NOVI SAD J. MATH.
VOL. 29, NO. 3, 1999, 109–119
XII Yugoslav Geometric Seminar
Novi Sad, October 8–11, 1998

RELATING ALGEBRAIC PROPERTIES OF THE CURVATURE TENSOR TO GEOMETRY *

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Abstract

Riemannian manifolds where the Jacobi operator $J(X)$ has globally constant eigenvalues are said to be Osserman; there is a generalization of this condition due to Stanilov. Manifolds where the skew-symmetric curvature operator $R(\pi)$ has pointwise constant eigenvalues are said to be Ivanov-Petrova or simply IP. We generalize these concepts to the complex setting and present examples of complex Osserman and complex IP algebraic curvature tensors and metrics.

AMS Mathematics Subject Classification (1991): 53B20.
Key words and phrases: Algebraic curvature tensor, Curvature operator, Ivanov-Petrova, Jacobi operator, Osserman.

1 Introduction

In this brief note, we describe two different areas where algebraic properties of the Riemann curvature tensor give rise to very rigid geometries. We shall describe some of the known results and discuss possible generalizations to the complex setting. We say that a 4 tensor R is an *algebraic curvature tensor* if R has the following symmetries:

$$R(X,Y,Z,W) = -R(Y,X,Z,W) \tag{1}$$
$$R(X,Y,Z,W) = R(Z,W,X,Y) \tag{2}$$
$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0. \tag{3}$$

The curvature tensor of a Riemannian metric defines an algebraic curvature tensor at each point of the manifold. Conversely, fix a metric g_0 at a point P

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of M and suppose given an algebraic curvature tensor $R_0 \in \otimes^4 T_P M$. Then there exists a Riemannian metric g on M so that $g(P) = g_0$ and so that the Riemann curvature tensor of g at P is given by R_0. Thus the algebraic curvature tensors are important in Riemannian geometry. Although all the questions we shall be considering have analogues for Lorentzian metrics and metrics of higher signature, we shall restrict to positive definite metrics in the interests of brevity; we have included a fairly extensive bibliography in the general setting.

Let $Gr_k(m) = Gr_k(\mathbb{R}^m)$ be the Grassmannian of unoriented k planes in $\mathbb{R}^m$ and let R be an algebraic curvature tensor. If X is a unit vector in $\mathbb{R}^m$, then the *Jacobi operator* $J(X)$ is the symmetric endomorphism of $\mathbb{R}^m$ which sends Y to $R(Y,X)X$. More generally, if $\{X_1, ..., X_k\}$ is an orthonormal basis for the unoriented k plane π in $Gr_k(m)$, then we define $J(\pi) := \sum_i J(X_i)$; this is independent of the particular orthonormal basis which is chosen. The following definition is due to Stanilov and extends a concept originally introduced by Osserman [19]. We assume $1 \leq k \leq m-1$ as the cases $k = 0$ and $k = m$ are trivial.

Definition 1.1 *Let R be an algebraic curvature tensor. We say R is k Osserman if the eigenvalues of $J(\pi)$ are constant on $Gr_k(\mathbb{R}^m)$. We say that a Riemannian manifold (M, g) is k-Osserman if the eigenvalues of $J(\pi)$ are constant on $Gr_k(TM)$; we* **do not** *permit the eigenvalues to vary from point to point.*

Let $Gr_2^+(m) = Gr_2^+(\mathbb{R}^m)$ be the Grassmanian of oriented 2 planes in $\mathbb{R}^m$, let $\mathfrak{so}(m)$ be the Lie algebra of the orthogonal group, and let R be an algebraic curvature tensor. Let $\{X, Y\}$ be an *oriented* orthonormal basis for an oriented 2 plane π. The skew symmetric curvature operator $R(\pi) := R(X,Y) \in \mathfrak{so}(m)$ is independent of the oriented orthonormal basis chosen. In contrast to the Jacobi operator which does not depend upon the orientation, the skew-symmetric curvature operator is sensitive to the orientation chosen.

