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NOTE ON INTERSECTIONS OF MAXIMAL CLONES

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Abstract. We investigate some interesting properties of the intersections of maximal clones which partially describe lattice of clones in P_k for $k \geq 3$. These intersections are very important for investigation of relative completeness with respect to $min(x, y)$ and $\overline{x} = k - 1 - x$.

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1. Notation and preliminaries

Denote by $\mathbf{N}$ the set $\{1, 2, \ldots\}$ of positive integers. For $k, n \in \mathbf{N}$, $E_k = \{0, 1, \ldots, k-1\}$, denote by $P_k^{(n)}$ the set of all maps $E_k^n \to E_k$, and $P_k = \bigcup_{n \in \mathbf{N}} P_k^{(n)}$. We say that f is an *i-th projection of arity n* $(1 \leq i \leq n)$ if $f \in P_k^{(n)}$ and f satisfies the identity $f(x_1, \ldots, x_n) \approx x_i$. We say that $f \in P_k^{(n)}$ is *essential* if it depends on at least two variables and it takes all values from E_k. Let π_i^n denote the i-th projection of arity n, and let Π_k denote the set of all the projections over E_k. For $n, m \geq 1, f \in P_k^{(n)}$ and $g_1, \ldots, g_n \in P_k^{(m)}$, *the superposition of* f and $g_1, \ldots, g_n$, denoted by $f(g_1, \ldots, g_n)$, is defined by $f(g_1, \ldots, g_n)(a_1, \ldots, a_m) = f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$ for all $(a_1, \ldots, a_m) \in E_k^m$. A set $F \subseteq P_k$ is *a clone of operations on* E_k (or *clone* for short) if $\Pi_k \subseteq F$ and F is closed with respect to the superposition. For $F \subseteq P_k$, $\langle F \rangle_{\text{CL}}$ stands for the clone generated by F. We say that the clone F is *maximal* if there is no clone G such that $F \subset G \subset P_k$. $F \subseteq P_k$ is *complete* if $\langle F \rangle_{\text{CL}} = P_k$.

Let $\varrho \subseteq E_k^h$ be an h-ary relation and $f \in P_k^{(n)}$. We say that f *preserves* ϱ if for all h-tuples $(a_{11}, \ldots, a_{1h}), \ldots, (a_{n1}, \ldots, a_{nh})$ from ϱ we have $(f(a_{11}, \ldots, a_{n1}), \ldots, f(a_{1h}, \ldots, a_{nh})) \in \varrho$. $\operatorname{Pol} \varrho$ is the set of all $f \in P_k$ which preserve ϱ. For $F \subseteq P_k$, $\operatorname{Inv} F$ denotes the set of all the relations preserved by each $f \in F$.

It is interesting to consider the followig problem: What are the maximal clones on a finite universe not containing a given clone C; or, equivalently, what are operations to add to C to make it complete (or primal)?

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The following concept of relative completeness was introduced in [3].

Let C be a clone on E_k and $F \subseteq P_k$. F is *complete relative* to C (or *C-complete*) if $\langle F \cup C\rangle_{\mathrm{CL}} = P_k$.

The following theorem gives a necessary and sufficient condition for F to be C-complete. It is analogous to the Post completeness criterion.

Theorem 1.1. [3] *Let C be a clone on E_k. $F \subseteq P_k$ is complete relative to C if and only if $F \setminus M \neq \emptyset$ for every maximal clone M containing C.*

Therefore, the problem of determining whether a set F is complete relative to C, reduces to determining all the maximal clones that contain C.

This paper heavily depends upon the famous Rosenberg characterization of maximal clones. The following special sets of relations are considered:

R_1 – the set of all bounded partial orders on E_k;

R_2 – the set of selfdual relations, i.e. relations of the form $\{(x, s(x)) : x \in E_k\}$, where s is a fixed point free permutation of prime order (i.e. $s^p = \mathrm{id}$ for some prime p);

R_3 – the set of affine relations, i.e. relations of the form $\{(a, b, c, d) \in E_k^4 : a * b = c * d\}$, where $(E_k, *)$ is a p-elementary Abelian group (p prime);

R_4 – the set of all nontrivial equivalence relations on E_k;

R_5 – the set of all central relations on E_k;

R_6 – the set of all h-regular relations on E_k ($h \geq 3$).

