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A LEFT COMPATIBLE COEQUALITY RELATION ON A SEMIGROUP WITH APARTNESS

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Abstract

This investigation is in Constructive Algebra in the sense of Bishop, Richman, Ruitenburg, Troelstra and van Dalen. The main subject of this paper is the characterization of the left compatible coequality relation q on semigroup with apartness by means of special right consistent subsets $L_{(a)} = \{x \in S : x \in Sa\}$ and by filled product of relations. Also, we give a construction of a quasi-antiorder c such that $q = c \cup c^{-1}$ and some descriptions of classes of relations c and q. We also present an assertion on the left class $A(a)$ $(a \in A)$ of the relation c in which we prove that $A(a)$ is a maximal strongly extensional right consistent subset of S such that $a \in A(a)$. Also, we give an assertion on the right class $B(a)$ $(a \in S)$ of the relation c in which we prove that $B(a)$ is a maximal strongly extensional left ideal of S such that $A \in B(a)$. Besides, we provide some descriptions of families $\{A(a) : a \in S\}$ and $\{B(a) : a \in S\}$.

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1. Preliminaries and Introduction

Let S be a set with apartness in the sense of Bishop [1], Mines [5], Troelstra and van Dalen [13]. A relation q on S is a coequality relation ([3],[10]) on S if and only if it is consistent, symmetric and cotransitive. The filled product ([9],[10]) of a relation f and a relation g on S is the relation $g * f$ defined by $\{(a,c) \in S \times S : (\forall b \in S)((a,b) \in f \vee (b,c) \in g)\}$. By $c(f)$ we denote the intersection $\cap_{n \in \mathbf{N}}({}^n f)$ and called cotransitive closure of f ([10]). If h is a relation on S, by h' we denote the relation $\{(x,y) \in S \times S : (x,y) \# h)\}$. More on coequality relations the reader may find in the papers [3],[7], [8], and on filled products in the papers [9],[10],[11].

Let $(S, =, \neq, 1, \cdot)$ be a semigroup with apartness and where the semigroup operation is strongly extensional in the following sense:

$$(\forall a, b, x, y \in S)((ay \neq by \Rightarrow a \neq b) \wedge (xa \neq xb \Rightarrow a \neq b)).$$

This is equivalent to the following condition: $(\forall a, b, x, y \in S)(ax \neq by \Rightarrow a \neq b \vee x \neq y)$. Throughout this paper Sa will denote a principal left ideal of a semigroup S. A subset T of S is a *right consistent subset* of S ([2]) or a *left coideal* of S if and only if

$$(\forall x, y \in S)(xy \in T \Rightarrow y \in T);$$

a subset T of S is a *left consistent subset* of S ([2]) or a *right coideal* of S if and only if

$$(\forall x, y \in S)(xy \in T \Rightarrow x \in T);$$

a subset T of S is a *consistent subset* of S ([2]) or a *coideal* of S if and only if

$$(\forall x, y \in S)(xy \in T \Rightarrow x \in T \wedge y \in T).$$

Let q be a coequality relation on a semigroup S such that

$$(\forall a, b, y \in S)((ay, by) \in q \Rightarrow (a,b) \in q).$$

Then we say that it is a *left cocongruence* on S or *left compatible with the semigroup operation* of S. If

$$(\forall a, b, x \in S)((xa, xb) \in q \Rightarrow (a,b) \in q)$$

holds, then q is a *right cocongruence* on S or *right compatible with the semigroup operation* of S. The coequality relation q on S is a *cocongruence* on S, or *compatible with semigroup operation* of S, if and only if it is both a left and a right cocongruence. A subset T of a semigroup S is *completely semiprime* if $x^2 \in T \Rightarrow x \in T$ for all $x \in S$. A subset T of a semigroup S is *completely prime* if and only if $xy \in T \Rightarrow x \in T \wedge y \in T$ for all $x, y \in S$ ([2]). An (left, right) ideal T is a *completely semiprime* (*completely prime*) (left, right) ideal of a semigroup S if it is a completely semiprime (prime) subset of S ([2]). Let T be a (left, right) coideal (consistent subset) of a semigroup S. T is a *completely semiprime* (*left, right*) *semifilter* if $x \in T \Rightarrow x^2 \in T$ for all $x \in T$ and T is a *completely prime* (*left, right*) *filter* if $x \in T \wedge y \in T \Rightarrow xy \in T$.

