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DISTINGUESHED LINEAR CONNECTIONS IN THE EINSTEIN-SCHRÖDINGER GEOMETRY OF THE SECOND ORDER

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Abstract

In 1945 A.Einstein [6] and E.Schrödinger [11] started form a generalized Riemann space, that is, a space M associated with a nonsymmetric tensor $G_{ij}(x)$ and desired to find the set of all linear connections $\Gamma^i_{jk}(x)$ compatible with such a metric : $G_{ij/k} = 0$ (see also [1]).The geometry of this space (M.G_{ij}) is called **the Einstein - Schrödinger's geometry** [3], [4].

The purpose of this paper is to discuss a nonsymmetric tensor field $G_{ij}(x, y^{(1)}, y^{(2)})$,where $(x, y^{(1)}, y^{(2)})$ is a point of the osculator bundle of the second order $(Osc^2 M, \pi, M)$ and to obtain the results for the Einstein - Schrödinger's geometry of the order two in a natural case.

The fundamental notions and notations concerning the osculator bundle of the order two are in the papers [2] [5] [7] [8].

We give, shortly, at the begining the notations used.

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1 Preliminaries

A transformation of coordinates $(x^i, y^{(1)i}, y^{(2)i}) \to (\tilde{x}^1, \tilde{y}^{(1)i}, \tilde{y}^{(2)i})$ on Osc^2M is given by

$$\begin{cases} \tilde{x}^i = \tilde{x}^i(x^1, ..., x^n), \qquad rank\left\| \dfrac{\partial \tilde{x}^i}{\partial x^j} \right\| = n \\ \tilde{y}^{(1)i} = \dfrac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j} \\ 2\tilde{y}^{(2)i} = \dfrac{\partial \tilde{y}^{(1)i}}{\partial x^j} y^{(1)j} + 2\dfrac{\partial \tilde{y}^{(1)i}}{\partial y^{(1)j}} y^{(2)j} \end{cases} \tag{1}$$

The point $u \in Osc^2M$ of coordinats $(x^i, y^{(1)i}, y^{(2)i})$ will be noted, also, with $u = (x^i, y^{(1)i}, y^{(2)i})$.

The bundle of the 1-jets Osc^1M can be identified with the tangent bundle TM.

A non linear connection N on $E = Osc^2M$ is characterized by the functions $\underset{(\alpha)}{N^i{}_j}\ (x, y^{(1)}, y^{(2)})$ $(\alpha = 1, 2)$ called the coefficients of N which to a transformation of coordinates on E has as effect the rules:

$$\begin{cases} \underset{(1)}{\tilde{N}^i{}_m} \dfrac{\partial \tilde{x}^m}{\partial x^j} = \underset{(1)}{\tilde{N}^m{}_j} \dfrac{\partial \tilde{x}^i}{\partial x^m} - \dfrac{\partial \tilde{y}^{(1)i}}{\partial x^j} \\ \underset{(2)}{\tilde{N}^i{}_m} \dfrac{\partial \tilde{x}^m}{\partial x^j} = \underset{(2)}{\tilde{N}^m{}_j} \dfrac{\partial \tilde{x}^i}{\partial x^m} + \underset{(1)}{\tilde{N}^m{}_j} \dfrac{\partial \tilde{y}^{(1)i}}{\partial x^m} - \dfrac{\partial \tilde{y}^{(2)i}}{\partial \tilde{x}^j} \end{cases} \tag{2}$$

We obtain the direct decomposition:

$$T_uE = N_0(u) \oplus N_1(u) \oplus V_2(u) \oplus, \ \forall u \in E \ (N_0 = N) \tag{3}$$

with the local basis adapted to this

$$\left\{ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \frac{\delta}{\delta y^{(2)i}} \right\} \quad , (i = 1, ..., n) \tag{4}$$

given by

$$\begin{cases} \dfrac{\delta}{\delta x^i} = \dfrac{\partial}{\partial x^i} - \underset{(1)}{N^j{}_i} \dfrac{\partial}{\partial y^{(1)j}} - \underset{(2)}{N^j{}_i} \dfrac{\partial}{\partial y^{(2)j}} \\ \dfrac{\delta}{\delta y^{(1)i}} = \dfrac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N^j{}_i} \dfrac{\partial}{\partial y^{(2)j}} \end{cases} \tag{5}$$

The fields of geometrical objects wich are important on E are introduced with respect to the direct decomposition (3).

