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COMPLETENESS, FUNCTIONAL COMPLETENESS AND RELATIVE COMPLETENESS

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Abstract

In this brief survey we give an overview of questions which concern maximal clones on finite sets containing a given clone, classifications of functions of k-valued logics according to their membership in these clones, the number of functionally complete algebras, and other related problems.

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1. Notation and Preliminaries

For positive integers k and n, let $E_k = \{0, 1, \ldots, k-1\}$, $P_k^{(n)}$ the set of all maps $E_k^n \to E_k$, and

$$P_k = \bigcup_{n \in \mathbf{N}} P_k^{(n)}.$$

For $1 \le i \le n$ we define the *i-th n-ary projection* $p_i^n \in P_k^{(n)}$ by setting $p_i^n(a_1, \ldots, a_n) := a_i$ for all $(a_1, \ldots, a_n) \in E_k$. Let Π_k denote the set of all projections over E_k. We say that $f \in P_k^{(n)}$ is essential if it depends on at least two variables and it takes all the values from E_k. For $n, m \geq 1, f \in P_k^{(n)}$ and $g_1, \ldots, g_n \in P_k^{(m)}$, *the superposition of* f and $g_1, \ldots, g_n$, denoted by $f(g_1, \ldots, g_n)$, is defined by

$$f(g_1, \ldots, g_n)(a_1, \ldots, a_n) = f(g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n)),$$

for all $(a_1, \ldots, a_n) \in E_k^m$. A set $F \subseteq P_k$ is a *clone of operations on* E_k (or *clone* for short) if $\Pi_k \subseteq F$ and F is closed with respect to superposition. For a subset F of P_k, $\langle F \rangle_{\mathrm{CL}}$ stands for the clone generated by F. We say that the clone F is *maximal* if there is no clone G such that $F \subset G \subset P_k$.

A subset F of P_k is *complete* and the algebra $\mathcal{A} = (E_k; F)$ is *primal* if

$$\langle F \rangle_{\mathrm{CL}} = P_k.$$

A subset F of P_k is *functionally complete*, or complete with constants and the algebra $\mathcal{A} = (E_k; F)$ is *functionally complete* if

$$\langle F \cup \{c_a : a \in E_k\} \rangle_{\mathrm{CL}} = P_k,$$

where c_a stands for the unary constant operation with the value a.

Let $\varrho \subseteq E_k^h$ be an h-ary relation and $f \in P_k^{(n)}$. We say that f *preserves* ϱ if for all h-tuples $(a_{11}, \ldots, a_{1h}), \ldots, (a_{n1}, \ldots, a_{nh})$ from ϱ we have

$$(f(a_{11}, \ldots, a_{n1}), \ldots, f(a_{1h}, \ldots, a_{nh})) \in \varrho.$$

$\mathrm{Pol}\,\varrho$ is the set of all $f \in P_k$ which preserve ϱ. For $F \subseteq P_k$, $\mathrm{Inv}\,F$ denotes the set of all the relations preserved by each $f \in F$.

Let C be a clone on E_k and $F \subseteq P_k$. F is *complete relative to* C (or C*-complete*) if $\langle F \cup C \rangle_{\mathrm{CL}} = P_k$.

Thus, the relative completeness is a generalization of the weak completeness, introduced in [14]. On the other hand, the relative completeness is a generalization of the usual completeness, because every complete set F is Π_k-complete.

A relative maximal clone [17] with respect to clone C is the maximal clone which contains C.

A relative complete set F in P_k is called a *relative base of* P_k *with respect to* C if no proper subset of F is relatively complete in P_k with respect to C.

The following theorem gives a necessary and sufficient condition for F to be C-complete. It is analogous to Post's completeness criterion.

Theorem 1.1. [17] *Let C be a clone and $\{M_1, \ldots, M_s\}$ be the set of all the maximal clones containing C:*

$$\{M_1, \ldots, M_s\} = \{D \in \mathcal{L}_k : D \text{ is maximal and } D \supseteq C\}.$$

$F \subseteq P_k$ is complete relative to C iff $F \setminus M_i \neq \emptyset$ for all $i \in \{1, \ldots, s\}$.

Therefore, the problem of determining relative completeness of a set F reduces to determining all the maximal clones that contain C.

The problem of relative completeness is studied also in [3, 11, 14, 16, 17].