Definition 1.2 *Let R be an algebraic curvature tensor. We say that R is IP if the eigenvalues of $R(\pi)$ are constant on $Gr_2^+(m)$. We say that a Riemannian manifold (M, g) is IP if the curvature tensor is IP at every point of M; in contrast to the Jacobi operator, we* **do** *permit the eigenvalues to vary from point to point.*

Let $m = 2\bar{m}$. Let $\mathfrak{J}$ be an almost complex structure on $\mathbb{R}^m$ preserving the standard inner product. Let $Gr_k(\bar{m}; \mathbb{C}) \subset Gr_{2k}(m)$ be the set of $\mathfrak{J}$ invariant real $2k$ planes. If $\pi \in Gr_k(\bar{m};\mathbb{C})$, then $\mathfrak{J}$ induces a natural orientation and complex structure on π; we shall say that π is a complex k plane. Let $U(\bar{m}) \subset SO(m)$ and let $\mathfrak{u}(\bar{m}) \subset \mathfrak{so}(m)$ be the associated unitary group and the Lie algebra.

Definition 1.3 *Let R be an algebraic curvature tensor. We say that R is complex if $\mathfrak{J}R(X,Y) = R(X,Y)\mathfrak{J}$ for all X and Y i.e. $R(\pi) \in \mathfrak{u}(\bar{m})$ for all $\pi \in Gr_2^+(m)$. We say that R is weakly complex if $\mathfrak{J}R(X,\mathfrak{J}X) = R(X,\mathfrak{J}X)\mathfrak{J}$ for all X i.e. $R(\pi) \in \mathfrak{u}(\bar{m})$ for all $\pi \in Gr_2(\bar{m};\mathbb{C})$.*

Note that if $(M,g,\mathfrak{J})$ is a Kähler manifold, then $\mathfrak{J}\nabla = \nabla\mathfrak{J}$ so the associated Riemann curvature tensor is complex. We shall see presently there are algebraic curvature tensors which are weakly complex but which are not complex. We generalize Definitions 1.1 and 1.2 to the complex setting in the algebraic context; they have an obvious extension to almost complex Riemannian manifolds.

Definition 1.4 *Let R be an algebraic curvature tensor which is complex (resp. weakly complex). We say that R is complex (resp. weakly complex) k Osserman if the eigenvalues of $J(\pi)$ are constant on $Gr_k(\bar{m};\mathbb{C})$.*

Definition 1.5 *Let R be an algebraic curvature tensor which is complex (resp. weakly complex). We say that R is complex (resp. weakly complex) IP if the eigenvalues of $R(\pi)$ are constant on $Gr_1(\bar{m};\mathbb{C})$.*

In §2, we review some results for k Osserman manifolds. We construct examples of complex and weakly complex k Osserman curvature tensors; see Theorem 2.3 for details. In §3, we review some results for IP manifolds. We construct examples of complex and weakly complex IP curvature tensors; see Theorem 3.4 for details. The case $m = 4$ is exceptional. In §4, we exhibit a weakly complex curvature tensor R which is IP of rank 4 and 1 Osserman; see Theorem 4.1 for details.

2 Osserman Manifolds

If (M,g) is flat or is locally a rank 1 symmetric space, then the set of local isometries of (M,g) acts transitively on $Gr_1(TM)$ and hence (M,g) is 1 Osserman. Osserman [19] conjectured that the converse implication might hold, i.e. that if (M,g) is 1 Osserman, then (M,g) is flat or is locally a rank 1 symmetric space. This conjecture has been established by Chi [6] if m is odd, if $m \equiv 2 \bmod 4$, or if $m = 4$. Thus the algebraic condition that the Jacobi operator has constant eigenvalues imposes great restrictions on the permissible geometries.

On the other hand, there are algebraic curvature tensors which are 1 Osserman but which are not the curvature tensors of a rank 1 symmetric space. Let $\{c_i\}_{1 \leq i \leq r}$ be skew-symmetric matrices which satisfy the Clifford commutation relations

$$c_i c_j + c_j c_i = -2\delta_{ij}.$$

One should think of the c_i as an anti-commuting family of almost complex structures. For example, if $m \equiv 0 \bmod 4$, we can identify $\mathbb{R}^m$ with $\mathbb{H}^{m/4}$ and take $\{c_1 = i, c_2 = j, c_3 = k\}$ to be defined by quaternion multiplication. Let $(\cdot,\cdot)$ be the standard inner product on $\mathbb{R}^m$. We define the curvature operators

$$\begin{aligned} &R_0(X,Y)Z := (Y,Z)X - (X,Z)Y \text{ and} \\ &R_i(X,Y)Z := (Y,c_iZ)c_iX - (X,c_iZ)c_iY - 2(X,c_iY)c_iZ. \end{aligned} \tag{4}$$

One checks easily that the associated 4 tensors satisfy the curvature symmetries of equations (1), (2), and (3) so that these define algebraic curvature tensors. The tensor R_0 is the curvature tensor for a metric of constant sectional curvature; the tensor $R_0 - R_i$ is the curvature tensor for a metric of constant holomorphic sectional curvature relative to the almost complex structure c_i.