The transformation (1) implies:

$$\frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{x}^j}, \ \frac{\delta}{\delta y^{(1)i}} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{y}^{(1)j}}, \ \frac{\delta}{\delta y^{(2)i}} = \frac{\partial \tilde{x}^j}{\partial x^i}\frac{\delta}{\delta \tilde{y}^{(2)j}}$$

If we consider the projectors h, v_1, v_2, determined by (3) and denote $v_\alpha X = X^{V_\alpha}$, $(\alpha = 1, 2)$ we can uniquely write

$$X = X^H + X^{v_1} + X^{v_2}, \quad \forall X \in \mathcal{X}(\mathcal{E}) \tag{6}$$

Thus we have

$$X^H = X^{(0)i}\frac{\delta}{\delta x^i}, \quad X^{v_1} = X^{(1)i}\frac{\delta}{\delta y^{(1)i}}, \quad X^{v_2} = X^{(2)i}\frac{\delta}{\delta y^{(2)i}}$$

The coordinates $X^{(\alpha)i}$, $(\alpha = 0, 1, 2)$ change under (1) as follows:

$$\tilde{X}^{(\alpha)i} = \frac{\partial \tilde{x}^i}{\partial x^j} X^{(\alpha)j}, \quad (\alpha = 0, 1, 2).$$

Each of them is called a distinguished vector field, shortly a d-vector field. Let us consider the dual basis of (4):

$$\{dx^i, \ \delta y^{(1)i}, \ \delta y^{(2)i}\}, \quad (i = 1, ..., n) \tag{7}$$

Then for a field of 1-form ω on E we put:

$$\omega = \omega^H + \omega^{v_1} + \omega^{v_2}, \tag{8}$$

where

$$\omega^H = \omega_i^{(0)} dx^i \ , \quad \omega^{v_1} = \omega_i^{(1)} \delta y^{(1)i} \ , \quad \omega^{v_2} = \omega_i^{(2)} \delta y^{(2)i}$$

and with respect to (1) we have:

$$\omega_i^{(\alpha)} = \frac{\partial \tilde{x}^j}{\partial x^i} \tilde{\omega}_j^{(\alpha)} \ , \quad (\alpha = 0, 1, 2)$$

Now , we can define a distinguised tensor field on E of type (r,s) (shortly a d-tensor field) as an element $T \in T_s^r(E)$ with the property:

$$T(\underset{1}{X}, ..., \underset{s}{X}, \overset{1}{\omega}, ..., \overset{r}{\omega}) = T(\underset{1}{X^H}, ..., \underset{s}{X^{v_2}}, \overset{1}{\omega^H}, ..., \overset{r}{\omega^{v_2}}) \tag{9}$$

$$\forall \ \underset{1}{X}, ..., \underset{s}{X} \in \mathcal{X}(\mathcal{E}) \ , \quad \forall \ \overset{\infty}{\omega}, ..., \overset{\nabla}{\omega} \in \overset{*}{\mathcal{X}} (\mathcal{E})$$

Then in adapted basis (4), (7) we obtain:

$$T = T^{i_1,...,i_r}_{j_1,...,j_s}(x, y^{(1)}, y^{(2)}) \frac{\delta}{\delta x^{i_1}} \otimes .. \otimes \frac{\partial}{\partial y^{(2)i_r}} \otimes dx^{j_1} \otimes .. \otimes \delta y^{(2)j_s}$$

and with respect to (1) we get:

$$\tilde{T}^{i_1,...,i_r}_{j_1,...,j_s} = \frac{\partial \tilde{x}^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial \tilde{x}^{i_r}}{\partial x^{m_r}} \frac{\partial x^{q_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{q_s}}{\partial \tilde{x}^{j_s}} T^{m_1,...,m_r}_{q_1,...,q_s}$$

Consequently we can give a d-tensor field T, of type (r,s) by its local components $T^{i_1,...,i_r}_{j_1,...,j_s}(x, y^{(1)}, y^{(2)})$.

