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CERTAIN COMPARATIVE EXAMINATIONS OF PLANE GEOMETRIES ACCORDING TO CAYLEY-KLEIN *

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Abstract

Referring to the axiomatic bases of two-dimensional geometries according to Cayley-Klein and their projective interpretations, we have examined parallelism of an isotropic and ideal line in geometries which have the same projective metrices on a line or a pencil of straight lines. In terms of these examinations for autodual HH-geometry (a geometry in which the projective metrices on a line and a pencil of lines is hyperbolical, and whose axiomatics is not known to us) we have concluded the following:

a) in the HH-geometry the Lobatchewskian axiom is true; and

b) besides the line as the starting notion, the HH-geometry contains both isotropic and ideal lines which are to be either considered starting notions during the axiomatic development of this geometry or defined.

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Introduction

It is known that according to Ceyley-Klein's project (see [4], [5]) there exist nine twodimensional geometries with projective metrics. These geometrices differ, among others, according to the type of projective metrics on a line and on a pencil of lines. Therefore, each of these geometries could be marked with two letters: PE (euclidean geometry), EP (dual geometry to euclidean

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geometry), HE (hyperbolical geometry - Lobatchewskian geometry), EH (dual geometry to hyperbolical geometry), PH (pseudoeuclidean geometry), HP (dual geometry to pseudoeuclidean geometry), EE (elliptical geometry), PP (parabolic geometry) and HH-geometry, where the first letter stands for the character of the projective metrics on a line (P-parabolic, H-hyperbolic, E-elliptic), and the second letter - in a pencil of lines (see [7]). Hereinafter, for the sake of simplicity, we shall mark these geometries with two letters. Each of the geometries could be interpreted in a projective plane. For that kind of interpretation it is necessary to determine an absolute (a second-other or a second-class curve). The choice of the absolute and basic objects in the projective interpretation of these geometries is as follows:

- for PE-geometry the absolute is a pair of conjugate complex points and a real projective line l which is incident to them; the 'points' of PE-geometry have been interpreted by way of projective points which are not incident to the line l, and the 'lines' - by way of projective lines which intersect the line l without an intersection point;

- for EP-geometry the absolute is a pair of conjugate complex lines and a real projective point Q of their intersection; the 'points' of EP-geometry have been interpreted by way of projective points, except for the point Q, and the 'lines' - by way of projective lines which are not incident to the absolute point Q;

- for HE-geometry the absolute is an arbitrary undegenerate real second-order curve; the 'points' of HE-geometry have been interpreted by way of the points of the projective plane inside the absolute, and the 'lineas' - by way of the parts of the projective lines which intersect the absolute and belong to its inside;

- for EH-geometry the absolute is an undegenerate real second-order curve; the 'points' of EH-geometry have been interpreted by way of the points of the projective plane which are outer in relation to the absolute, and the 'lines' - by way of the projective lines which do not intersect the absolute;

- for PH-geometry the absolute is two real and different projective points P and Q and a real line l which is incident to them; the 'points' of PH-geometry have been interpreted by way of the projective points which are not incident to the line l, and the 'lines' - by way of the projective lines which intersect the determined open projective segment PQ, without the intersection point;

- for HP-geometry the absolute is a pair of real projective lines p and q which intersect in the real point Q; the 'points' of HP-geometry have been interpreted by way of the projective points which belong to the inside of one of projective angles pq which is determined by the absolute, the 'lines' - by way of the parts of the projective lines, which are not incident to the point Q and belong to the noticed projective angle pq;

- for EE-geometry the absolute is an undegenerate imaginary second-order curve; the 'points' and 'lines' of EE-geometry have been interpreted by way of the points and lines of the projective plane;

- for PP-geometry the absolute is a double real point Q and a double real projective line l which is incident to it; the 'points' of PP-geometry have been interpreted by way of projective points which are not incident to the line l, and the 'lines' - by way of projective lines which intersect the absolute line l outside the point Q, without that intersection point;

- for HH-geometry the absolute is an undegenerate real second-order curve; the 'points' of HH-geometry have been interpreted by way of outer projective points in relation to the absolute, and the 'lines' - by way of parts of the projective lines which intersect the absolute and belong to its outside.

To our knowledge, at least one system of axioms has been suggested for each of these geometries, except for HH-geometry.