Theorem 1.2. [2] *A clone M is maximal iff there is a $\varrho \in R_1 \cup \ldots \cup R_6$ such that $M = \mathrm{Pol}\,\varrho$.*

2. Some properties of intersections of maximal clones

Let $M_i = \mathrm{Pol}\varrho_i$, lrt C_i be the center of the relation $\varrho_i \in R_5^{(2)}$, $\overline{M} = P_k \backslash M$, $M_i M_j = M_i \cap M_j$, $N_k = \{1, 2, \ldots k\}$ and let $\varrho_1\varrho_2 = \varrho_1 \cap \varrho_2$.

Theorem 2.1. *If $\varrho_1\varrho_2 = \varrho_3$, $\varrho_1, \varrho_2, \varrho_3 \in R_1$, then $M_1M_2 \subset M_3$.*

Proof. Suppose that $f \in M_1M_2$ and let $c_i \in \varrho_3$ for all $i \in N_n$. Then, $\varrho_3 = \varrho_1\varrho_2 \;\Rightarrow\; c_i \in \varrho_1 \wedge c_i \in \varrho_2 \;\Rightarrow\; f(\mathbf{c}) \in \varrho_1 \wedge f(\mathbf{c}) \in \varrho_2 \;\Rightarrow\; f(\mathbf{c}) \in \varrho_1\varrho_2 = \varrho_3$ i.e. $f \in M_3$. □

Theorem 2.2. *If $\varrho_1, \varrho_2 \in R_5^{(2)}$, $\varrho_3 \in R_5^{(1)}$, $\varrho_3 \subset C_2$ and*

$$(1) \quad (a, b) \notin \varrho_1 \Rightarrow (\forall c \in E_k \backslash \varrho_3)(a, c) \notin \varrho_2 \vee (b, c) \notin \varrho_2, \text{ then}$$

$$\overline{M}_1 M_2 \subset M_3$$

Proof. Suppose that $f \in \overline{M}_1 M_2 \overline{M}_3$. Because $f \in \overline{M}_1$ there are $\mathbf{a}, \mathbf{b} \in E_k^n$ such that $(a_i, b_i) \in \varrho_1$ for all $i \in N_n$ and $(f(\mathbf{a}), f(\mathbf{b})) \notin \varrho_1$. If $f \in \overline{M}_3$, then there exists $\mathbf{c} \in E_k^n$ such that $c_i \in \varrho_3$ for each $i \in N_n$ and $f(\mathbf{c}) \notin \varrho_3$. On the other hand, $\varrho_3 \subset C_2$ and $f \in M_2$, which implies

$$(a_i, c_i) \in \varrho_2 \ \wedge \ (b_i, c_i) \in \varrho_2 \ \Rightarrow \ (f(\mathbf{a}), f(\mathbf{c})) \in \varrho_2 \ \wedge \ (f(\mathbf{b}), f(\mathbf{c})) \in \varrho_2$$

for all $i \in N_n$, which is a contradiction with (1) □

Theorem 2.3. *If the relation $\varrho_2 \in R_5^{(1)}$ is a union of some equivalence classes of $\varrho_1 \in R_4$, $\varrho_3 \subset \varrho_2$, $\varrho_3 \in R_5^{(1)}$ and $(\forall a_i \in \varrho_2)$ $(\exists b_i \in \varrho_3)$ $(a_i, b_i) \in \varrho_1$, then $M_1 M_3 \subset M_2$.*

Proof. Suppose that $f \in M_1 M_3 \overline{M}_2$. $f \in \overline{M}_2$ implies that there exists $\mathbf{a} \in E_k^n$ such that $a_i \in \varrho_2$ for all $i \in N_n$ and $f(\mathbf{a}) \notin \varrho_2$. Now, from $\varrho_3 \subset \varrho_2$ follows $f(\mathbf{a}) \notin \varrho_3$.