Let us consider the $F(E)$-linear map $J : \mathcal{X}(\mathcal{E}) \to \mathcal{X}(\mathcal{E})$ given on the natural basis of $\mathcal{X}(\mathcal{E})$ by:

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}} \ , \quad J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}} \ , \quad J\left(\frac{\partial}{\partial y^{(2)i}}\right) = 0 \ , \tag{10}$$

$$(i = 1, .., n)$$

We define a N-linear connection on E as a connection D on E wich preserves by parallelism the horizontal distribution N and wich is compatible with the structure J (i.e. $D_X J = 0, \forall X \in \mathcal{X}(\mathcal{E})$).

In the adapted basis (4) it is sufficient to give

$$D_{\frac{\delta}{\delta x^j}} \frac{\delta}{\delta y^{(\alpha)i}} = L^m_{ij} \frac{\delta}{\delta y^{(\alpha)m}}, \quad D_{\frac{\delta}{\delta y^{(\beta)j}}} \frac{\delta}{\delta y^{(\alpha)i}} = \underset{(\beta)}{C^m_{ij}} \frac{\delta}{\delta y^{(\alpha)m}}, \tag{11}$$

$$(\alpha = 0,\ 1,\ 2, \quad \beta = 1,\ 2, \quad y^{(0)i} = x^i)$$

in order to obtain all the coefficients $D\Gamma(N) = (L^i_{jm}, \underset{(1)}{C^i_{jm}}, \underset{(2)}{C^i_{jm}})$ of a N-linear connection D.

With respect to (1) we have for the coefficients $\underset{(\alpha)}{C^i_{jm}}(x, y^{(1)}, y^{(2)})$ the transformation of the d-tensor field of type (1,2) and for the coefficients $L^i_{jm}(x, y^{(1)}, y^{(2)})$ the transformation law of an object of connection:

$$\tilde{L}^i_{pq} \frac{\partial \tilde{x}^p}{\partial x^r} \frac{\partial \tilde{x}^q}{\partial x^s} = L^m_{rs} \frac{\partial \tilde{x}^i}{\partial x^m} - \frac{\partial^2 \tilde{x}^i}{\partial x^r \partial x^s}$$

The h-covariant derivative noted with $|$ and the v_α**-covariant** derivative noted with $\overset{(\alpha)}{|}$ $(\alpha = 1, 2)$ in the algebra of the d-tensor act, for exemple, for a d-tensor field $K^i_j(x, y^{(1)}, y^{(2)})$ of the type (1,1) as:

$$\begin{cases} K^i_{j|m} = \frac{\delta K^i_j}{\delta x^m} + L^i_{rm} K^r_j - L^s_{jm} K^i_s \\ K^i{}_{j \overset{(\alpha)}{|} m} = \frac{\delta K^i_j}{\delta y^{(\alpha)m}} + \underset{(\alpha)}{C^i_{rm}} K^r_j - \underset{(\alpha)}{C^s_{jm}} K^i_s, \ (\alpha = 1, 2) \end{cases} \tag{12}$$

If $D\Gamma(N) = (L^m_{ij}, \underset{(1)}{C^m_{ij}}, \underset{(2)}{C^m_{ij}})$ are the local components of a N-linear connection D on E, then the identities of Ricci holds, written for a d-vector field $X^m(x, y^{(1)}, y^{(2)})$:

$$X^m{}_{|p|q} - X^m{}_{|q|p} = X^r R_r{}^m{}_{pq} - T^r{}_{pq} X^m{}_{|r}$$

$$- \overset{(1)}{\underset{(0)}{R}}{}^{r}{}_{pq} X^m \overset{(1)}{|}_r - \overset{(2)}{\underset{(0)}{R}}{}^{r}{}_{pq} X^m \overset{(2)}{|}_r$$

