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INFLATION OF t-SEMIRINGS

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Abstract. Inflations of some classes of algebras are studied in [2,3,4]. In the present paper the construction for an n-inflation of the t-semirings class is given. In particular, for $t = 1$ the studied class is the class of semirings.

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1. A nonempty set S with two binary associative operations "+" and " $\cdot$ " for which the distributive law holds:

$$x \cdot (y+z) = xy + xz, \quad (x+y) \cdot z = xz + yz$$

for every $x, y, z \in S$, is a semiring. A subsemiring T of a semiring S is an ideal of S if T is an additive and multiplicative ideal of S. The semiring T is with a zero element if for every $a \in T$:

$$a + 0 = 0 + a = 0 = a \cdot 0 = 0 \cdot a.$$

Let T and K be two disjoint semirings and let K have a zero element. A semiring S is an ideal extension of a semiring T by a semiring K if T is an ideal of S and the Rees factor semiring S/T is isomorphic to K. Let S be an extension of T. Then S is a retract extension if there exists a homomorphism φ of S onto T and $\varphi(x) = x$ for every $x \in T$. In this case we call φ a retraction.

Proposition 1.1. [2] *Let* $(T, +, \cdot)$ *be a semiring. With each* $a \in T$ *we associate a set* Y_a *such that*

$$a \in Y_a, \quad Y_a \cap Y_b = \emptyset \quad if \quad a \neq b. \tag{1.1}$$

Let

$$\varphi^{(a_1,a_2)} \; : \; Y_{a_1} \times Y_{a_2} \longrightarrow Y_{a_1+a_2},$$
$$\psi^{(a_1,a_2)} \; : \; Y_{a_1} \times Y_{a_2} \longrightarrow Y_{a_1 \cdot a_2},$$

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be functions for which

$$\begin{aligned} \varphi^{(a_1,a_2)}(x_1,a_2) &= \varphi^{(a_1,a_2)}(a_1,x_2) = a_1+a_2, \\ \psi^{(a_1,a_2)}(x_1,a_2) &= \psi^{(a_1,a_2)}(a_1,x_2) = a_1\cdot a_2 \end{aligned} \tag{1.2}$$

$$\begin{aligned} &\varphi^{(a_1+a_2,a_3)}\left(\varphi^{(a_1,a_2)}(x_1,x_2),x_3\right) = \varphi^{(a_1,a_2+a_3)}\left(x_1,\varphi^{(a_2,a_3)}(x_2,x_3)\right), \\ &\psi^{(a_1\cdot a_2,a_3)}\left(\psi^{(a_1,a_2)}(x_1,x_2),x_3\right) = \psi^{(a_1,a_2\cdot a_3)}\left(x_1,\psi^{(a_2,a_3)}(x_2,x_3)\right), \end{aligned} \tag{1.3}$$

$$\begin{aligned} \psi^{(a_1,a_2+a_3)}\left(x_1,\varphi^{(a_2,a_3)}(x_2,x_3)\right) &= \varphi^{(a_1\cdot a_2,a_1\cdot a_3)}\left(\psi^{(a_1,a_2)}(x_1,x_2),\psi^{(a_1,a_3)}(x_1,x_3)\right), \\ \psi^{(a_1+a_2,a_3)}\left(\varphi^{(a_1,a_2)}(x_1,x_2),x_3\right) &= \varphi^{(a_1\cdot a_3,a_2\cdot a_3)}\left(\psi^{(a_1,a_3)}(x_1,x_3),\psi^{(a_2,a_3)}(x_2,x_3)\right). \end{aligned} \tag{1.4}$$

On the set $S=\bigcup_{a\in T} Y_a$ *we define the operations "$*$" and "$\circ$" by:*

$$\begin{aligned} x_1 * x_2 &= \varphi^{(a_1,a_2)}(x_1,x_2) \\ x_1 \circ x_2 &= \psi^{(a_1,a_2)}(x_1,x_2) \end{aligned}$$

if $x_1\in Y_{a_1}, x_2\in Y_{a_2}$.
Then $(S,*,\circ)$ *is a semiring and* $(S,*,\circ)$ *is a retract extension of* $(T,+,\cdot)$.

Conversely, every retract extension S *of a semiring* T *can be so constructed.*

2. The next lemma enables us to introduce the notion of an n-inflation of a semiring. Let us denote $max(i,j)+1$ by $m(i,j)$ i.e. $m(i,j)=max(i,j)+1$, $(i,j\in N)$.