In what follows, we give Rosenberg's characterization of maximal clones. First we introduce some special sets of relations:

R_1 – the set of all bounded partial orders on E_k;

R_2 – the set of selfdual relations, i.e. relations of the form

$$\{(x, s(x)) : x \in E_k\},$$

where s is a fixed point free permutation of prime order (i.e. $s^p = \mathrm{id}$ for some prime p);

R_3 – the set of affine relations, i.e. relations of the form $\{(a, b, c, d) \in E_k^4 : a * b = c * d\}$, where $(E_k, *)$ is a p-elementary Abelian group (p prime);

R_4 – the set of all nontrivial equivalence relations on E_k;

R_5 – the set of all central relations on E_k;

R_6 – the set of all h-regular relations on E_k ($h \geq 3$).

Theorem 1.2. [12] *A clone M is maximal iff there is a $\varrho \in R_1 \cup \ldots \cup R_6$ such that $M = \mathrm{Pol}\, \varrho$.*

2. Relative Completeness With Respect to Transpositions

Consider the following transpositions (see [7, Theorem 8, p.54]) on E_k:

$$g_i(x) = \begin{cases} i, & x = 0 \\ 0, & x = i \\ x, & \text{otherwise.} \end{cases}$$

and the clone generated by them:

$$C = \langle \bigcup_{i=1}^{k-1} g_i \rangle_{\mathrm{CL}}.$$

There are two relative maximal clones with respect to C if $k = 3$, 4 and there is one relative maximal clone with respect to C if $k > 4$, contrary to the completeness which gives an increasing number of maximal clones if k increase.

Theorem 2.1. [19]

(a) If $k > 4$ then there is exactly 1 relative maximal clone with respect to C.

(b) If $k \in \{3,4\}$ then there are exactly 2 relative maximal clones with respect to C.

Corollary 2.2. *If $k > 4$ then F is relatively complete with respect to C iff it contains an essential function.*

3. Relative Completeness With Respect to Two Negations

Consider the following two negations:

$$\begin{aligned} x^{\sim} &= x+1 \pmod{k} \\ x^{(k-1)} &= \begin{cases} k-1, & x = k-1 \\ 0, & x \neq k-1 \end{cases} \end{aligned}$$

and the clone generated by these negations:

$$U = \langle x^{\sim}, x^{(k-1)} \rangle_{\mathrm{CL}}.$$

Theorem 3.1. [5, 17] *If $k \leq 15$ then there exist exactly*

$$\mathrm{card}(\{h : h|k\} \setminus \{1,2\})$$

maximal clones which contain the clone U.

It is proved in [17] for $k \leq 8$ and in [5] for $k \leq 15$. We conclude with an open problem that could clarify the obscurity of the case R_6.

Problem 1. *Is there a $\varrho \in R_6$ such that $\mathrm{ar}(\varrho) > 1$ and $x^{\sim}$ preserves ϱ?*

4. Relative Completeness With Respect to Minimum and Complement

Let $K = \langle \min, \bar{x} \rangle_{\text{CL}}$, where $\bar{x} = (k-1) - x$.

Theorem 4.1. [17] *There exist exactly* $2^{\lfloor \frac{k}{2} \rfloor} + 2^{\lceil \frac{k}{2} \rceil} + 2^{\lfloor \frac{k-1}{2} \rfloor} - 5$ *maximal clones which contain the clone* K.

The functions from P_k may be partitioned into nonempty equivalence classes according to their membership in the relative maximal clones. It enables us to discuss the completeness properties in terms of these classes instead of individual functions: if a set is complete, then replacing a function in the set by any function in the corresponding equivalence class yields another complete set.

Let $M_1, ..., M_m$ be relative maximal clones. Define the map $\phi : P_k \to \{0,1\}^m$ by setting $\phi(f) := a_1...a_m$ where $a_i = 0$ if $f \in M_i$ and $a_i = 1$ if $f \notin M_i$. We call $\phi(f)$ *the characteristic vector of f*. We put $f \varrho g$ if $f, g \in P_k$ have the same characteristic vector, i.e. if $\phi(f) = \phi(g)$. It means that for all $j \in \{1, 2, ..., n\}$, either $\{f, g\} \subset M_j$ or $\{f, g\} \cap M_j = \emptyset$. Clearly, ϱ is an equivalence relation on P_k and so it partitions P_k into pairwise disjoint nonempty sets called *(equivalence) classes.*

To $a_1, ..., a_m \in \{0,1\}^m$ we associate $A = \{i : a_i = 1\}$ and if $A_1, ..., A_l$ are the subsets of $\{1, ..., m\}$ corresponding to the characteristic vectors, the relative completeness problem is reduced to the listing of subsets of $\{A_1, ..., A_l\}$ covering $\{1, ..., m\}$.

All maximal clones in P_k containing the functions *min* and *complement* are determined in [17].

