrac{2p_1p_3}{p_2^2}\, \overline{A}_{31},$$
$$D_{21} = -\tfrac{p_3}{p_1}\, D_{32}, \quad \overline{D}_{21} = -\tfrac{p_3}{p_1}\, \overline{D}_{32}.$$

Since

$$\begin{aligned}
A_{22} &= \|A_2\|^2 - \|B_2\|^2 = a_{23}^2 + a_{24}^2 - b_{23}^2 - b_{24}^2,\\
A_{31} &= \langle A_3, A_1\rangle - \langle B_3, B_1\rangle = 2\mu a_{11},\\
\overline{A}_{22} &= 2\langle A_2, B_2\rangle = 2(a_{23}b_{23} + a_{24}b_{24}),\\
\overline{A}_{31} &= \langle A_3, B_1\rangle + \langle A_1, B_3\rangle = 2\mu a_{12},\\
D_{21} &= \langle A_2, A_1\rangle + \langle B_2, B_1\rangle = \nu(a_{23} + b_{24}),\\
D_{32} &= \langle A_3, A_2\rangle + \langle B_3, B_2\rangle = 2\mu\, a_{21},\\
\overline{D}_{21} &= \langle A_2, B_1\rangle - \langle A_1, B_2\rangle = \nu(a_{24} - b_{23}),\\
\overline{D}_{32} &= \langle A_3, B_2\rangle - \langle A_2, B_3\rangle = -2\,\mu a_{22},
\end{aligned}$$

a substitution immediately gives the equations (1)–(6).

Since the converse is immediate, we completed the proof. □

Proposition 7 (Case (10^0)) ($p_1 : p_2 : p_3 = 1 : 2 : 3$). *In this case $\gamma(s)$ is of type 3 if and only if, in a coordinate system, we have*

$$\begin{aligned}
&A_1 = (a_{11}, a_{12}, a_{13}, a_{14}), \quad B_1 = (b_{11}, b_{12}, b_{13}, b_{14}),\\
&A_2 = (a_{21}, a_{22}, a_{23}, a_{24}), \quad B_2 = (-a_{22}, a_{21}, b_{23}, b_{24}),\\
&A_3 = (\mu, 0, 0, 0), \quad B_3 = (0, \mu, 0, 0),
\end{aligned}$$

where $\mu \neq 0$ and

$$a_{21}\, a_{11} + a_{22}\, a_{12} + a_{23}\, a_{13} + a_{24}\, a_{14} = -a_{22}\, b_{11} + a_{21}\, b_{12} + b_{23}\, b_{13} + b_{24}\, b_{14}, \quad (1)$$

$$a_{21}\, b_{11} + a_{22}\, b_{12} + a_{23}\, b_{13} + a_{24}\, b_{14} = a_{22}\, a_{11} - a_{21}\, a_{12} - b_{23}\, a_{13} - b_{24}\, a_{14}, \quad (2)$$

$$a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{14}^2 - b_{11}^2 - b_{12}^2 - b_{13}^2 - b_{14}^2 = 3\,\mu\,(a_{11} + b_{12}), \quad (3)$$

$$a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + a_{14}b_{14} = 1.5\,\mu\,(b_{11} - a_{12}), \quad (4)$$

$$a_{21}\, a_{11} + a_{22}\, a_{12} + a_{23}\, a_{13} + a_{24}\, a_{14} = -3\,\mu\, a_{21}, \quad (5)$$

$$a_{21}\, b_{11} + a_{22}\, b_{12} + a_{23}\, b_{13} + a_{24}\, b_{14} = 3\,\mu\, a_{22}, \quad (6)$$

$$a_{23}^2 + a_{24}^2 - b_{23}^2 - b_{24}^2 = -1.5\,\mu\,(a_{11} - b_{12}), \quad (7)$$

$$a_{23}\, b_{23} + a_{24}\, b_{24} = -\frac{3}{4}\,\mu\,(b_{11} + a_{12}). \quad (8)$$

Remark. The system above has 8 equations and 13 parameters. So 5 of these parameters are "free" while the other 8 can be expressed by them.

This system always has at least one solution, since the corresponding case for the space E^3 is contained in the case 2.2 by D. Blair.