If $\mathbf{b} \in E_k^n$ such that $b_i \in \varrho_3$ for all $i \in N_n$ and $(a_i, b_i) \in \varrho_1$ (it is possible because $(\forall a_i \in \varrho_2)(\exists b_i \in \varrho_3)$ $(a_i, b_i) \in \varrho_1$), then $f \in M_3$ implies $f(\mathbf{b}) \in \varrho_3 \subset \varrho_2$. Now, $f(\mathbf{a}) \notin \varrho_2$ and $f(\mathbf{b}) \in \varrho_2$ implies $(f(\mathbf{a}), f(\mathbf{b})) \notin \varrho_1$ (because ϱ_2 is a union of the equivalence classes of ϱ_1), but we get a contradiction since $(a_i, b_i) \in \varrho_1$ implies $(f(\mathbf{a}), f(\mathbf{b})) \in \varrho_1$. □

Theorem 2.4. *If the relations $\varrho_1, \varrho_2 \in R_5^{(2)}$, $\varrho_1 \subset \varrho_2$, $\varrho_3 \in \varrho_4$ and $\varrho_4 \in R_5^{(1)}$ satisfying the following condition:*

(1) $(a, b) \notin \varrho_1 \Rightarrow \{a, b\} \not\subset C_2$,
(2) $(a, b) \in \varrho_1 \wedge a \notin \varrho_4 \Rightarrow (\exists c \in \varrho_4)\, (a, c) \in \varrho_3 \wedge (b, c) \in \varrho_2$ *and*
(3) $(a, b) \notin \varrho_1 \wedge (a, c) \in \varrho_3 \wedge c \in \varrho_4 \wedge b \notin C_2 \Rightarrow (c, b) \notin \varrho_2$

then $M_2 M_3 M_4 \subset M_1$.

Proof. Suppose that $f \in M_2 M_3 M_4 \overline{M}_1$. From $f \in \overline{M}_1$ follows that there exist $\mathbf{a}, \mathbf{b} \in E_k^n$ such that $(a_i, b_i) \in \varrho_1$ for all $i \in N_n$ and $(f(\mathbf{a}), f(\mathbf{b})) \notin \varrho_1$. By (1) $f(\mathbf{a}) \notin C_2$ or $f(\mathbf{b}) \notin C_2$. Let $f(\mathbf{b}) \notin C_2$ (The case $f(\mathbf{b}) \notin C_2$ is analogous). We shall define vector the $\mathbf{c} \in E_k^n$ in the following way:

$$(a_i, b_i) \in \varrho_1 \subset \varrho_2 \ \wedge \ a_i \in \varrho_4 \ \Rightarrow \ c_i = a_i$$

$$(a_i, b_i) \in \varrho_1 \ \wedge \ a_i \notin \varrho_4 \ \overset{(2)}{\Rightarrow} \ (\exists c_i \in \varrho_4)\, (a_i, c_i) \in \varrho_3 \ \wedge \ (b_i, c_i) \in \varrho_2$$

for all $i \in N_n$.

In this way we obtain that $(a_i, c_i) \in \varrho_3$, $(b_i, c_i) \in \varrho_2$, and $c_i \in \varrho_4$ for all $i \in N_n$. Now, $f \in M_2 M_3 M_4$ implies

$$(f(\mathbf{a}), f(\mathbf{c})) \in \varrho_3, \quad (f(\mathbf{b}), f(\mathbf{c})) \in \varrho_2 \ \text{and} \ f(\mathbf{c}) \in \varrho_4$$

However, $(f(\mathbf{a}), f(\mathbf{b})) \notin \varrho_1$, $(f(\mathbf{a}), f(\mathbf{c})) \in \varrho_3$, $f(\mathbf{c}) \in \varrho_4$, $f(\mathbf{b}) \notin C_2$ and the condition (3) implies $(f(\mathbf{c}), f(\mathbf{b})) \notin \varrho_2$, which is a contradiction with $(f(\mathbf{c}), f(\mathbf{b})) \in \varrho_2$ □

Theorem 2.5. *If for the relations* $\varrho_1, \varrho_2 \in R_5^{(1)}$, $\varrho_2 \subset \varrho_1$, $\varrho_3, \varrho_4 \in R_5^{(2)}$ $\varrho_2 \subset C_4$ *holds*

(1) $(a,b) \notin \varrho_3 \Rightarrow (\forall c \in \varrho_1 \backslash \varrho_2)\ (a,c) \notin \varrho_4 \ \vee\ (b,c) \notin \varrho_4$ *then* $M_1\overline{M}_2\overline{M}_3 \subset \overline{M}_4$.