1. Parallelism in twodimensional geometries

In order to examine parallelism in twodimensional geometries, we shall first examine the geometries in which distance metrics (projective metrics on a line) is parabolic. In other words, we shall deal with PE, PH and PP geometries. It is well known that, in 1899, Hilbert (see [3]) provided the first completely narrow axiomatic foundation of PE-geometry, and, in 1966, Kropina (see [6]) provided a projective interpretation of Hilbert's system of axioms. In 1970, Proskurina (see [15]) provided a system of axioms of PH-geometry, and in 1983 we provided a projective model of this axiomatics. In 1956, Parnasskiy (see [13]) provided a system of axioms of threedimensional parabolic geometry, and of PP-geometry accordingly, whereby he used a projective model as the proof for the uncontradictoriness of axiomatics.

Looking into the systems of axioms of these geometrices, one could directly notice that euclidean axiom of parallelism is true there. This fact is to be explained from the projective point of view.

Imagine a line a and a point B which is not incident to a in any of the following geometries: PE, PH or PP. In view of the fact that projective metrics on a line is parabolic, the absolute for this metrics then consists of a double real point (see [7], p.275). We could mark this point as A_0. The point A_0 is an uncharacteristic point of these geometries, and in the projective model it is obtained in the intersection of a projective line by way of which the line a and the absolutes of a relevant geometry have been interpreted (the absolutes are enumerated in the Introduction). A unique line is defined in the projective plane by the point A_0 and projective point B. A line b, which is incident to the point B and parallel to the line a, has been interpreted in the projective model by way of this line excluding the point A_0.

We shall examine, then, twodimensional geometries whese projective metrics on a line is elliptic, i.e. EP, EH and EE geometries. In 1989, in the doctoral dissertation, we suggested a system of axioms of EP-geometry and a projective model of that axiomatics (see [8], [11], [12]). In 1973, Parnasskaya (see [14]) suggested a system of axioms of a threedimensional geometry which is dual to a hyperbolical geometry, and in 1923 Bogomolov (see [2]) provided a system of axioms of a threedimensional elliptic geometry. In the geometries EP, EH and EE we notice that there are no parallel lines, but any two different lines do intersect at a certain point. This fact could also be explained from the projective point of view. Namely, in each of these geometries projective metrics on a line is elliptic. The absolute of this metrics consists of a pair of conjugationally complex points, which are obtained in the intersection of the projective line by way of which, in the model, we have interpreted the line in these geometrices and the absolutes of the relevant geometry. So, the 'line' in those geometries has been interpreted in the projective model by way of the whole projective line. Since any two different projective lines intersected, the two lines in those geometries intersect as well.

Finally, we shall examine threedimensional geometries in which projective metrics on a line is hyperbolic, i.e. the geometries HE, HP and HH. It is well known that the axiomatics of HE-geometry differs from the euclidean geometry axiomatics (PE-geometry) only in the parallelism axiom, and that Klein gave a projective model of this geometry. In 1970, Aleksandrova (see [1]) suggested a system of axioms of HP-geometry, and in 1984 we gave a projective model of this axiomatics (see [10]). Having a direct insight into the systems of axiom is true therein. The result of this axiom in those geometrices is the following: through a point B which is not incident to a line a there exist two lines parallel to the line a. This fact could be explained from the projective point of view. Namely, in those geometries projective metrics on a line is hyperbolic. The absolute of this metrics consists of two real and different points, which, in the projective model, have been obtained in the intersection of the projective line, by way of which we have interpreted the line a, and the absolutes of the relevant geometry. We shall mark these points as P and Q respectively. Two projective lines which are incident to the point B have been obtained by way of the projective points B, P and Q. In the projective model, lines of those geometries which are incident to the point B and at the same time parallel to the line a have been interpreted by way of projective segments which are obtained in the intersection of the projective lines $p(B, P)$ and $p(B, Q)$ and the absolute of the relevant geometry.

The autodual geometry HH, to our knowledge, is not axiomatically built. However, in the Introduction we stated that this geometry too could be interpreted in a projective plane, so that it is to be noticed that the examinations above could apply to this geometry. Now, with the axiomatic foundation of HH-geometry one is to include Lobatchewskian axiom into the system of axioms.

2. Isotropic sequence of points (isotropic lines) in twodimensional geometries

Examining axiomaticly founded twodimensional geometries whose angular metrics (projective metrics in a pencil of lines) is parabolic, i.e. the geometries EP, HP and PP, we notice that parallel points have been introduced. In other words, the following is to be defined as: *two points are parallel* if there does not exist a line which is incident to them; then, the following definition is to be introduced: a set of points a which contains the given point A and all the points parallel to that point is called an *isotropic sequence of points* (*an isotropic line*) with the basis A. It is proved that the following is true:

Theorem 1 *Any point A determines a unique isotropic sequence of points.*

This result could also be explained from the projective point of view. Namely, let us imagine an arbitrary point A in any of the geometies of EP, HP and PP. If the point A represents the apex of an arbitrary angle of the relevant geometry, then it is necessary to choose the absolute in order to define parabolic projective metrics in a pencil of lines with the centre in the point A. Since we here discuss parabolic metrics, the absolute, therefore, consists of a dual real line (see [7], p.276). Specifically, we discuss a tangent from the point A to the absolute of the relevant geometry. This tangent is incident to an absolute point Q (see the Introduction) of the relevant geometry. So, Q is an uncharacteristic point to those geometries. A unique projective line has been given to the points A and Q. In the projective model, an isotropic sequence of points with the basis A has been interpreted by way of this line, excluding the point Q.