Semigroups with apartness were first defined and studied by Heyting. After that, several authors have worked on this important topic, for example Ruitenburg [12], Troelstra and van Dalen [13], Johnstone [4], Mulvey [6] and the author. There are some general problems on semigroups with apartness in the constructive algebra. In this paper we are interested in the following problems:

(1) Is there a coequality relation q on a semigroup S with apartness such that it is a left cocongruence on S?

(2) If such q exists, describe q-classes.

For the first question we have the following answer. Let f be a relation on a semigroup S with apartness. Then the cotransitive closures $a = c((f \cup f^{-1}) \cap \neq)$ and $b = c((f \cap f^{-1}) \cap \neq)$ ([9]) are coequality relations on S ([3],[10]) and the relations $*a = \{(u, v) \in S \times S : (\exists x \in S)((ux, vx) \in a)\}$ and $*b$ are left cocongruences on S. In this paper we will give (Theorem 2.5) a construction of a left cocongruence q on a semigroup S with apartness by using principal left consistent subsets $L(a)$, $a \in S$, of semigroup S constructed in Theorem 2.1. Also, we will describe classes $Z(a)$, $a \in S$, of the so constructed left cocongruence (Theorem 3.5). Besides, we will define a new notion, quasi-antiorder relation c on a semigroup S such that $q = c \cup c^{-1}$ and we describe the classes $A(a)$ and $B(a)$ ($a \in S$) of the relation c (Section 3). We will prove that the left class $A(a)$ is a strongly extensional right consistent subset, while the right class $B(a)$ is a strongly extensional left ideal of S such that $a \# A(a)$ and $a \# B(a)$ (Theorems 3.2 and 3.4). In the last section we give descriptions of the semigroup S/q in the following cases:

(i) $A(a)$ is a left semifilter for every $a \in S$;

(ii) $A(a)$ is a consistent subset of S for every $a \in S$.

For undefined notions and notations we refer to [1, 2, 5, 7, 8, 10, 12, 13].

2. Construction of a left cocongruence

Let $(S, =, \neq, \cdot, 1)$ be a semigroup with apartness. In this section we introduce the notion of a principal right consistent subset (left coideal) of a semigroup with apartness. Also, we introduce a relation defined by these sets and describe their basic properties. We start with the following theorem whose proof is omitted.

Theorem 2.1. *Let a be an element of S. Then the set $L_{(a)} = \{x \in S : x\#Sa\}$ is a right consistent subset (left coideal) of S such that:*

(i) $a\#L_{(a)}$;

(ii) $L_{(a)} = \emptyset \Rightarrow 1 \in L_{(a)}$;

(iii) let a be an invertible element of S; then $L_{(a)} = \emptyset$;

(iv) $(\forall x \in S)(L_{(a)} \subseteq L_{(xa)})$;

(v) $(\forall n \in \mathbf{N})(L_{(a)} \subseteq L_{(a^n)})$.

We introduce the following notation:

$$(a, b) \in \ell \Leftrightarrow b \in L_{(a)}.$$

The properties of this relation are described by the following

Theorem 2.2.

(vi) ℓ is a consistent relation on S;

(vii) $(a, b) \in \ell \Leftrightarrow (\forall x \in S)((xa, xb) \in \ell)$;

(viii) $(a, b) \in \ell \Leftrightarrow (\forall n \in \mathbf{N})((a^n, b) \in \ell)$;

(ix) $(\forall x \in S)((a, xb) \in \ell \Rightarrow (xa, b) \in \ell)$;

(x) $(\forall x \in S)(a, xa) \notin \ell$;

(xi) $(\forall y \in S)((ay, by) \in \ell \Rightarrow (a, b) \in \ell)$.