Proof. Suppose that $f \in M_1\overline{M}_2\overline{M}_3$. If $f \in \overline{M}_3$, then there are $\mathbf{a}, \mathbf{b} \in \mathrm{E}_\mathrm{k}^\mathrm{n}$ such that $(a_i, b_i) \in \varrho_3$ for all $i \in N_n$ and $(f(\mathbf{a}), f(\mathbf{b})) \notin \varrho_3$. If $f \in \overline{M}_2$, then there exists $\mathbf{c} \in \mathrm{E}_\mathrm{k}^\mathrm{n}$,such that $c_i \in \varrho_2$ for all $i \in N_n$ and $f(\mathbf{c}) \notin \varrho_2$. From $\varrho_2 \subset \varrho_1$ and $f \in M_1$ follows that $f(\mathbf{c}) \in \varrho_1$, i.e. $f(\mathbf{c}) \in \varrho_1 \backslash \varrho_2$. The condition (1) implies $(f(\mathbf{a}), f(\mathbf{c})) \notin \varrho_4$ or $(f(\mathbf{b}), f(\mathbf{c})) \notin \varrho_4$, while $\varrho_2 \subset C_4$ implies that $(a_i, c_i) \in \varrho_4$ and $(b_i, c_i) \in \varrho_4$. It means that $f \in \overline{M}_4$. □

Theorem 2.6. *If* $\varrho_2 \in R_5^{(1)}$ *is an equivalence class of* $\varrho_1 \in R_4$, $C_4 \cap \varrho_2 \neq \emptyset$ *and if for the relations* ϱ_3, $\varrho_4 \in R_5^{(2)}$ *holds:*

$$(1) \qquad (a,b) \notin \varrho_3 \ \Rightarrow\ (\forall c \in E_k \backslash \varrho_2)\ (a,c) \notin \varrho_4 \ \vee\ (b,c) \notin \varrho_4$$

then $M_1\overline{M}_2\overline{M}_3 \subset \overline{M}_4$.

Proof. Suppose that $f \in M_1\overline{M}_2\overline{M}_3$. If $f \in \overline{M}_2$ then there exists $\mathbf{a} \in \mathrm{E}_\mathrm{k}^\mathrm{n}$ such that $a_i \in \varrho_2$ for all $i \in N_n$ and $f(\mathbf{a}) \notin \varrho_2$. If we choose $b_i \in \varrho_2 \cap C_4 \neq \emptyset$, from $f \in M_1$ follows that for all $\mathbf{b} \in \mathrm{E}_\mathrm{k}^\mathrm{n}$ such that $b_i \in \varrho_2$, $i \in N_n$ we have $(f(\mathbf{a}), f(\mathbf{b})) \in \varrho_1$. But $f(\mathbf{a}) \notin \varrho_2$ and $(f(\mathbf{a}), f(\mathbf{b})) \in \varrho_1$, implies $f(\mathbf{b}) \notin \varrho_2$. Since $f \in \overline{M}_3$ there are $\mathbf{c}, \mathbf{d} \in \mathrm{E}_\mathrm{k}^\mathrm{n}$ such that $(c_i, d_i) \in \varrho_3$ for all $i \in N_n$ and $(f(\mathbf{c}), f(\mathbf{d})) \notin \varrho_3$. The condition (1) implies $(f(\mathbf{c}), f(\mathbf{b})) \notin \varrho_4$ or $(f(\mathbf{d}), f(\mathbf{b})) \notin \varrho_4$. So, $f \in \overline{M}_4$ because $(c_i, b_i) \in \varrho_4$, and $(d_i, b_i) \in \varrho_4$. □

Theorem 2.7. *If for the relations* $\varrho_1, \varrho_2, \varrho_3 \in R_5^{(2)}, \varrho_2 \subset \varrho_1$; $\varrho_4 \in R_5^{(1)}$ *holds:*

$$\begin{array}{lll} (1) & (a,b) \notin \varrho_2 & \Rightarrow \quad a \in \varrho_4 \vee b \in \varrho_4 \\ (2) & (a,b) \in \varrho_2 & \Rightarrow \quad (\exists c \in \varrho_4)(b,c) \in \varrho_3 \wedge (a,c) \in \varrho_1 \\ (3) & (a,b) \in \varrho_1 \backslash \varrho_2 & \textit{implies:} \end{array}$$