Lemma 2.1. *Let* T *be a semiring. With each* $a\in T$ *we associate a family of sets* X_i^a, $(i=1,2,\dots,n)$ *where* $a\in X_r^a$ *for some* $1\le r\le n$, *such that:*

$$\begin{aligned} X_i^a\cap X_j^a &= \emptyset \quad if \quad i\neq j, \\ X_i^a\cap X_j^b &= \emptyset \quad if \quad a\neq b, \end{aligned} \tag{2.1}$$

and let

$$\varphi_{(i_1,i_2)}^{(a_1,a_2)}: X_{i_1}^{a_1}\times X_{i_2}^{a_2}\longrightarrow \bigcup_{\nu=m(i_1,i_2)}^{n} X_\nu^{a_1+a_2} \quad if \quad m(i_1,i_2)\le n,$$

$$\varphi_{(i_1,i_2)}^{(a_1,a_2)}(x_1,x_2)=a_1+a_2 \quad if \quad m(i_1,i_2)=n+1,$$

$$\varphi^{(a_1,a_2)}_{(i_1,i_2)}(a_1,x_2)=\varphi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,a_2)=a_1+a_2,$$

$$(2.2)\qquad \psi^{(a_1,a_2)}_{(i_1,i_2)}:X^{a_1}_{i_1}\times X^{a_2}_{i_2}\longrightarrow \bigcup_{\nu=m(i_1,i_2)}^{n} X^{a_1\cdot a_2}_{\nu}\quad if\quad m(i_1,i_2)\le n,$$

$$\psi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,x_2)=a_1\cdot a_2\quad if\quad m(i_1,i_2)=n+1,$$

$$\psi^{(a_1,a_2)}_{(i_1,i_2)}(a_1,x_2)=\psi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,a_2)=a_1\cdot a_2$$

be the functions for which:

$$(\forall s\ge m(i_1,i_2))(\forall t\ge m(i_2,i_3))\varphi^{(a_1+a_2,a_3)}_{(s,i_3)}\left(\varphi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,x_2),x_3\right)=\varphi^{(a_1,a_2+a_3)}_{(i_1,t)}\left(x_1,\varphi^{(a_2,a_3)}_{(i_2,i_3)}(x_2,x_3)\right),$$

$$(2.3)\qquad (\forall s\ge m(i_1,i_2))(\forall t\ge m(i_2,i_3))\psi^{(a_1\cdot a_2,a_3)}_{(s,i_3)}\left(\psi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,x_2),x_3\right)=\psi^{(a_1,a_2\cdot a_3)}_{(i_1,t)}\left(x_1,\psi^{(a_2,a_3)}_{(i_2,i_3)}(x_2,x_3)\right)$$

for all $a_1,a_2,a_3\in T$, *where* $m(i_1,i_2)\le n$ *and* $m(i_2,i_3)\le n$,

$$(\forall s\ge m(i_1,i_2))(\forall t\ge m(i_2,i_3))(\forall l\ge m(i_1,i_3))\psi^{(a_1,a_2+a_3)}_{(i_1,t)}\left(x_1,\varphi^{(a_2,a_3)}_{(i_2,i_3)}(x_2,x_3)\right)=\varphi^{(a_1a_2,a_1a_3)}_{(s,l)}\left(\psi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,x_2),\psi^{(a_1,a_3)}_{(i_1,i_3)}(x_1,x_3)\right),$$

$$(2.4)\qquad (\forall s\ge m(i_1,i_2))(\forall t\ge m(i_2,i_3))(\forall l\ge m(i_1,i_3))\psi^{(a_1+a_2,a_3)}_{(s,i_3)}\left(\varphi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,x_2),x_3\right)=\varphi^{(a_1a_3,a_2a_3)}_{(l,t)}\left(\psi^{(a_1,a_3)}_{(i_1,i_3)}(x_1,x_3),\psi^{(a_2,a_3)}_{(i_2,i_3)}(x_2,x_3)\right)$$

for all $a_1,a_2,a_3\in T$, *where* $m(i_1,i_2)\le n$, $m(i_2,i_3)\le n$ *and* $m(i_1,i_3)\le n$.
Let $Y_a=\bigcup_{i=1}^{n}X^a_i$ *and define the operations "$*$" and "$\circ$" on* $S=\bigcup_{a\in T}Y_a$ *by:*
$x_1\in Y_{a_1}$,
$x_2\in Y_{a_2}$,

$$x_1 * x_2 = \varphi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,x_2)$$
$$x_1\circ x_2 = \psi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,x_2)$$

if $x_1\in X^{a_1}_{i_1}$, $x_2\in X^{a_2}_{i_2}$, $1\le i_1,i_2\le n$.
Then $(S,*,\circ)$ *is a semiring.*