As a consequence of this theorem we obtain

Theorem 2.3 *Let S be a semigroup with apartness. The relation $c(\ell)$ has the following properties:*

(xii) $c(\ell)$ *is a consistent relation on* S;

(xiii) $c(\ell)$ *is a cotransitive relation on* S;

(xiv) $(\forall y \in S)((ay, by) \in c(\ell) \Rightarrow (a, b) \in c(\ell))$;

(xv) $(\forall y \in S)((y, xy) \# c(\ell))$;

(xvi) $(\forall y \in S)(\forall n \in \mathbf{N})((y, y^n) \# c(\ell))$;

(xvii) $(\forall x \in S)((a, b) \in c(\ell) \Rightarrow (xa, b) \in c(\ell))$;

(xviii) $(\forall n \in \mathbf{N})((a, b) \in c(\ell) \Rightarrow (a^n, b) \in c(\ell))$.

Proof. *(xii)* and *(xiii)* follow immediately from the definition of $c(l)$ and the previous theorem.

(xiv)
$$\begin{aligned} (ay, by) \in c(\ell) &\Rightarrow (ay, by) \in \ell \\ &\Rightarrow (a, b) \in \ell \quad (\text{by } (xi)) \\ (ay, by) \in c(\ell) &\Rightarrow (ay, by) \in {}^2\ell \\ &\Rightarrow (\forall s \in S)((ay, sy) \in \ell \vee (sy, by) \in \ell) \\ &\Rightarrow (\forall s \in S)((a, s) \in \ell \vee (s, b) \in \ell) \quad (\text{by } (xi)) \\ &\Leftrightarrow (a, b) \in {}^2\ell. \end{aligned}$$

Assume that for some $n \in \mathbf{N}$ we have

$$(ay, by) \in c(\ell) \Rightarrow (ay, by) \in {}^n\ell \Rightarrow (a, b) \in {}^n\ell.$$

Then
$$\begin{aligned} (ay, by) \in c(\ell) &\Rightarrow (ay, by) \in {}^{n+1}\ell \\ &\Rightarrow (\forall s \in S)((ay, sy) \in {}^n\ell \vee (sy, by) \in \ell) \\ &\Rightarrow (\forall s \in S)((a, s) \in {}^n\ell \vee (s, b) \in \ell) \\ &\Leftrightarrow (a, b) \in {}^{n+1}\ell. \end{aligned}$$

So, if $(ay, by) \in c(\ell)$, then

$$(a, b) \in \bigcap_{n \in \mathbf{N}} {}^n\ell = c(\ell).$$

(xv) Let (u, v) be an arbitrary element of $c(\ell)$. Then

$$\begin{aligned}(u,v)\in c(\ell) &\Rightarrow (u,v)\in c(\ell)\vee(y,xy)\in c(\ell)\vee(xy,v)\in c(\ell)\\ &\Rightarrow u\neq y\vee(y,xy)\in c(\ell)\vee xy\neq v\\ &\Rightarrow (u,v)\neq(y,xy)\vee(xy,xy)\in c(\ell)\quad\text{(by } (vii))\\ &\Rightarrow (u,v)\neq(y,xy).\end{aligned}$$

(xvi) Follows from *(xv)*.

(xvii) $(a,b)\in c(\ell)\Rightarrow(x,ax)\in c(\ell)\vee(xa,b)\in c(\ell)\Rightarrow(xa,b)\in c(\ell)$ (by *(xv)*).

(xviii) Follows from *(xvii)*. □

Corollary 2.4. *The relation $c(\ell)'$ is a quasi-order on S such that*

$$(\forall a,b,y\in S)((a,b)\in c(\ell)'\Rightarrow(ay,by)\in c(\ell)').$$

Proof. It is clear that $c(\ell)'$ is reflexive. Let $(x,y),(y,z)\in c(\ell)'$ and let (u,v) be an arbitrary element of $c(\ell)$. Then

$$\begin{aligned}(u,v)\in c(\ell) &\Rightarrow (u,x)\in c(\ell)\vee(y,z)\in c(\ell)\vee(z,v)\in c(\ell)\\ &\Rightarrow u\neq x\vee z\neq v\quad\text{(by } (xii))\\ &\Leftrightarrow (u,v)\neq(x,z).\end{aligned}$$

So, $(x,z)\in c(\ell)'$.