Proof. The only equalities in this case are

$$2p_1 = p_3 - p_1, \quad 2p_2 = p_1 + p_3, \quad p_2 - p_1 = p_3 - p_2.$$

Assuming that $\gamma(s)$ is of type 3, we find that in a coordinate system A_2, B_2, A_3, B_3 have the mentioned form. Next by relations $I(p_1+p_2), \overline{I}(p_1+p_2)$ we have $A_{12} = \overline{A}_{12} = 0$.

Hence we get the equations

$$\langle A_2, A_1\rangle = \langle B_2, B_1\rangle, \quad \langle A_2, B_1\rangle = -\langle A_1, B_2\rangle,$$

thus we find equations (1) and (2).

By equations $I(2p_1), \overline{I}(2p_1), I(p_2 - p_1), \overline{I}(p_2 - p_1)$ we also find:

$$A_{11} = 3\,D_{31}, \quad \overline{A}_{11} = 3\,\overline{D}_{31}, \quad D_{21} = -3\,D_{32}, \quad \overline{D}_{21} = -3\,\overline{D}_{32},$$

while by relations $I(2p_2), \overline{I}(2p_2)$ we obtain

$$A_{22} = -1.5\,A_{31}, \quad \overline{A}_{22} = -1.5\overline{A}_{31}.$$

Since, taking in account (1) and (2)

$$\begin{aligned}
A_{11} &= ||A_1||^2 - ||B_1||^2, \quad \overline{A}_{11} = 2\,\langle A_1, B_1\rangle,\\
D_{31} &= \langle A_3, A_1\rangle + \langle B_3, B_1\rangle = \mu\,(a_{11} + b_{12}),\\
\overline{D}_{31} &= \langle A_3, B_1\rangle - \langle A_1, B_3\rangle = \mu\,(b_{11} - a_{12}),\\
D_{21} &= \langle A_2, A_1\rangle - \langle A_1, B_2\rangle = 2\langle A_2, A_1\rangle,\\
\overline{D}_{21} &= \langle A_2, B_1\rangle - \langle A_1, B_2\rangle = 2\,\langle A_2, B_1\rangle,\\
D_{32} &= \langle A_3, A_2\rangle + \langle B_3, B_2\rangle = 2\,\mu\,a_{21},\\
\overline{D}_{32} &= \langle A_3, B_2\rangle - \langle A_2, B_3\rangle = -2\,\mu\,a_{22},\\
A_{22} &= ||A_2||^2 - ||B_2||^2, \quad \overline{A}_{22} = 2\,\langle A_2, B_2\rangle,\\
A_{31} &= \langle A_3, A_1\rangle - \langle B_3, B_1\rangle = \mu\,(a_{11} - b_{12}),\\
\overline{A}_{31} &= \langle A_3, B_1\rangle + \langle A_1, B_3\rangle = \mu\,(a_{12} + b_{11}),
\end{aligned}$$

a substitution in the last 6 equations give the equations (3)–(8).

Since the converse is immediate, the proof is completed. □

Proposition 8 (Case (11^0)) $(p_1 : p_2 : p_3 = 1 : 3 : 9)$. *In this case a curve $\gamma(s)$ is of type 3 if and only if in a coordinate system we have*

$$\begin{aligned} A_1 &= (0,0,a_{13},a_{14}), \quad B_1 = (0,0,b_{13},b_{14}),\\ A_2 &= (a_{21},a_{22},a_{23},a_{24}), \quad B_2 = (-a_{22},a_{21},b_{23},b_{24}),\\ A_3 &= (\mu,0,0,0), \quad B_3 = (0,\mu,0,0), \end{aligned}$$

where $\mu \neq 0$ and

$$a_{23}\,a_{13} + a_{24}\,a_{14} = b_{23}\,b_{13} + b_{24}\,b_{14}, \tag{1}$$
$$a_{23}\,b_{13} + a_{24}\,b_{14} = -b_{23}\,a_{13} - b_{24}\,a_{14}, \tag{2}$$
$$a_{13}^2 + a_{14}^2 - b_{13}^2 - b_{14}^2 = 6\,(a_{23}\,a_{13} + a_{24}\,a_{14}), \tag{3}$$
$$a_{13}b_{13} + a_{14}b_{14} = 3\,(a_{23}\,b_{13} + a_{24}\,b_{14}), \tag{4}$$
$$a_{23}^2 + a_{24}^2 - b_{23}^2 - b_{24}^2 = 6\,\mu\,a_{21}, \tag{5}$$
$$a_{23}\,b_{23} + a_{24}\,b_{24} = -3\,\mu\,a_{22}. \tag{6}$$