$$(\forall c \in \varrho_4)\ a \in \varrho_4 \ \Rightarrow\ \Big((b,c) \in \varrho_3 \wedge (a,c) \notin \varrho_1\Big) \vee \Big((a,c) \in \varrho_1 \wedge (b,c) \notin \varrho_3\Big),$$

$$\textit{then} \qquad M_1\overline{M}_2M_4 \subset \overline{M}_3$$

Proof. Suppose that $f \in M_1\overline{M}_2M_4M_3$ from $f \in \overline{M}_2$ follows that there exist $\mathbf{a}, \mathbf{b} \in \mathrm{E}_\mathrm{k}^\mathrm{n}$ such that $(a_i, b_i) \in \varrho_2$ for all $i \in N_n$ and $(f(\mathbf{a}), f(\mathbf{b})) \notin \varrho_2$. However, from $\varrho_2 \subset \varrho_1$ and $f \in M_1$ follows $(f(\mathbf{a}), f(\mathbf{b})) \in \varrho_1$, i.e. $(f(\mathbf{a}), f(\mathbf{b})) \in \varrho_1 \backslash \varrho_2$.

By (1) $f(\mathbf{a}) \in \varrho_4 \ \vee$ f$(\mathbf{b}) \in \varrho_4$. Let $f(\mathbf{a}) \in \varrho_4$ (analogous for $f(\mathbf{b}) \in \varrho_4$).

Now we shall define the vector $\mathbf{c} \in \mathrm{E}_\mathrm{k}^\mathrm{n}$ in the following way:

$$(a_i, b_i) \in \varrho_2 \overset{(2)}{\Rightarrow} (\exists c_i \in \varrho_4)\ \ (b_i, c_i) \in \varrho_3 \wedge (a_i, c_i) \in \varrho_1 \quad \text{for all } i \in N_n.$$

From $f \in M_1M_3$, we have $(f(\mathbf{b}), f(\mathbf{c})) \in \varrho_3 \ \wedge\ (f(\mathbf{a}), f(\mathbf{c})) \in \varrho_1$. But now we have that $(f(\mathbf{a}), f(\mathbf{b})) \in \varrho_1 \backslash \varrho_2$, $f(\mathbf{c}) \in \varrho_4$, $f(\mathbf{a}) \in \varrho_4, (f(\mathbf{b}), f(\mathbf{c})) \in \varrho_3$ and $(f(\mathbf{a}), f(\mathbf{c})) \in \varrho_1$, which is a contradiction with the condition (3). □

Theorem 2.8. *If for the relation* $\varrho_1, \varrho_2 \in R_5^{(2)}$, $\varrho_1 \subset \varrho_2$, $\varrho_3 \in R_5^{(1)}$ *holds:*

$$(1)\quad (a,b) \in \varrho_2 \backslash \varrho_1 \ \vee \ \Big(a \notin \varrho_3 \ \wedge \ b \notin \varrho_3 \ \wedge \ (a,b) \in \varrho_2\Big) \Rightarrow$$
$$(\exists c \in \varrho_3) \ \Big((a,c) \in \varrho_1 \ \wedge \ (b,c) \in \varrho_1\Big)$$
$$(2)\quad (a,b) \notin \varrho_2 \Rightarrow \Big((\forall c \in \varrho_3) \ (a,c) \in \varrho_1 \Rightarrow (b,c) \notin \varrho_1\Big)$$

$$\textit{then} \qquad M_1 M_3 \subset M_2.$$

Proof. Suppose that $f \in M_1 \overline{M}_2 M_3$. From $f \in \overline{M}_2$ follows that there exist $\mathbf{a}, \mathbf{b} \in \mathrm{E}_\mathrm{k}^\mathrm{n}$ such that $(a_i, b_i) \in \varrho_2$ for all $i \in N_n$ and $(f(\mathbf{a}), f(\mathbf{b})) \notin \varrho_2$.