Let $(a,b)\in c(\ell)'$ and let (u,v) be an arbitrary element of $c(\ell)$. Then

$$\begin{aligned}(u,v)\in c(\ell) &\Rightarrow (u,ay)\in c(\ell)\vee(ay,by)\in c(\ell)\vee(by,v)\in c(\ell)\\ &\Rightarrow u\neq ay\vee(a,b)\in c(\ell)\vee by\neq v\\ &\Leftrightarrow (u,v)\neq(ay,by).\end{aligned}$$

Therefore, $(ay,by)\in c(\ell)'$. □

The above properties of the relation $c(\ell)$ provide a motivation to define a new notion. A relation r on a set with apartness X is a *quasi-antiorder* if and only if it is consistent and cotransitive relation on X. Let S be a semigroup with apartness and assume that

$$(\forall a,b,y\in S)((ay,by)\in r\Rightarrow(a,b)\in r).$$

In that case we say that r is a *left compatible* with the semigroup operation. It is easy to see that if r is a left compatible quasi-antiorder on a semigroup S, then the relation $r\cup r^{-1}$ is a left cocongruence on S. This assertion is the matter of the following theorem.

Theorem 2.5. *The relation* $q = c(\ell) \cup (c(\ell))^{-1}$ *is a left cocongruence on* S.

Proof.

$$\begin{array}{lll}
(a,b) \in q & \Leftrightarrow & (a,b) \in c(\ell) \vee (b,a) \in c(\ell) \\
 & \Rightarrow & (\forall s \in S)((a,b) \neq (s,s)). \\
(a,b) \in q & \Leftrightarrow & (a,b) \in c(\ell) \vee (b,a) \in c(\ell) \\
 & \Leftrightarrow & (b,a) \in q. \\
(a,c) \in q & \Leftrightarrow & (a,c) \in c(\ell) \vee (c,a) \in c(\ell) \\
 & \Rightarrow & (a,b) \in c(\ell) \vee (b,c) \in c(\ell) \vee (c,b) \in c(\ell) \vee (b,a) \in c(\ell) \\
 & \Leftrightarrow & (a,b) \in q \vee (b,c) \in q. \\
(ay,by) \in q & \Leftrightarrow & (ay,by) \in c(\ell) \vee (by,ay) \in c(\ell) \\
 & \Rightarrow & (a,b) \in c(\ell) \vee (b,a) \in c(\ell) \\
 & \Leftrightarrow & (a,b) \in q. \quad \square
\end{array}$$

On the other hand, we can construct a consistent and symmetric relation $t = \ell \cup \ell^{-1}$, i.e.

$$(a,b) \in t \Leftrightarrow a \in L_{(b)} \vee b \in L_{(a)}$$

and the coequality relation $c(t)$. At the end of this section we formulate the following questions:

Question 1. Is the relation $c(\ell \cup \ell^{-1})$ a cocongruence on an arbitrary semigroup S?

Question 2. What is the connection between the relations q and $c(\ell \cup \ell^{-1})$?

3. The classes of relations $c(\ell)$ and q

For an element a of a semigroup S and for $n \in \mathbf{N}$ we introduce the following notation:

$$\begin{array}{rcl}
A_n(a) & = & \{x \in S : (a,x) \in^n \ell\}, \\
B_n(a) & = & \{y \in S : (y,a) \in^n \ell\}, \\
A(a) & = & \{x \in S : (a,x) \in c(\ell)\}, \\
B(a) & = & \{y \in S : (y,a) \in c(\ell)\}.
\end{array}$$

The following results show some of the basic properties of these sets.

Theorem 3.1 *Let a be an element of a semigroup S. Then*

(xix) $A_1(a) = L_{(a)}$;

(xx) $A_{n+1}(a) \subseteq A_n(a)$;

(xxi) $A_{n+1}(a) = \{x \in S : \; S = A_n(a) \cup B_1(x)\}$;

(xxii) $A(a) = \bigcap_{n \in \mathbf{N}} A_n(a)$;

(xxiii) $a \# A(a)$;

(xxiv) $A(a)$ *is a strongly extensional right consistent subset of* S;

(xxv) $(\forall x \in S)(A(a) \subseteq A(xa))$;

(xxvi) $(\forall n \in \mathbf{N})(A(a) \subseteq A(a^n))$.