$$X^m{}_{|p} \overset{(\beta)}{|}_q - X^m \overset{(\beta)}{|}_{q|p} = X^r \underset{(\beta)}{P}{}_r{}^m{}_{pq} - \underset{(\beta)}{C}{}^r_{pq} X^m{}_{|r} - \overset{(1)}{\underset{(\beta)}{P}}{}^r{}_{pq} X^m \overset{(1)}{|}_r - \overset{(2)}{\underset{(\beta)}{P}}{}^r{}_{pq} X^m \overset{(2)}{|}_r \quad (\beta = 1, 2)$$

$$X^m \overset{(1)}{|}_p \overset{(2)}{|}_q - X^m \overset{(2)}{|}_q \overset{(1)}{|}_p = X^r \underset{(1)(2)}{P}{}_r{}^m{}_{pq} - (\underset{(2)}{C}{}^r_{pq} X^m \overset{(1)}{|}_r - \underset{(1)}{C}{}^r_{qp} X^m \overset{(2)}{|}_r) - \overset{(2)}{\underset{(1)(2)}{P}}{}^r{}_{pq} X^m \overset{(2)}{|}_r$$

$$X^m \overset{(1)}{|}_p \overset{(1)}{|}_q - X^m \overset{(1)}{|}_q \overset{(1)}{|}_p = X^r \underset{(1)}{S}{}_r{}^m{}_{pq} - \underset{(1)}{S}{}_r{}^m{}_{pq} X^m \overset{(1)}{|}_r - \overset{(1)}{\underset{(1)}{R}}{}^r{}_{pq} X^m \overset{(2)}{|}_r$$

$$X^m \overset{(2)}{|}_p \overset{(2)}{|}_q - X^m \overset{(2)}{|}_q \overset{(2)}{|}_p = X^r \underset{(2)}{S}{}_r{}^m{}_{pq} - \underset{(2)}{S}{}^r{}_{pq} X^m \overset{(2)}{|}_r$$

where the tensor fields of torsion $T^r{}_{pq}$, $\overset{(1)}{\underset{(0)}{R}}{}^r{}_{pq}$, $\overset{(2)}{\underset{(0)}{R}}{}^r{}_{pq}$, $\underset{(1)}{C}{}^r{}_{pq}$, $\underset{(2)}{C}{}^r{}_{pq}$, $\overset{(1)}{\underset{(1)}{P}}{}^r{}_{pq}$, $\overset{(1)}{\underset{(2)}{P}}{}^r{}_{pq}$, $\overset{(2)}{\underset{(2)}{P}}{}^r{}_{pq}$, $\overset{(2)}{\underset{(1)}{P}}{}^r{}_{pq}$, $\overset{(2)}{\underset{(1)(2)}{P}}{}^r{}_{pq}$, $\underset{(1)}{S}{}^r{}_{pq}$, $\underset{(2)}{S}{}^r{}_{pq}$, $\overset{(1)}{\underset{(1)}{R}}{}^r{}_{pq}$

and the tensor fields of curvature $R_r{}^m{}_{pq}$, $\underset{(1)}{P}{}_r{}^m{}_{pq}$, $\underset{(2)}{P}{}_r{}^m{}_{pq}$, $\underset{(1)(2)}{P}{}_r{}^m{}_{pq}$,

$\underset{(1)}{S}{}_r{}^m{}_{pq}$, $\underset{(2)}{S}{}_r{}^m{}_{pq}$ appear.

2 N-linear connections compatible with an asymmetric metric $G_{ij}(x, y^{(1)}, y^{(2)})$

For a nonsymmetric tensor field $G_{ij}(x, y^{(1)}, y^{(2)})$ on Osc^2M, we have a symmetric tensor field $g_{ij}(x, y^{(1)}, y^{(2)})$ and a skew-symmetric one $a_{ij}(x, y^{(1)}, y^{(2)})$ from the spliting

$$G_{ij} = g_{ij} + a_{ij}, \tag{13}$$

where we suppose that

$$det \parallel g_{ij}(x, y^{(1)}, y^{(2)})\parallel \cdot \parallel a_{ij}(x, y^{(1)}, y^{(2)}) \parallel \neq 0 \tag{14}$$

and dim M = n = 2n'.