Now we shall define the vector $\mathbf{c} \in \mathrm{E}_\mathrm{k}^\mathrm{n}$ in the following way:

$$(a_i, b_i) \in \varrho_1 \ \wedge \ a_i \in \varrho_3 \quad \Rightarrow \quad c_i = a_i$$
$$(a_i, b_i) \in \varrho_1 \ \wedge \ b_i \in \varrho_3 \quad \Rightarrow \quad c_i = b_i$$
$$(a_i, b_i) \in \varrho_2 \backslash \varrho_1 \ \vee \ (a_i \notin \varrho_3 \ \wedge \ b_i \notin \varrho_3) \quad \overset{(1)}{\Rightarrow} \quad (\exists c_i \in \varrho_3) \ (a_i, c_i) \in \varrho_1 \ \wedge \ (b_i, c_i) \in \varrho_1$$

for all $i \in N_n$.

However, $(a_i, c_i) \in \varrho_1$, $(b_i, c_i) \in \varrho_1$, $c_i \in \varrho_3$ for all $i \in N_n$ and $f \in M_1 M_3$ implies $(f(\mathbf{a}), f(\mathbf{c})) \in \varrho_1$, $(f(\mathbf{b}), f(\mathbf{c})) \in \varrho_1$ and $f(\mathbf{c}) \in \varrho_3$, which is a contradiction with the condition (2). □

Theorem 2.9. *If for the relations* $\varrho_1, \varrho_2 \in R_4$ *and* $\varrho_3 \in R_5^{(2)}$ *holds:*

$$(1)\quad (a,b) \in \varrho_3 \ \Rightarrow \ \Big((\exists c, d \in E_k) \ (a,c) \in \varrho_1 \ \wedge \ (b,d) \in \varrho_1 \ \wedge \ (c,d) \in \varrho_2\Big)$$
$$(2)\quad \Big((a,b) \notin \varrho_3 \ \wedge \ (a,c) \in \varrho_1 \ \wedge \ (b,d) \in \varrho_1\Big) \ \Rightarrow \ (c,d) \notin \varrho_2,$$

$$\textit{then} \qquad M_1 M_2 \subset M_3.$$

Proof. Suppose that $f \in M_1 M_2 \overline{M}_3$. From $f \in \overline{M}_3$ follows that there exist $\mathbf{a}, \mathbf{b} \in \mathrm{E}_\mathrm{k}^\mathrm{n}$ such that $(a_i, b_i) \in \varrho_3$ for all $i \in N_n$ and $(f(\mathbf{a}), f(\mathbf{b})) \notin \varrho_3$. From the condition (1) implies that there exist $c_i, d_i \in E_k$ such that $(a_i, c_i) \in \varrho_1$, $(b_i, d_i) \in \varrho_1$ and $(c_i, d_i) \in \varrho_2$ for all $i \in N_n$.

However, from $f \in M_1 M_2$ we have $(f(\mathbf{a}), f(\mathbf{c})) \in \varrho_1$, $(f(\mathbf{b}), f(\mathbf{d})) \in \varrho_1$ and $(f(\mathbf{c}), f(\mathbf{d})) \in \varrho_2$, which is contradiction with condition (2). □

Theorem 2.10. *If the relations* $\varrho_1 \in R_4$ *and* $\varrho_2, \varrho_3 \in R_5^{(2)}$ *satisfy the conditions*

$$(1)\quad (a,b) \in \varrho_3 \backslash \varrho_2 \quad \Rightarrow \quad (\exists c, d \in E_k)(a,c) \in \varrho_1 \ \wedge \ (b,d) \in \varrho_1 \ \wedge \ (c,d) \in \varrho_2$$
$$(2)\quad (a,b) \notin \varrho_3 \quad \Rightarrow \quad (\forall c, d \in E_k)(a,c) \notin \varrho_1 \ \vee \ (b,d) \notin \varrho_1 \ \vee \ (c,d) \notin \varrho_2$$

$$\textit{then} \qquad M_1 M_2 \subset M_3.$$

Proof. Suppose that $f \in M_1M_2\overline{M}_3$. From $f \in \overline{M}_3$ follows that there exist $\mathbf{a}, \mathbf{b} \in \mathrm{E}_k^n$ such that $(a_i, b_i) \in \varrho_3$ for all $i \in N_n$ and $(f(\mathbf{a}), f(\mathbf{b})) \notin \varrho_3$.