Theorem 2.1 *Let $R := \lambda_0 c_0 + \sum_i \lambda_i R_i$. Then R is an algebraic curvature tensor which is 1 Osserman.*

Proof: We follow the discussion in [11]. We use equation (4) to see:

$$\begin{aligned} &J_R(X)X = 0,\ J_R(X)c_iX = (\lambda_0 - 3\lambda_i)c_iX, \text{ and} \\ &J_R(X)Y = \lambda_0 Y \text{ if } Y \perp \{X, c_1X, ...\}. \ \square \end{aligned} \tag{5}$$

If R is the algebraic curvature tensor of a rank 1 symmetric space, then J_R has two eigenvalues μ_1 and μ_2 where $\mu_1 = 4\mu_2$. Thus if $\lambda_i + \lambda_0 \neq 0$ for some i, then the curvature defined in Theorem 2.1 is not the algebraic curvature tensor of a rank 1 symmetric space. Thus the classification of 1 Osserman algebraic curvature tensors is likely to be much more complicated than the corresponding classification of 1 Osserman metrics if $m \equiv 0 \bmod 4$.

We note that there are 4 dimensional manifolds whose curvature tensor is Osserman at each point; as the eigenvalues vary from point to point, these manifolds are not Osserman. We refer to [17] for details. There is a great deal of work which has been done to study 1 Osserman manifolds both in the Riemannian and in the pseudo-Riemannian setting. We refer to [2, 3, 4, 5, 6, 7, 9, 17, 19, 20, 22] for other related results.

Relatively little is known about the k Osserman condition for $2 \leq k \leq m-2$. Let σ_π be orthogonal projection on a k plane π. We use equation (5) to see that

$$J_{R_0(\pi)} = k \cdot I - \sigma_\pi \text{ and } J_{R_i(\pi)} = -3\sigma_{c_i\pi}. \tag{6}$$

This shows that the curvature tensors R_0 and R_i are k Osserman for any k; in the Riemannian setting, these are the only known examples of algebraic curvature tensors which are k Osserman for $2 \leq k \leq m-2$. (For metrics of signature (p,p), if $\mathfrak{J}$ is a para-complex structure, then $R_{\mathfrak{J}}$ is k Osserman for any k. It is known that k Osserman implies constant sectional curvature for a metric of signature $(1, m-1)$ if $2k \neq m$.) Equation (6) also shows that if $\lambda_0\lambda_1 \neq 0$ and if $2 \leq k \leq m-2$, then $\lambda_0 R_0 + \lambda_1 R_1$ is not k Osserman. The spaces of constant sectional curvature are the only rank 1 symmetric spaces which are k Osserman for $2 \leq k \leq m-2$.

We refer to [12, 21, 15] for the proof of the following result which summarizes most of the known results about k Osserman curvature tensors in the Riemannian setting where $2 \leq k \leq m-2$.

Theorem 2.2 *Let $2 \leq k \leq m-2$ and let R be an algebraic curvature tensor which is k Osserman.*

1. *The tensor R is $m-k$ Osserman and Einstein.*
2. *If $m \neq 2k$, then R is 2 Stein - i.e. $Tr(J(X)^2)$ is constant on $Gr_2^+(m)$.*
3. *Suppose $k = 2$. If m is odd, then $R = \lambda_0 R_0$. If $m \equiv 2 \mod 4$, then either $R = \lambda_0 R_0$ or $R = \lambda_1 R_1$ for some unitary almost complex structure c_1.*

We can exhibit the following examples in the complex setting:

Theorem 2.3 *Let $\{c_i\}$ be Clifford matrices on $\mathbb{R}^m$. Use c_1 to give $\mathbb{R}^m$ an almost complex structure.*

1. *We have that $\lambda_0 R_0 + \sum_i \lambda_i R_i$ is weakly complex 1 Osserman.*
2. *We have that $\lambda_0 R_0 + \lambda_1 R_1$ is weakly complex k Osserman for any k.*