Proof. *(xx)*
$$\begin{aligned} x \in A_{n+1}(a) &\Leftrightarrow (a,x) \in {}^{n+1}\ell \\ &\Leftrightarrow (\forall s \in S)((a,s) \in {}^{n}\ell \vee (s,x) \in \ell) \\ &\Rightarrow (a,x) \in {}^{n}\ell \\ &\Leftrightarrow x \in A_n(a). \end{aligned}$$

(xxi)
$$\begin{aligned} x \in A_{n+1}(a) &\Leftrightarrow (a,x) \in {}^{n+1}\ell \\ &\Leftrightarrow (\forall s \in S)((a,s) \in {}^{n}\ell \vee (s,x) \in \ell) \\ &\Leftrightarrow (\forall s \in S)(s \in A_n(a) \vee s \in B(a)) \\ &\Leftrightarrow S = A_n(a) \cup B(a). \end{aligned}$$

(xxiii)
$$\begin{aligned} x \in A(a) &\Leftrightarrow (a,x) \in c(\ell) \\ &\Rightarrow (a,x) \neq (x,x) \quad (\text{by } (xii)) \\ &\Leftrightarrow a \neq x. \end{aligned}$$

(xxiv)
$$\begin{aligned} xy \in A(a) &\Leftrightarrow (a,xy) \in c(\ell) \\ &\Rightarrow (a,y) \in c(\ell) \vee (y,xy) \in c(\ell) \\ &\Rightarrow (a,y) \in c(\ell) \quad (\text{by } (xv)) \\ &\Leftrightarrow y \in A(a). \end{aligned}$$

Let x be an arbitrary element of S and let $y \in A(a)$. Then

$$(a,y) \in c(\ell) \Rightarrow (a,x) \in c(\ell) \vee (x,y) \in c(\ell) \Rightarrow x \in A(a) \vee x \neq y.$$

(xxv)
$$\begin{aligned} x \in A(a) &\Leftrightarrow (a,x) \in c(\ell) \\ &\Rightarrow (\forall s \in S)((sa,x) \in c(\ell)) \quad (\text{by } (xvii)) \\ &\Leftrightarrow (\forall s \in S)(x \in A(sa)) \end{aligned}$$

(xxvi) Follows from *(xxv)*. □

Theorem 3.2 *Let a be an arbitrary element of a semigroup S with apartness. Then the set $A(a)$ is a maximal strongly extensional right consistent subset of S such that $a\#A(a)$.*

Proof. Let T be a strongly extensional right consistent subset of S such that $a\#T$. Let t be an arbitrary element of T. Then $t \neq sa \vee sa \in T$ for every $s \in S$. As $sa \in T$ implies $a \in T$, which is impossible, we have $t \neq sa$ for all $s \in S$. So, $t \in L_{(a)}$. Assume that $T \subseteq A_n(a)$ and let $t \in T$ be an arbitrary element of T and let z be an arbitrary element of S. Then $t \neq sz \vee sz \in T$ for all $s \in S$. Therefore, $t \neq sz (s \in S) \vee z \in T \subseteq A_n(a)$, i.e. $z \in B_1(t) \vee z \in A_n(a)$. Hence $S = B_1(t) \cup A_n(a)$, i.e. $t \in A_{n+1}(a)$ and finally, $T \subseteq A_{n+1}(a)$. By induction, we have $T \subseteq \bigcap_{n\in \mathbf{N}} A_n(a) = A(a)$. Hence, $A(a)$ is a maximal strongly extensional right consistent subset of S such that $a\#A(a)$. □

Symmetrically, we have

Theorem 3.3 *Let a be an element of a semigroup S. Then*

(xxvii) $B_1(a) = \{y \in S : \; b \in L_{(y)}\}$;

(xxviii) $B_{n+1}(a) \subseteq B_n(a)$;

(xxix) $B_{n+1}(a) = \{y \in S : \; S = B_n(a) \cup A_1(y)\}$;

(xxx) $B(a) = \bigcap_{n\in \mathbf{N}} B_n(a)$;

(xxxi) $b\#A(b)$;

(xxxii) $B(b)$ *is a strongly extensional left ideal of* S;

(xxxiii) $(\forall x \in S)(B(xb) \subseteq B(b))$;

(xxxiv) $(\forall n \in \mathbf{N})(B(b^n) \subseteq B(b))$.