We denote

$$\|g_{ij}(x, y^{(1)}, y^{(2)})\|^{-1} = \|g^{ij}(x, y^{(1)}, y^{(2)})\| \ ,$$
$$\|a_{ij}(x, y^{(1)}, y^{(2)})\|^{-1} = \|a^{ij}(x, y^{(1)}, y^{(2)})\|$$

We have from $G_{ij|k} = 0 \quad , \quad G_{ij} \overset{(\alpha)}{|}_k = 0 \quad (\alpha = 1, 2)$
the following equations :

$$g_{ij|k} = 0 \ , \ g_{ij} \overset{(\alpha)}{|}_k = 0 \ , \ a_{ij|k} = 0 \ , \ a_{ij} \overset{(\alpha)}{|}_k = 0 \quad (\alpha = 1, 2) \ , \tag{15}$$

which is equivalent to

$$g^{ij}{}_{|k} = 0 \ , \ g^{ij} \overset{(\alpha)}{|}_k = 0 \ , \ a^{ij}{}_{|k} = 0 \ , \ a^{ij} \overset{(\alpha)}{|}_k = 0 \quad (\alpha = 1, 2). \tag{16}$$

We investigate the set of all N-linear connections $D\Gamma(N) = (L^i_{jk}, \ \underset{(\alpha)}{C^i}{}_{jk} \ )$ $(\alpha = 1, 2)$ for which we have (15) in the form

$$L^i_{jk} \ = \ \overset{\circ}{L^i_{jk}} + A^i_{jk} \ , \quad \underset{(\alpha)}{C^i}{}_{jk} \ = \ \underset{(\alpha)}{\overset{\circ}{C^i}}{}_{jk} \ + B^i_{jk} \quad (\alpha = 1, 2) \ ,$$

where $D \overset{\circ}{\Gamma} (N) \ = \ (\overset{\circ}{L^i_{jk}}, \ \underset{(\alpha)}{\overset{\circ}{C^i_{jk}}} \ ) \ (\alpha = 1, 2)$ is a fixed N-linear connection on Osc^2M and A^i_{jk} , $\underset{(\alpha)}{B^i_{jk}}$ are arbitrary tensor fields of type (1,2).

We obtain for A and for $\underset{(\alpha)}{B}$ the equations

$$A^r_{ik}g_{rj} + A^r_{jk}g_{ir} = g_{ij\overset{o}{|}k}\ ,\ A^r_{ik}a_{rj} + A^r_{jk}a_{ir} = a_{ij\overset{o}{|}k}, \tag{17}$$

$$\begin{cases} \underset{(\alpha)}{B^r_{ik}}\ g_{rj} + \underset{(\alpha)}{B^r_{jk}}\ g_{ir} = g_{ij}\ \overset{(\alpha)}{\overset{o}{|}}_k, \\ \underset{(\alpha)}{B^r_{ik}}\ a_{rj} + \underset{(\alpha)}{B^r_{jk}}\ a_{ir} = a_{ij}\ \overset{(\alpha)}{\overset{o}{|}}_k\ (\alpha = 1, 2) \end{cases} \tag{18}$$

We do not know the general solution of the equation system (17) and (18)

We give a solution for these equations in the following special case.

Definition 2.1 *An assymmetric metric (13) is called* **natural** *if we have*

$$\Lambda^{rk}_{is}\Phi^{hs}_{rj} = \Phi^{rk}_{is}\Lambda^{hs}_{rj} \tag{19}$$

where

$$\Lambda^{kh}_{ij} = \frac{1}{2}(\delta^k_i\delta^h_j - g_{ij}g^{kh})\ ,\ \Phi^{kh}_{ij} = \frac{1}{2}(\delta^k_i\delta^h_j - a_{ij}a^{kh}). \tag{20}$$

Theorem 2.1 *An assymmetric metric $G_{ij}(x, y^{(1)}, y^{(2)})$ on Osc^2M is natural if and only if there exist a function $\mu(x, y^{(1)}, y^{(2)})$ on Osc^2M such that*

$$g_{ir}g_{js}a^{rs} = \mu g_{ij}. \tag{21}$$

Examples.