Proof. By Lemma 2.1. [1], $(S,*)$ and $(S,\circ)$ are semigroups. Let $m(i_1,i_2)\le n$, $m(i_2,i_3)\le n$ and $m(i_1,i_3)\le n$. Then

$$\begin{aligned} x_1\circ(x_2*x_3) &= x_1\circ\varphi^{(a_2,a_3)}_{(i_2,i_3)}(x_2,x_3)\\ &= \psi^{(a_1,a_2+a_3)}_{(i_1,t_1)}\left(x_1,\varphi^{(a_2,a_3)}_{(i_2,i_3)}(x_2,x_3)\right),\end{aligned}$$

where $\varphi^{(a_2,a_3)}_{(i_2,i_3)}(x_2,x_3)\in X^{a_2+a_3}_{t_1}$, $m(i_2,i_3)\le t_1\le n$,

$$\begin{aligned} (x_1\circ x_2)*(x_1\circ x_3) &= \psi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,x_2)*\psi^{(a_1,a_3)}_{(i_1,i_3)}(x_1,x_3)\\ &= \varphi^{(a_1a_2,a_1a_3)}_{(s_2,l)}\left(\psi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,x_2),\psi^{(a_1,a_3)}_{(i_1,i_3)}(x_1,x_3)\right),\end{aligned}$$

where $\psi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,x_2)\in X^{a_1a_2}_{s_2}$, $\psi^{(a_1,a_3)}_{(i_1,i_3)}(x_1,x_3)\in X^{a_1a_3}_{l}$ $m(i_1,i_2)\le s_2\le n$, $m(i_1,i_3)\le l\le n$,

$$\begin{aligned} (x_1*x_2)\circ x_3 &= \varphi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,x_2)\circ x_3\\ &= \psi^{(a_1+a_2,a_3)}_{(s_1,i_3)}\left(\varphi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,x_2),x_3\right),\end{aligned}$$

where $\varphi^{(a_1,a_2)}_{(i_1,i_2)}(x_1,x_2)\in X^{a_1+a_2}_{s_1}$, $m(i_1,i_2)\le s_1\le n$,

$$\begin{aligned} (x_1\circ x_3)*(x_2\circ x_3) &= \psi^{(a_1,a_3)}_{(i_1,i_3)}(x_1,x_3)*\psi^{(a_2,a_3)}_{(i_2,i_3)}(x_2,x_3)\\ &= \varphi^{(a_1a_3,a_2a_3)}_{(l,t_2)}\left(\psi^{(a_1,a_3)}_{(i_1,i_3)}(x_1,x_3),\psi^{(a_2,a_3)}_{(i_2,i_3)}(x_2,x_3)\right),\end{aligned}$$

where $\psi^{(a_1,a_3)}_{(i_1,i_3)}(x_1,x_3)\in X^{a_1a_3}_{l}$ $,\psi^{(a_2,a_3)}_{(i_2,i_3)}(x_2,x_3)\in X^{a_2a_3}_{t_2}$, $m(i_1,i_3)\le l\le n$, $m(i_2,i_3)\le t_2\le n$, and by (2.4) the distributive law of the operation "$\circ$" to the operation "$*$" holds. In other cases the distributive law reduces to the distributive law for the operation "$\cdot$" to "$+$". Therefore, $(S,*,\circ)$ is a semiring.

Definition 2.1. *The semiring S constructed in Lemma 2.1. is called an n-inflation of the semiring T.*

Definition 2.2. *Let $(S,+,\cdot)$ be a semiring and $t\in N$. $(S,+,\cdot)$ is called a t-semiring if for all i_1,i_2 for which $m(i_1,i_2)\le t+1$ it holds:*

$$(i_1S\cup S^{i_1})+(i_2S\cup S^{i_2})\subset m(i_1,i_2)S\cup S^{m(i_1,i_2)},$$

$$(t)\qquad (i_1S\cup S^{i_1})\cdot(i_2S\cup S^{i_2})\subset m(i_1,i_2)S\cup S^{m(i_1,i_2)}.$$

For the class of t-semirings we consider the n-inflations where $n\le t$.