Proof. *(xxvii)–(xxxi)* is analogous to *(xix)–(xxiii)* from Theorem 3.1.

$$\begin{aligned} xxxii \quad y \in B(b) &\Leftrightarrow b \in A(y) \\ &\Rightarrow (\forall x \in S)(b \in A(xy)) \quad \text{(by } (xxv)) \\ &\Leftrightarrow (\forall x \in S)(xy \in B(b)). \end{aligned}$$

Let x be an arbitrary element of S and let $y \in B(a)$. Then

$$(y, a) \in c(\ell) \vee (x, a) \in c(\ell) \Rightarrow y \neq x \vee x \in B(a).$$

$$
\begin{aligned}
\textit{(xxxiii)} \quad y \in B(xb) \quad &\Leftrightarrow \quad xb \in A(y) \\
&\Rightarrow \quad b \in A(y) \quad \text{(by } \textit{(xxiv)}\text{)} \\
&\Leftrightarrow \quad y \in B(b).
\end{aligned}
$$

(xxxiv) Follows immediately from *(xxxiii)*. □

Theorem 3.4 *Let a be an arbitrary element of a semigroup S with apartness. Then the set $B(a)$ is a maximal strongly extensional left ideal of S such that $a\#B(a)$.*

Proof. Let J be a strongly extensional left ideal of S such that $a\#J$. Then $a\#SJ$, i.e. $J \subseteq B_1(a)$. Assume that $J \subseteq B_n(a)$. Let z be an arbitrary element of S and let $y \in J$. Then $z \neq sy \vee z \in J$ for all $s \in S$, because the left ideal J is a strongly extensional subset of S. So, $z \in A_1(y) \vee z \in B_n(a)$. This means that $J \subseteq B_{n+1}(a)$. By induction we have $J \subseteq B(a)$. Therefore, $B(a)$ is a maximal strongly extensional left ideal of S such that $a\#B(a)$. □

Theorem 3.5 *Let S be a semigroup with apartness and let a be an element of S. The left cocongruence q has the set $Z(a) = A(a) \cup B(a)$ as the q-class of a.*

Proof.
$$
\begin{aligned}
x \in Z(a) \quad &\Leftrightarrow \quad (a,x) \in q \\
&\Leftrightarrow \quad (a,x) \in c(\ell) \vee (x,a) \in c(\ell) \\
&\Leftrightarrow \quad x \in A(a) \vee x \in B(a) \\
&\Leftrightarrow \quad x \in A(a) \cup B(a). \quad \square
\end{aligned}
$$

4. Some properties of classes $A(a)$ and $B(a)$

In the following two theorems we are going to describe some properties of the left and right classes of the quasi-antiorder $c(\ell)$. In the first one we describe some conditions under which the right consistent subset $A(a)$ (left ideal $B(a)$) is a semifilter (completely semiprime ideal) for all $a \in S$. In the second theorem, we describe conditions which ensure that the right consistent subset $A(a)$ (left ideal $B(a)$) is a consistent subset (ideal) of S for all $a \in S$.

Theorem 4.1 *Let S be a semigroup with apartness. Then the following conditions on S are equivalent:*

(1) $(\forall a \in S)(A(a) = A(a^2))$;

(2) $(\forall b \in S)(B(b) = B(b^2))$;

(3) *$A(a)$ is a left semifilter of S for each $a \in S$;*

(4) *$B(b)$ is a completely semiprime left ideal of S for each $B \in S$;*

Proof. (1)⇒(4) $y^2 \in B(b) \Leftrightarrow b \in A(y^2) = A(y) \Leftrightarrow y \in B(b)$.