1. Let $f^i_j(x, y^{(1)}, y^{(2)})$ be a tensor field of type (1,1) which gives an almost complex d-structure on Osc^2M : $f^2 = -\delta$. If we put:

$$a_{ij} = f^r_i g_{rj}, \tag{22}$$

then $a_{ij}(x, y^{(1)}, y^{(2)})$ is alternating and $G_{ij} = g_{ij} + a_{ij}$ is an asymmetric metric on Osc^2M. In this case $\mu = -1$.

2. Let $q^i_j(x, y^{(1)}, y^{(2)})$ be a tensor field of type (1,1) which gives an almost product d-structure on Osc^2M : $q^2 = +\delta$. If we put:

$$a_{ij} = q^r_i g_{rj} \tag{23}$$

then $a_{ij}(x, y^{(1)}, y^{(2)})$ is alternate and $G_{ij} = g_{ij} + a_{ij}$ is an asymmetric metric on Osc^2M. In this case $\mu = +1$.

Theorem 2.2 *If there exist a N-linear connection on Osc^2M compatible with a natural asymmetric metric $G_{ij}(x, y^{(1)}, y^{(2)})$, then the function μ is constant.*

Definition 2.2 *A natural asymmetric metric (13) is called* **elliptic** *if $\mu = -c^2$ and* **hyperbolic** *if $\mu = c^2$, where c is a positive constant.*

The converse of Theorem (2.2) holds as follows:

Theorem 2.3 *If a natural asymmetric metric (13) is elliptic or hyperbolic, then there exist N-linear connections $D\tilde{\Gamma}(N) = (\tilde{L}^i_{jk}, \underset{(\alpha)}{\tilde{C}^i{}_{jk}})$ compatible with $G_{ij}(x, y^{(1)}, y^{(2)})$. Let $D\overset{\circ}{\Gamma}(N) = (\overset{\circ}{L}{}^i_{jk}, \underset{(\alpha)}{\overset{\circ}{C}{}^i{}_{jk}})$ be a given N-linear connection, then in the elliptic case we have*

$$\begin{cases} \tilde{L}^i_{jk} = \overset{\circ}{L}{}^i_{jk} + \frac{1}{4}\{g^{ir} g_{\underset{\circ}{rj|k}} + a^{ir} a_{\underset{\circ}{rj|k}} + f^r_j f^i_{\underset{\circ}{r|k}}\} \\ \underset{(\alpha)}{\tilde{C}^i{}_{jk}} = \underset{(\alpha)}{\overset{\circ}{C}{}^i{}_{jk}} + \frac{1}{4}\{g^{ir} g_{rj} \overset{(\alpha)}{\underset{\circ}{|}}_k + a^{ir} a_{rj} \overset{(\alpha)}{\underset{\circ}{|}}_k + f^r_j f^i_r \overset{(\alpha)}{\underset{\circ}{|}}_k\} \end{cases} \tag{24}$$

$(\alpha = 1, 2)$ *, and in the hyperbolic case we have*

$$\begin{cases} \tilde{L}^i_{jk} = \overset{\circ}{L}{}^i_{jk} + \frac{1}{4}\{g^{ir} g_{\underset{\circ}{rj|k}} + a^{ir} a_{\underset{\circ}{rj|k}} - q^r_j q^i_{\underset{\circ}{r|k}}\} \\ \underset{(\alpha)}{\tilde{C}^i{}_{jk}} = \underset{(\alpha)}{\overset{\circ}{C}{}^i{}_{jk}} + \frac{1}{4}\{g^{ir} g_{rj} \overset{(\alpha)}{\underset{\circ}{|}}_k + a^{ir} a_{rj} \overset{(\alpha)}{\underset{\circ}{|}}_k - q^r_j q^i_r \overset{(\alpha)}{\underset{\circ}{|}}_k\} \end{cases} \tag{25}$$