3. We have that $R_0 - R_1$ is complex k Osserman for any k.

Proof: Let $\pi = \text{Span}\{X, c_1X\}$ be a complex line; orthogonal projection σ_π is complex. Let $Z \perp \pi$. We show that R_0 and R_1 are weakly complex with respect to the structure c_1 by using equation (4) to compute:

$$\begin{array}{ll} R_0(X,c_1X)X = -c_1X, & R_1(X,c_1X)X = 3c_1X, \\ R_0(X,c_1X)c_1X = X, & R_1(X,c_1X)c_1X = -3X, \\ R_0(X,c_1X)Z = 0, & R_1(X,c_1X)Z = 2c_1Z, \\ R_0(\pi) = -c_1\sigma_\pi, & R_1(\pi) = 2c_1 + c_1\sigma_\pi \end{array} \qquad (7)$$

If $i > 1$, then $c_i\pi$ is a complex line which is perpendicular to π. Let $Z \perp c_i\pi$. We compute:

$$\begin{array}{ll} R_i(X,c_1X)c_iX = -c_1c_iX, & R_i(X,c_1X)c_1c_iX = c_iX, \\ R_i(X,c_1X)Z = 0, & R_i(\pi) = -c_1\sigma_{c_i\pi}. \end{array} \qquad (8)$$

This shows that $R_i(X, c_1X)$ is complex; the first asssertion now follows from Theorem 2.1. We use equation (5) to prove assertion (2) by computing:

$$\begin{array}{l} J(\pi) = 2\lambda_0 \cdot I - (\lambda_0 + 3\lambda_1)\sigma_\pi \text{ if } \pi \in Gr_1(\bar{m}; \mathbb{C}) \\ J(\pi) = 2k\lambda_0 \cdot I - (\lambda_0 + 3\lambda_1)\sigma_\pi \text{ if } \pi \in Gr_k(\bar{m}; \mathbb{C}). \end{array}$$

To prove assertion (3), we need only show that $R_0 - R_1$ is complex. We have already shown $(R_0 - R_1)(X, c_1X) \in U(\bar{m})$. Let $Y \perp \pi$. We compute:

$$\begin{array}{ll} (R_0 - R_1)(X,Y)X = -Y, & (R_0 - R_1)(X,Y)c_1X = -c_1Y, \\ (R_0 - R_1)(X,Y)Y = X, & (R_0 - R_1)(X,Y)c_1Y = c_1X, \\ (R_0 - R_1)(X,Y)Z = 0 \text{ for } Z \perp \{X, Y, c_1X, c_1Y\}. & \square \end{array}$$

The curvature tensor $\lambda_0 R_0 + \lambda_1 R_1$ is weakly complex; it is complex if and only if $\lambda_0 + \lambda_1 = 0$. Thus there exist algebraic curvature tensors which are weakly complex but not complex. Suppose $m \equiv 0 \bmod 4$. The tensor $R_1 + R_2$ is weakly complex 1 Osserman; it is not weakly complex k Osserman for any $1 < k < \bar{m} - 1$.

We note that the curvature tensor $R_0 - R_1$ has constant holomorphic sectional curvature $+1$ and modulo a suitable normalizing constant is the curvature tensor of the Fubini-Study metric on $\mathbb{CP}^n$. The group of complex isometries of $\mathbb{CP}^n$ is the projective unitary group; this acts transitively on the set of complex k planes in the tangent bundle. This gives a geometric proof that the algebraic curvature tensor $R_0 - R_1$ is complex k Osserman for any k.

3 The skew symmetric curvature operator

We summarize below what is known for IP algebraic curvature tensors and for IP metrics. We assume $m \geq 4$ as the situation if $m = 3$ is quite different. Only the case $m = 7$ is open. We refer to [16, 18] for the proof of the following result:

Theorem 3.1

1. *Let R be a 4 tensor satisfying equation (1) so that* $\text{Rank}\,(R(\pi))$ *is constant. If $m \geq 5$ and if $m \neq 7$, then* $\text{Rank}\,(R) \leq 2$.

2. *If $m = 7$, then there exists a 4 tensor satisfying the symmetries given in equations* (1) *and* (2) *so that $R(\pi)$ has constant eigenvalues and so that* $\text{Rank}\,(R(\pi)) > 2$.

3. *If $m = 4$, then the following curvature tensor is an IP algebraic curvature tensor of rank 4 and this is the only such curvature tensor up to a change of basis for $\mathbb{R}^4$:*

$$R_{1212} = a_2,\ R_{1313} = a_2,\ R_{2424} = a_2, R_{1414} = a_1,\ R_{2323} = a_1,$$
$$R_{3434} = a_2,\ R_{1234} = a_1,\ R_{1324} = -a_1,\ R_{1423} = a_2,\ a_2 + 2a_1 = 0.$$

We refer to [13, 16] for the proof of assertion (1); like Chi's proof of the Osserman conjecture for m odd and $m \equiv 2 \bmod 4$, the proof uses substantial amounts of algebraic topology. We refer to [14, 16] for the proof of assertion (2). We refer to [18] for the proof of assertion (3); we shall give a different derivation in Theorem 4.1 below. We do not know if there exists an algebraic IP curvature tensor in dimension 7 of higher rank. In the pseudo-Riemannian setting, Zhang has shown that assertion (1) holds if $m \geq 12$ and if the metric has signature $(1, m-1)$ or if $m \geq 13$, if $m \not\equiv 0 \bmod 8$, and if the metric has signature $(2, m-2)$.