(4)⇒(1) $x \in A(a^2) \Leftrightarrow a^2 \in B(x) \Rightarrow a \in B(x) \Leftrightarrow x \in A(a)$, because $B(x)$ is a completely semiprime left ideal of S.

(2)⇒(3) $x \in A(a) \Leftrightarrow a \in B(x) = B(x^2) \Leftrightarrow x^2 \in A(a)$.

(3)⇒(2) $y \in B(b) \Leftrightarrow b \in A(y) \Rightarrow b^2 \in A(y) \Leftrightarrow y \in B(b^2)$, because $A(y)$ is a left semifilter of S.

(1)⇒(3)
$$\begin{aligned} x \in A(a) &\Leftrightarrow (a, x) \in c(\ell) \\ &\Rightarrow (a, x^2) \in c(\ell) \vee (x^2, x) \in c(\ell) \\ &\Leftrightarrow x^2 \in A(a) \vee x \in A(x^2) = A(x) \\ &\Rightarrow x^2 \in A(a) \quad \text{(by } (xxiii)\text{)}. \end{aligned}$$

(2)⇒(4)
$$\begin{aligned} y^2 \in B(b) &\Leftrightarrow (y^2, b) \in c(\ell) \\ &\Rightarrow (y^2, y) \in c(\ell) \vee (y, b) \in c(\ell) \\ &\Leftrightarrow y^2 \in B(y) = B(y^2) \vee y \in B(b) \\ &\Rightarrow y \in B(b) \quad \text{(by } (xxx)\text{)}. \quad \square \end{aligned}$$

Theorem 4.2 *The following conditions on a semigroup S are equivalent:*

(1) $(\forall a, b \in S)((a, ab)\#c(\ell))$;

(2) $(\forall a, b \in S)(B(ab) \subseteq B(a) \cap B(b))$;

(3) *for all $b \in S$, $B(b)$ is an ideal of S;*

(4) $(\forall a, b \in S)(A(a) \cup A(b) \subseteq A(ab))$;

(5) *for all $a \in S$, $A(a)$ is a consistent subset of S.*

Proof. (1)⇒(2)

$$\begin{aligned} y \in B(ab) &\Leftrightarrow (y, ab) \in c(\ell) \\ &\Rightarrow ((y, a) \in c(\ell) \vee (a, ab) \in c(\ell)) \wedge \\ &\qquad \wedge((y, b) \in c(\ell) \vee (b, ab) \in c(\ell)) \\ &\Rightarrow (y, a) \in c(\ell) \wedge (y, b) \in c(\ell) \quad \text{(by (1) and } (xi)\text{)} \\ &\Leftrightarrow y \in B(a) \wedge y \in B(b) \\ &\Leftrightarrow y \in B(a) \cap B(b). \end{aligned}$$

$$\begin{aligned}(1)\Rightarrow(3)\quad y \in B(b) &\Leftrightarrow (y,b) \in c(\ell)\\ &\Rightarrow (y,yx) \in c(\ell) \vee (yx,b) \in c(\ell)\\ &\Rightarrow (yx,b) \in c(\ell)\\ &\Leftrightarrow yx \in B(b).\end{aligned}$$

So, the left ideal $B(b)$ is an ideal of S.

(2)⇒(1) Let (u,v) be an arbitrary element of $c(\ell)$ and let $a,b \in S$. Then we have

$$\begin{aligned}(u,v) \in c(\ell) &\Rightarrow (u,a) \in c(\ell) \vee (a,ab) \in c(\ell) \vee (ab,v) \in c(\ell)\\ &\Rightarrow u \neq a \vee ab \neq v\\ &\Leftrightarrow (u,v) \neq (a,ab).\end{aligned}$$

$$\begin{aligned}(1)\Rightarrow(4)\quad x \in A(a) &\Leftrightarrow (a,x) \in c(\ell)\\ &\Rightarrow (a,ab) \in c(\ell) \vee (ab,x) \in c(\ell)\\ &\Rightarrow (ab,x) \in c(\ell)\\ &\Leftrightarrow x \in A(ab).\end{aligned}$$

Analogously, $x \in A(b) \Rightarrow x \in A(ab)$. Thus, $A(a) \cup A(b) \subseteq A(ab)$.