Theorem 2.1. *A t-semiring S is an n-inflation of a semiring $(T,+,\cdot)$ if and only if $(n+1)S \subset T$, $S^{n+1} \subset T$ and S is a retract extension of T.*

Proof. Let $(S,*,\circ)$ be an n-inflation of a semiring $(T,+,\cdot)$. Then, by (2.2). T is an ideal of S. As the semigroup $(S,*)$ is the n-inflation of the semigroup $(T,+)$ and the semigroup $(S,\circ)$ is the n-inflation of the semigroup $(T,\cdot)$ then by [1] $(n+1)S \subset T$ and $S^{n+1} \subset T$. Define the function $\varphi : S = \bigcup_{a\in T} Y_a \longrightarrow T$ by $\varphi(Y_a) = a$. According to [1] for all $x,y \in S$ we have: $\varphi(x*y) = \varphi(x) + \varphi(y)$, $\varphi(x\circ y) = \varphi(x)\cdot\varphi(y)$. It is clear that $\varphi(x) = x$ for every $x \in T$. Therefore S is a retract extension of T.

Conversely, let n be the smallest positive integer such that $(n+1)S \subset T$, $S^{n+1} \subset T$ and let φ be a retraction of S onto T. An arbitrary $a \in T$ belongs to one of the following sets: $S\backslash(2S\cup S^2)$, $(2S\cup S^2)\backslash(3S\cup S^3), \ldots, ((n-1)S\cup S^{n-1})\backslash(nS\cup S^n)$, $nS\cup S^n$. If $Y_a = \varphi^{-1}(a)$, let us define the sets:

$$\begin{aligned} X_1^a &= Y_a \cap (S\backslash(2S\cup S^2)) \\ X_2^a &= Y_a \cap ((2S\cup S^2)\backslash(3S\cup S^3)) \\ &\ \ \vdots \\ X_{n-1}^a &= Y_a \cap (((n-1)S\cup S^{n-1})\backslash(nS\cup S^n)) \\ X_n^a &= Y_a \cap (nS\cup S^n)). \end{aligned}$$

It is clear that the conditions (2.1) hold for every X_i^a and X_j^b $(1 \le i,j \le n;\ a,b \in T)$. From the defined sets follows that $Y_a = \bigcup_{i=1}^{n} X_i^a$ $(a \in T)$. Also, $S = \bigcup_{a\in T} Y_a$. For $x_1, x_2 \in S$ there exist $a_1, a_2 \in T$ such that $x_1 \in Y_{a_1}$, $x_2 \in Y_{a_2}$. Then from Corollary 1.1 we have: $Y_{a_1} * Y_{a_2} \subset Y_{a_1+a_2}$, $Y_{a_1}\circ Y_{a_2} \subset Y_{a_1\cdot a_2}$. Let $x_1 \in X_{i_1}^{a_1}$, $x_2 \in X_{i_2}^{a_2}$, where $1 \le i_1, i_2 \le n$. Then $x_1 \in X_{i_1}^{a_1} \subset i_1S \cup S_{i_1}$, $x_2 \in X_{i_2}^{a_2} \subset i_2S\cup S^{i_2}$ and so

$$\begin{aligned} x_1 * x_2 \in (i_1S\cup S^{i_1}) * (i_2S\cup S^{i_2}) &\subset m(i_1,i_2)S\cup S^{m(i_1,i_2)} \\ x_1 \circ x_2 \in (i_1S\cup S^{i_1}) \circ (i_2S\cup S^{i_2}) &\subset m(i_1,i_2)S\cup S^{m(i_1,i_2)}. \end{aligned}$$

For $m(i_1,i_2) \le n$ we have that:

$$x_1 * x_2 \in \bigcup_{\nu=m(i_1,i_2)}^{n} X_\nu^{a_1+a_2}, \quad x_1\circ x_2 \in \bigcup_{\nu=m(i_1,i_2)}^{n} X_\nu^{a_1\cdot a_2}.$$

For $m(i_1,i_2) = n+1$ we have that: $x_1 * x_2 = a_1 + a_2 \in T$, $x_1\circ x_2 = a_1\cdot a_2 \in T$. Also ($T$ is an ideal of S) we have that: $x_1 * a_2 = a_1 * x_2 = a_1 + a_2$, $x_1 \circ a_2 =$

$a_1 \circ x_2 = a_1 \cdot a_2$. In this way the functions: $\varphi_{(i_1,i_2)}^{(a_1,a_2)}$, $\psi_{(i_1,i_2)}^{(a_1,a_2)}$ from Lemma 2.1 are defined and the conditions (2.3) and (2.4) hold.

Corollary 2.1. *A semiring S is 1-inflation of a semiring T if and only if $2S \subset T$, $S^2 \subset T$ and S is a retract extension of T.*

Proof. The proof immediately follows from the fact that each 1-semiring is at the same time a semiring.

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