Let R be an IP algebraic curvature tensor. If $m \geq 5$ and if $m \neq 7$, then Theorem 3.1 shows that $\text{Rank}\,(R) \leq 2$. The IP algebraic curvature tensors were classified if $m = 4$ by Ivanov and Petrova [18] so we assume $m \geq 5$. Let $\phi \in O(m)$. We twist the curvature operator for a metric of constant sectional curvature to define

$$R_\phi(X,Y)Z := R_0(\phi X, \phi Y)Z = (\phi Y, Z)\phi X - (\phi X, Z)\phi Y. \tag{9}$$

We refer to [16] for the proof of the following classification result:

Theorem 3.2 *Let $m \geq 5$.*

1. *If $\phi^2 = id$, then R_ϕ is an IP algebraic curvature tensor.*

2. *If R is an IP algebraic curvature tensor of rank 2, then $R = cR_\phi$ for some constant c and for some isometry ϕ with $\phi^2 = id$.*

The Osserman metrics have been classified by Chi if m is odd and if $m \equiv 2 \bmod 4$. The IP metrics have been classified if $m \geq 4$ and $m \neq 7$. Both structures are very rigid. We refer to [13, 16, 18] for the proof of:

Theorem 3.3 *Let (M, g) be a Riemannian manifold of dimension $m \geq 4$ which is IP. For $m = 7$, assume that R has rank at most 2. Then either (M, g) has constant sectional curvature or (M, g) is locally isometric to a warped product of the form $ds^2 = dt^2 + f(t)ds_N^2$ on $(t_0, t_1) \times N$ where $f(t) := \frac{1}{2}(\mathcal{K}t^2 + At + B) > 0$ and where ds_N^2 has constant sectional curvature $\mathcal{K}$.*

We have the following examples of complex and weakly complex IP algebraic curvature tensors.

Theorem 3.4

1. *Let $\phi \in U(\bar{m})$ and let $\phi^2 = id$. Then cR_ϕ is a complex IP algebraic curvature tensor.*

2. *The tensor $\lambda_0 R_0 + \lambda_1 R_1 + \lambda_2 R_2$ is a weakly complex IP algebraic curvature tensor.*

3. *The tensor $R_0 - R_1$ is a complex IP algebraic curvature tensor.*

Proof: Since $R_\phi(\pi) = \sigma_{\phi\pi}$, R_ϕ is complex. The first asssertion now follows from Theorem 3.1. We use equations (7) and (8) to see that if π belongs to $Gr_1(\bar{m}; \mathbb{C})$, then

$$(\lambda_0 R_0 + \lambda_1 R_1 + \lambda_2 R_2)(\pi) = 2\lambda_1 c_1 + (\lambda_1 - \lambda_0)c_1\sigma_\pi - \lambda_2 c_1 \sigma_{c_2\pi}.$$

Assertion (2) now follows since π and $c_2\pi$ are perpendicular complex lines. By Theorem 2.3, $R_0 - R_1$ is complex; assertion (3) now follows. □

4 Four dimensional geometry

The dimension $m = 4$ is exceptional.

Theorem 4.1 *If $m = 4$, then the curvature tensor $R_1 + 2R_0$ is weakly complex, IP of rank 4, 1 Osserman; it is not Ricci flat.*