The classes of functions according to their membership in the relative maximal clones with respect to clone K and classes of relative bases for the cases $k = 3$ and $k = 4$ are determined in [18].

The following results were proved in [18].

Theorem 4.2. P_3 *has* 3 *relative maximal clones with respect to clone* K*:* $T_1 = Pol(1), T_{02} = Pol(02)$ *and* $B_1 = Pol(E_3^2 - P_{02})$.

The classes of functions in P_3 are determined in [15]. The number of different classes is 406. Among 406 classes of functions of three-valued logic there are 8 different classes according to their membership in the clones T_1, T_{02} and B_1.

Theorem 4.3. *P_3 has 8 nonempty classes of functions according to their membership in the relative maximal clones with respect to K.*

Theorem 4.4. *P_3 has 8 classes of relative bases with respect to clone K.*

Theorem 4.5. *P_4 has 5 relative maximal clones with respect to clone K.*

Theorem 4.6. *P_4 has 28 nonempty classes of functions according to their membership in the relative maximal clones with respect to K.*

Theorem 4.7. *P_4 has 255 classes of relative bases with respect to clone K.*

From [17] it follows:

Theorem 4.8. *P_5 has 11 relative maximal clones with respect to clone K:*

$$\begin{aligned}
Q_1 &= Pol\begin{pmatrix}012340134\\012341043\end{pmatrix},\\
Q_2 &= Pol\begin{pmatrix}01234121323\\01234213132\end{pmatrix},\\
Q_3 &= Pol(0134),\\
Q_4 &= Pol(04),\\
Q_5 &= Pol(123),\\
Q_6 &= Pol(13),\\
Q_7 &= Pol(024),\\
Q_8 &= Pol(2),\\
Q_9 &= Pol\begin{pmatrix}0123401020312131423243 4\\01234102030213141324243\end{pmatrix},\\
Q_{10} &= Pol\begin{pmatrix}01234010212132324 34\\01234102021313242 43\end{pmatrix},\\
Q_{11} &= Pol\begin{pmatrix}01234010212232434\\01234102021324243\end{pmatrix}.
\end{aligned}$$

From Theorems 1–14 of [6], for above relative maximal clones Q_i, we have

Theorem 4.9.

1. $Q_5Q_7 \subset Q_8$, $Q_3Q_5 \subset Q_6$, $Q_3Q_7 \subset Q_4$.
2. $\overline{Q}_9Q_{10} \subset Q_8$, $\overline{Q}_9Q_{11} \subset Q_8$, $\overline{Q}_{10}Q_{11} \subset Q_8$.
3. $Q_2Q_6 \subset Q_5$, $Q_2Q_8 \subset Q_5$, $Q_1Q_4 \subset Q_3$, $Q_1Q_6 \subset Q_3$.
4. $Q_1Q_7Q_9 \subset Q_{10}$, $Q_1Q_7Q_{10} \subset Q_{11}$.
5. $Q_7\overline{Q}_8\overline{Q}_{10} \subset \overline{Q}_{11}$, $Q_7\overline{Q}_8\overline{Q}_{11} \subset \overline{Q}_{10}$, $Q_7\overline{Q}_8\overline{Q}_9 \subset \overline{Q}_{10}$, $Q_7\overline{Q}_8\overline{Q}_9 \subset \overline{Q}_{11}$.
6. $Q_2\overline{Q}_5\overline{Q}_{11} \subset \overline{Q}_{10}$, $Q_2\overline{Q}_5\overline{Q}_9 \subset \overline{Q}_{10}$, $Q_2\overline{Q}_5\overline{Q}_9 \subset \overline{Q}_{11}$.
7. $Q_4Q_9\overline{Q}_{10} \subset \overline{Q}_{11}$
8. $Q_3Q_{10} \subset Q_9$, $Q_6Q_{10} \subset Q_9$, $Q_4Q_{10} \subset Q_{11}$.
9. $Q_1Q_2 \subset Q_9$.
10. $Q_2Q_{10} \subset Q_9$, $Q_2Q_{11} \subset Q_9$.
11. $Q_2Q_{11} \subset Q_{10}$.
12. $Q_1Q_2\overline{Q}_5 \subset Q_9$, $Q_1Q_2\overline{Q}_5 \subset Q_{10}$, $Q_1Q_2\overline{Q}_5 \subset Q_{11}$.
13. $Q_1\overline{Q}_5\overline{Q}_9 \subset \overline{Q}_{10}$, $Q_1\overline{Q}_5\overline{Q}_{11} \subset \overline{Q}_{10}$.
14. $Q_1Q_2\overline{Q}_5 \subset Q_3$.