In PH-geometry Proskurina defined the so called *plus-parallel* and *minus-parallel* points for the point A, and then a *first-order isotropic sequence* and a *second-order isotropic sequence* with the bases A (see [15], def.19 and def.21). According to that, in this geometry two isotropic sequences of points with the basis A were given. From the projective point of view it could be explained in the following way: the absolute of hyperbolic metrics in a pencil of lines with the centre A consists of a pair of real and different lines which are incident to the point A (see [7], p.276), i.e. these are the tangents from the point A to the absolute of PH-geometry (in the projective model the tangents are incident respectively to the points P and Q of the absolute of this geometry). So, in the projective model, isotropic sequences of points with the basis A have been interpreted by way of the projective lines $p(A,P)$ and $p(A,Q)$, excluding the points P and Q.

In the projective models of EH and HH geometries we could, also, construct two different tangents to the absolute from any point A (see the Introduction). In the projective model of the relevant geometry, isotropic sequences of points with the basis A have been interpreted by way of these tangents, excluding the points of tangency.

Consequently, in twodimensional geometries with hyperbolic projective metrics in a pencil of lines, i.e. in PH, EH and HH geometries, two isotropic sequences of points with the basis A have been given by any point A. We stress that Parnasskaya in her axiomatic foundation of threedimensional geometry which is dual to a hyperbolic geometry defined in a plane α points parallel and divergent to the point A for the point A which is not incident to the plane α (see [14], p.121), but she did not define isotropic sequences of points with the basis A. We hold that with the axiomatic foundation of EH and HH geometries isotropic sequences of points are to be either taken as the starting notions or defined in the way Proskurina did for PH-geometry.

In geometries in which the projective metrics in a pencil of lines is elliptic, i.e. in PE, HE and EE geometries, there exist no parallel points to the given point A, and, therefore, there exist no isotropic sequences of points with the basis A either. Nevertheless, for any two different points there is a unique line which is incident to them. From the projective point of view it could be explained in the following way: for elliptic projective metrics in a pencil of lines with the centre in the point A the absolute consists of a pair of conjugationally complex lines which are incident to the point A. In other words, these are the tangents from the point A on the absolute of the relevant geometry. Accordingly, we notice that in the projective model the line of the relevant geometry has been interpreted by way of every real projective line (or its part), which is incident to the point A. Thus, in the projective model of the relevant geometry, a unique projective line, by way of which the former in those geometries was interpreted, has been given by every point M, which is not incident to A, and by the point A.

3. Ideal sequence of point (ideal line) in twodimensional geometries

In PH-geometry, Proskurina defined *divergent points* (see [15], def.25), and then an *ideal sequence of points* (see [15], def.26). The following theorem was proved true:

Theorem 2 *Any two divergent points A and B determine a unique ideal sequence of points (an ideal line).*

We could interpret this statement in the projective model too. Namely, from an arbitrary point A in the projective model of this geometry, as it was said above, we could construct tangents $p(A,P)$ and $p(A,Q)$ on the absolute of this geometry. The lines $p(A,P)$ and $p(A,Q)$ determine two projective angles one of which comprises the open projective segment PQ, and the other comprises a complement to this segment. The point B which is inside the projective angle which does not contain the projective segment PQ (the segment PQ is a projective segment on the absolute l; see the Introduction) is divergent with the point A. To the points A and B a unique projective line $p(A,B)$ has been given. In the model, the ideal sequence of points has been interpreted by way of this line, excluding the point of intersection with the absolute line l.

On the basis of the projective models of EH and HH geometries, one could notice the existence of ideal sequences of points (ideal lines) in these geometries. The ideal sequences of points in those geometries are to be either defined on the basis of the suggested axiomatics or accepted as the starting notions. We stress that Parnasskaya in [14] did not define an ideal sequence of points.

As a result, in geometries with hyperbolic projective metrics in a pencil of lines, i.e. in PH, EH and HH geometries, besides a line (a straight sequence of points), there exist isotropic sequences of points (isotropic lines) and ideal sequences of points (ideal lines).

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