Remark. The system above obviously has 5 "free" parameters, while the remaining 6 can be expressed by them. It is important to say that this case is contradictory in the space E^3. Hence we give at least one solution in the space E^4. It is enough to take

$$\begin{aligned} &a_{13} = a_{14} = b_{13} = 1, \quad b_{14} = 0,\\ &a_{23} = 1/3,\ a_{24} = -1/6,\ b_{23} = 1/6,\ b_{24} = -1/2,\\ &a_{21} = 2,\ a_{22} = 1,\ \mu = -5/108. \end{aligned}$$

Proof. In this case the only equalities are

$$2p_1 = p_2 - p_1, \quad 2p_2 = p_3 - p_2.$$

Assuming that $\gamma(s)$ is of type 3 we find by relations $I(p_1+p_3), \overline{I}(p_1+p_3), I(p_3-p_1), \overline{I}(p_3-p_1)$ that

$$A_{31} = \overline{A}_{31} = D_{31} = \overline{D}_{31} = 0.$$

Hence $A_1, B_1 \perp A_3, B_3$ and consequently $A_1 = (0,0,a_{13},a_{14}), \quad B_1 = (0,0,b_{13},b_{14})$.

By relations $I(p_1+p_2), \overline{I}(p_1+p_2)$ we also get $A_{21} = \overline{A}_{21} = 0$, and hence

$$\langle A_2, A_1\rangle = \langle B_2, B_1\rangle, \quad \langle A_2, B_1\rangle = -\langle A_1, B_2\rangle.$$

Since $A_2 = (a_{21}, a_{22}, a_{23}, a_{24})$, $B_2 = (-a_{22}, a_{21}, b_{23}, b_{24})$, the last two equations give (1) and (2).

Finally, by relations $I(2p_1), \overline{I}(2p_1)$ $(2p_1 = p_2 - p_1)$, $I(2p_2), \overline{I}(2p_2)$ $(2p_2 = p_3 - p_2)$ we get

$$A_{11} = 3D_{21}, \quad \overline{A}_{11} = 3\overline{D}_{21}, \quad A_{22} = 3D_{32}, \quad \overline{A}_{22} = 3\overline{D}_{32}.$$

This, together with (1) and (2), gives the equations (3)–(6).

Since the converse is immediate, the proof is complete. □

References

[1] D. Blair, A classification of 3–type curves, Soochow J. Math. **21** (1995), 145–158.

[2] B. Y. Chen, On the total curvature of immersed manifolds, vi: Submanifolds of finite type and their applications, Bull. Inst. Math. Acad. Sinica **11** (1983), 309–328.

[3] ———, Total mean curvature and submanifolds of finite type, World Scientific, Singapore, 1984.

[4] ———, A report on submanifolds of finite type, Soochow J. Math. **22** (1996), 117–337.

[5] B. Y. Chen, J. Deprez, F. Dillen, L. Verstraelen, and L. Vrancken, Curves of finite type, Geometry and Topology of submanifolds, II, World Scientific, Singapore, 1990, pp. 76–110.

[6] B. Y. Chen, F. Dillen, and L. Verstraelen, Finite type space curves, Soochow J. Math. **12** (1986), 1–10.

[7] J. Deprez, F. Dillen, and L. Vrancken, Finite type curves on quadrics, Chinese J. Math. **18** (1990), 95–121.

[8] M. Petrović-Torgašev, L. Verstraelen, and L. Vrancken, 3–type curves on ellipsoids of revolution, Preprint series Dept. Math. Kath. Univ. Leuven 2, 1990.

[9] ———, 3–type curves on hyperboloids of revolution and cones of revolution, Publ. Inst. Math. (Beograd) **59** (1996), no. 73, 138–152.

[10] L. Verstraelen, Curves and surfaces of finite Chen type, Geometry and Topology of Submanifolds, III, World Scientific, Singapore, 1991, pp. 304–311.