Remark. It is important to remark that Theorem 4.9 is useful for classification of functions in P_5 (according to their membership in all maximal clones), which is still an open problem.

Theorem 4.10. *There are 391 nonempty classes of functions in P_5 with respect to the clone generated by $min(x, y)$ and $\overline{x} = 4 - x$.*

Problem 2. *Classify five-valued logic functions.*

5. The Number of Functionally Complete Algebras on the Three Element Set

Demetrovics and Hannák proved in [2] that there are continuum non-equivalent functionally complete algebras with the base set E_k, $k > 3$.

In the case $k = 2$ we have a complete description of all algebras (Post, [7]), and so we know that there are only finitely many non-equivalent functionally complete algebras with a two-element base set.

Theorem 5.1. [9] *There exists a continuum pairwise non-equivalent functionally complete algebras on* E_3.

Problem 3. *Determine the cardinality of the set of functionally complete algebras whose identities are not finitely based.*

References

[1] Burris, S. and Sankappanavar, H. P., A Course in Universal Algebra, Springer-Verlag, 1981.

[2] Demetrovics, J. and Hannák, L., On the number of functionally complete algebras, in: Proc. 12th Int. Symp. on Multiple-Valued Logic, pp. 329–330, IEEE, 1982.

[3] Demetrovics, J., Reischer, C., Simovici, D. and Stojmenović, I., Enumeration of functions and bases of three-valued set logic under compositions with Boolean functions, in: Proc. 24th Int. Symp. on Multiple-Valued Logic, pp. 164–171, IEEE, 1994.

[4] Doroslovački, R., Pantović, J. and Vojvodić, G., Classification of maps by their membership on maximal clones that contain minimum and complement, Matematički Vesnik (to appear).

[5] Doroslovački, R., Pantović, J., Tošić, R. and Vojvodić, G., Some remarks on relative completeness with respect to two negations, in: Proc. XII Conf. on Appl. Math. PRIM '97. (Palić, 1997), pp. 27–32, Novi Sad, 1998.

[6] Doroslovački, R., Pantović, J. and Vojvodić, G., Note on intersections of maximal clones, Novi Sad J. Math. (to appear).

[7] Jablonskiĭ, S. V., Introduction to Discrete Mathematics (in Russian), Nauka, Moskva, 1979.

[8] Ngom, A., Reischer, C., Simovici, D., Stojmenović, I., Set-valued logic algebra: a carrier computing foundation, Multiple Valued Logic, 2 (1996), 183–216.

[9] Pantović, J., Tošić, R. and Vojvodić, G., The cardinality of functionally complete algebras on a three element set, Algebra Universalis 38 (1997), 136–140.

[10] Pantović, J., Tošić, R. and Vojvodić, G., Relative completeness with respect to two unary functions, unpublished manuscript.

[11] Pogosyan, G., Nozaki, A., Miyakawa, M. and Rosenberg, I. G., Hereditary clones of multiple valued logic algebra, in: Proc. 24th Int. Symp. on Multiple-Valued Logic, pp. 306–313, IEEE, 1994.

[12] Rosenberg, I. G., Completeness properties of multiple-valued logic algebra, in: ed. D. C. Rhine, "Computer Science and Multiple-Valued Logic: Theory and Application", pp. 144–186, North-Holland, 1977.

[13] Rosenberg, I. G. and Simovici, D., Algebraic aspects of multiple-valued logic, in: Proc. 28th Int. Symp. on Multiple-Valued Logic, pp. 266–276, IEEE, 1998.

[14] Simovici, D., Stojmenović, I. and Tošić, R., Functional completeness and weak completeness in set logic, in: Proc. 23rd Int. Symp. on Multiple-Valued Logic, pp. 251–256, IEEE, 1993.

[15] Stojmenović, I., Classification of P_3 and the enumeration of bases of P_3, Rev. Research Fac. Sci. Novi Sad, Ser. Math. 14 (1) (1984), 73–80.

[16] Stojmenović, I., Completeness criteria in many-valued set logic under compositions with Boolean functions, in: Proc. 24th Int. Symp. on Multiple-Valued Logic, pp. 177–183, IEEE, 1994.

[17] Tošić, R., Vojvodić, G., Mašulović, D., Doroslovački, R. and Rosić, J., Two examples of relative completness, Multiple Valued Logic 2 (1996), 67–78.

[18] Tošić, R., Vojvodić, G. and Rosić, J., Some problems on relative completeness in k-valued logic, in: Proc. Conf. LIRA '95., Novi Sad, 1995.

[19] Vojvodić, G., Pantović, J. and Tošić, R., Relative completeness with respect to transpositions, submitted for publication.

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