$(\alpha = 1, 2)$

Theorem 2.4 *The set of all N-linear connections*

$$D\overset{*}{\Gamma}(N) = (\overset{*}{L}{}^i_{jk}, \underset{(\alpha)}{\overset{*}{C}{}^i{}_{jk}})$$

compatible with a natural asymmetric metric (13) on Osc^2M is given by

$$\overset{*}{L}{}^i_{jk} = \tilde{L}^i_{jk} + \Lambda^{pi}_{jq}\Phi^{rq}_{ps}Y^s_{rk}\,, \quad \overset{*}{\underset{(\alpha)}{C}}{}^i{}_{jk} = \underset{(\alpha)}{\tilde{C}}{}^i{}_{jk} + \Lambda^{pi}_{jq}\Phi^{rq}_{ps}\underset{(\alpha)}{Z}{}^s_{rk} \tag{26}$$

where $D\tilde{\Gamma}(N)$ is the N-linear connection in Theorem (2.3) and Y^i_{jk}, $\underset{(\alpha)}{Z}{}^i_{jk}$ $(\alpha = 1, 2)$ are arbitrary tensor fields on Osc^2M.

If we put $D\overset{o}{\Gamma}(N) = D\overset{c}{\Gamma}(N) = (\overset{c}{L}{}^i_{jk}, \overset{c}{\underset{(\alpha)}{C}}{}^i{}_{jk})$, $(\alpha = 1, 2)$ for $g_{ij}(x, y^{(1)}, y^{(2)})$, that is:

$$\begin{cases} \overset{c}{L}{}^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\delta g_{js}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s}\right), \\ \overset{c}{\underset{(1)}{C}}{}^i{}_{jk} = \frac{1}{2}g^{is}\left(\frac{\delta g_{js}}{\delta y^{(1)k}} + \frac{\delta g_{sk}}{\delta y^{(1)j}} - \frac{\delta g_{jk}}{\delta y^{(1)s}}\right), \\ \overset{c}{\underset{(2)}{C}}{}^i{}_{jk} = \frac{1}{2}g^{is}\left(\frac{\delta g_{js}}{\delta y^{(2)k}} + \frac{\delta g_{sk}}{\delta y^{(2)j}} - \frac{\delta g_{jk}}{\delta y^{(2)s}}\right), \end{cases} \tag{27}$$

the generalized Christoffel symbols (cf. with [8], pg.54) we have:

Theorem 2.5 *The canonical N-linear connection compatible with a natural asymmetric metric $G_{ij}(x, y^{(1)}, y^{(2)})$ is given in the elliptic case by:*

$$\begin{cases} L^i_{jk} = \overset{c}{L}{}^i_{jk} + \frac{1}{4}\{a^{ir}a_{rj\overset{c}{|}k} + f^r_j f^i_{r\overset{c}{|}k}\} \\ \underset{(\alpha)}{C}{}^i_{jk} = \overset{c}{C}{}^i_{jk} + \frac{1}{4}\{a^{ir}a_{rj}\overset{(\alpha)}{\overset{c}{|}}_k + f^r_j f^i_r\overset{(\alpha)}{\overset{c}{|}}_k\}\ (\alpha = 1, 2) \end{cases} \tag{28}$$

and in the hyperbolic case by

$$\begin{cases} L^i_{jk} = \overset{c}{L}{}^i_{jk} + \frac{1}{4}\{a^{ir}a_{rj\overset{c}{|}k} - q^r_j q^i_{r\overset{c}{|}k}\} \\ \underset{(\alpha)}{C}{}^i_{jk} = \overset{c}{\underset{(\alpha)}{C}}{}^i_{jk} + \frac{1}{4}\{a^{ir}a_{rj}\overset{(\alpha)}{\overset{c}{|}}_k - q^r_j q^i_r\overset{(\alpha)}{\overset{c}{|}}_k\}, \quad (\alpha = 1, 2) \end{cases} \tag{29}$$

Now, the Einstein equations, electromagnetic tensors, Maxwell equations for the Einstein-Schrödinger geometry of the second order can be studied using these N-linear connections.

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