Proof: Let $\{1, i, j, k\}$ be the usual basis for $\mathbb{H} = \mathbb{R}^4$, let $c_1 x = i \cdot x$, and let $\mathcal{R} = R_1 + 2R_0$. We use equation (4) to compute:

$$\begin{array}{llll}
R_0(1,i)1 = -i, & R_0(1,i)i = 1, & R_0(1,i)j = 0, & R_0(1,i)k = 0, \\
R_0(1,j)1 = -j, & R_0(1,j)i = 0, & R_0(1,j)j = 1, & R_0(1,j)k = 0, \\
R_0(1,k)1 = -k, & R_0(1,k)i = 0, & R_0(1,k)j = 0, & R_0(1,k)k = 1, \\
R_0(i,j)1 = 0, & R_0(i,j)i = -j, & R_0(i,j)j = i, & R_0(i,j)k = 0, \\
R_0(i,k)1 = 0, & R_0(i,k)i = -k, & R_0(i,k)j = 0, & R_0(i,k)k = i, \\
R_0(j,k)1 = 0, & R_0(j,k)i = 0, & R_0(j,k)j = -k, & R_0(j,k)k = j
\end{array}$$

$$\begin{array}{llll}
R_1(1,i)1 = 3i, & R_1(1,i)i = -3, & R_1(1,i)j = 2k, & R_1(1,i)k = -2j, \\
R_1(1,j)1 = 0, & R_1(1,j)i = k, & R_1(1,j)j = 0, & R_1(1,j)k = -i, \\
R_1(1,k)1 = 0, & R_1(1,k)i = -j, & R_1(1,k)j = i, & R_1(1,k)k = 0, \\
R_1(i,j)1 = -k, & R_1(i,j)i = 0, & R_1(i,j)j = 0, & R_1(i,j)k = 1, \\
R_1(i,k)1 = j, & R_1(i,k)i = 0, & R_1(i,k)j = -1, & R_1(i,k)k = 0, \\
R_1(j,k)1 = 2i, & R_1(j,k)i = -2, & R_1(j,k)j = 3k, & R_1(j,k)k = -3j.
\end{array}$$

$$\begin{array}{llll}
\mathcal{R}(1,i)1 = i, & \mathcal{R}(1,i)i = -1, & \mathcal{R}(1,i)j = 2k, & \mathcal{R}(1,i)k = -2j, \\
\mathcal{R}(1,j)1 = -2j, & \mathcal{R}(1,j)i = k, & \mathcal{R}(1,j)j = 2, & \mathcal{R}(1,j)k = -i, \\
\mathcal{R}(1,k)1 = -2k, & \mathcal{R}(1,k)i = -j, & \mathcal{R}(1,k)j = i, & \mathcal{R}(1,k)k = 2, \\
\mathcal{R}(i,j)1 = -k, & \mathcal{R}(i,j)i = -2j, & \mathcal{R}(i,j)j = 2i, & \mathcal{R}(i,j)k = 1, \\
\mathcal{R}(i,k)1 = j, & \mathcal{R}(i,k)i = -2k, & \mathcal{R}(i,k)j = -1, & \mathcal{R}(i,k)k = 2i, \\
\mathcal{R}(j,k)1 = 2i, & \mathcal{R}(j,k)i = -2, & \mathcal{R}(j,k)j = k, & \mathcal{R}(j,k)k = -j.
\end{array} \tag{10}$$

This tensor is weakly complex by Theorem 3.4. It is 1 Osserman by Theorem 2.1. We set $a_1 = 1$, $a_2 = -2$, $e_1 = 1$, $e_3 = j$, $e_2 = k$, and $e_4 = i$ to see that this tensor is equivalent to the one given in Theorem 3.1 (3).

Left and right quaternion multiplication commute. Thus the map $\xi \to \xi \cdot \eta$ commutes with c_1. Since the structures are preserved, $R(\pi)$ and $R(\pi \cdot \eta)$ have the same eigenvalues. Thus it suffices to show that the eigenvalues of $R(1, \xi)$ are independent of the purely imaginary unit quaternion ξ which is chosen. Let

$$T_\xi X := 3(i, X)i\xi - 3(i\xi, X)i - 2X\xi.$$

Then T is right multiplication by ξ on the plane spanned by $\{i, i\xi\}$ and T is right multiplication by -2ξ on orthogonal plane. Thus T has constant eigenvalues $\{\pm\sqrt{-1}, \pm 2\sqrt{-1}\}$. The map $\xi \to T_\xi$ is linear in ξ. We use equation (10) to see that

$$T(1,i) = \mathcal{R}(1,i),\ T(1,j) = \mathcal{R}(1,j),\ T(1,k) = \mathcal{R}(1,k).$$

It now follows that $\mathcal{R}(1,\xi) = T(\xi)$ for any purely imaginary unit vector ξ and that $\mathcal{R}$ is IP. □

Professor Alekseevsky [1] has informed us that he has proved the following result:

Theorem 4.2 *Let $m = 4$ and let R be an algebraic curvature tensor. If R is is Käehler Einstein and if R is complex IP, then $R = c(R_0 - R_1)$.*

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