(4)⇒(1) Let (u,v) be an arbitrary element of $c(\ell)$ and let $a,b \in S$. Then

$$\begin{aligned}(u,v) \in c(\ell) &\Rightarrow (u,a) \in c(\ell) \vee (a,ab) \in c(\ell) \vee (ab,v) \in c(\ell)\\ &\Rightarrow u \neq a \vee ab \in A(a) \subseteq A(ab) \vee ab \neq v\\ &\Rightarrow (u,v) \neq (a,ab).\end{aligned}$$

$$\begin{aligned}(2)\Rightarrow(5)\quad xy \in A(a) &\Leftrightarrow a \in B(xy) \subseteq B(x) \cap B(y)\\ &\Rightarrow a \in B(x) \wedge a \in B(y)\\ &\Leftrightarrow x \in A(a) \wedge y \in A(a).\end{aligned}$$

$$\begin{aligned}(5)\Rightarrow(2)\quad y \in B(ab) &\Leftrightarrow ab \in A(y)\\ &\Rightarrow a \in A(y) \wedge b \in A(y)\\ &\Leftrightarrow y \in B(a) \wedge y \in B(b)\\ &\Leftrightarrow y \in B(a) \cap B(b).\end{aligned}$$

(3)⇒(1) Let (u,v) be an arbitrary element of $c(\ell)$ and let $a,b \in S$. Then

$$\begin{aligned}(u,v) \in c(\ell) &\Rightarrow (u,a) \in c(\ell) \vee (a,ab) \in c(\ell) \vee (ab,v) \in c(\ell)\\ &\Rightarrow u \neq a \vee ab \in A(a) \subseteq A(a) \cup A(b) \subseteq A(ab) \vee ab \neq v\\ &\Rightarrow (u,v) \neq (a,ab). \quad \square\end{aligned}$$

Now we describe the situation in which the equality $A(ab) = A(a) \cup A(b)$ holds in S.

Theorem 4.3 *The following conditions on a semigroup S are equivalent:*

(1) $(\forall a, b \in S)(A(ab) = A(a) \cup A(b))$;

(2) for all $y \in S$, the set $B(y)$ is a completely prime ideal of S.

Proof. (2)⇒(1) Let $B(y)$ be a completely prime ideal of S for all $y \in S$. Then by the above theorem we have $A(ab) \supseteq A(a) \cup A(b)$ for all $a, b \in S$. Further, we have

$$\begin{aligned} x \in A(ab) &\Leftrightarrow ab \in B(x) \\ &\Rightarrow a \in B(x) \vee b \in B(x) \\ &\Leftrightarrow x \in A(a) \vee x \in A(b) \\ &\Leftrightarrow x \in A(a) \cup A(b). \end{aligned}$$

(1)⇒(2) Assume that (1) holds. By the above theorem, $B(b)$ is then an ideal of S for all $b \in S$. Further, we have

$$\begin{aligned} xy \in B(b) &\Leftrightarrow ab \in B(x) \\ &\Rightarrow b \in A(x) \vee b \in A(y) \\ &\Leftrightarrow x \in B(b) \vee y \in B(b). \quad \square \end{aligned}$$

Corollary 4.4 *Assume that S satisfies the equivalent conditions of the above theorem. Then $A(a)$ is a semifilter of S for all $a \in S$.*

Similarly as Theorem 4.3 we prove the assertion below, which characterizes semigroups in which $A(a)$ is a filter for all $a \in S$.

Theorem 4.5 *The following conditions on a semigroup S are equivalent:*

(1) $(\forall a, b \in S)(B(ab) = B(a) \cap B(b))$;

(2) for all $a \in S$, the set $A(a)$ is a filter of S.

Corollary 4.6 *Assume that S satisfies the equivalent conditions of the above theorem. Then $B(b)$ is a completely semiprime ideal of S for all $b \in S$.*

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