Now we shall define the vectors $\mathbf{c}, \mathbf{d} \in \mathrm{E}_k^n$ in the following way:

$$(a_i, b_i) \in \varrho_2 \quad \Rightarrow \quad c_i = a_i \ \wedge \ d_i = b_i$$
$$(a_i, b_i) \in \varrho_3 \backslash \varrho_2 \quad \overset{(1)}{\Rightarrow} \quad (\exists c_i, d_i \in E_k)(a_i, c_i) \in \varrho_1 \ \wedge \ (b_i, d_i) \in \varrho_1 \ \wedge \ (c_i, d_i) \in \varrho_2$$

for all $i \in N_n$.

However, $f \in M_1M_2$ implies $(f(\mathbf{a}), f(\mathbf{c})) \in \varrho_1$, $(f(\mathbf{b}), f(\mathbf{d})) \in \varrho_1$ and $(f(\mathbf{c}), f(\mathbf{d})) \in \varrho_2$, which is contradiction with the condition (2). □

Theorem 2.11. *If for the relations $\varrho_1 \in R_4$, $\varrho_2, \varrho_3 \in R_5^{(2)}$ holds:*

(1) $(a, b) \in \varrho_3 \backslash \varrho_2 \Rightarrow (\exists c \in E_k)(a, c) \in \varrho_1 \ \wedge \ (b, c) \in \varrho_2$

(2) $(a, b) \notin \varrho_3 \Rightarrow$
$$(\forall c \in E_k)((a, c) \notin \varrho_1 \ \vee \ (b, c) \notin \varrho_2) \ \vee \ (\forall c \in E_k)((b, c) \notin \varrho_1 \ \vee \ (a, c) \notin \varrho_2)$$

then $\quad M_1M_2 \subset M_3.$

Proof. Suppose that $f \in M_1M_2\overline{M}_3$. From $f \in \overline{M}_3$ follows that there exist $\mathbf{a}, \mathbf{b} \in \mathrm{E}_k^n$ such that $(a_i, b_i) \in \varrho_3$ for all $i \in N_n$ and $(f(\mathbf{a}), f(\mathbf{b})) \notin \varrho_3$. The condition (2) implies

$$(\forall c \in E_k)((f(\mathbf{a}), c) \notin \varrho_1 \vee (f(\mathbf{b}), c) \notin \varrho_2) \vee (\forall c \in E_4)((f(\mathbf{b}), c) \notin \varrho_1 \vee (f(\mathbf{a}), c) \notin \varrho_2)$$

Let $(\forall c \in E_k)(f(\mathbf{b}), c) \notin \varrho_1 \vee (f(\mathbf{b}), c) \notin \varrho_2$. (In the other case the proof is analogous).

Now we shall define the vector $\mathbf{c} \in \mathrm{E}_k^n$ in the following way:

$$(a_i, b_i) \in \varrho_2 \quad \Rightarrow \quad c_i = a_i$$
$$(a_i, b_i) \in \varrho_3 \backslash \varrho_2 \quad \overset{(1)}{\Rightarrow} \quad (\exists c_i \in E_k)(a_i, c_i) \in \varrho_1 \ \wedge \ (b_i, c_i) \in \varrho_2$$

for all $i \in N_n$.

However, $f \in M_1M_2$ implies $(f(\mathbf{a}), f(\mathbf{c})) \in \varrho_1$ and $(f(\mathbf{b}), f(\mathbf{c})) \in \varrho_2$ which is a contradiction. □

Theorem 2.12. *If $\varrho_3 \in R_5^{(1)}$ is an equivalent class of the relation $\varrho_2 \in R_4$, $\varrho_1 \in R_4$, $\varrho_4 \in R_5^{(2)}$, $\varrho_1 \subset \varrho_4$ and holds:*

(1) *C is equivalent class of $\varrho_1 \Rightarrow C \cap \varrho_3 \neq \emptyset$*

(2) $a \notin \varrho_3 \Rightarrow (\forall c \in E_k)(a, c) \notin \varrho_2 \ \vee \ (a, c) \in \varrho_1$

then $\quad M_1M_2\overline{M}_3 \subset M_4.$

Proof. Suppose that $f \in M_1 M_2 \overline{M}_3$.

From $f \in \overline{M}_3$ follows that there exist $\mathbf{a} \in \mathrm{E}_{\mathrm{k}}^{\mathrm{n}}$ such that $a_i \in \varrho_3$ for all $i \in N_n$ and $f(\mathbf{a}) \notin \varrho_3$.

Let $\mathbf{c}$ be a vector from E_k^n.

Now we shall define the vector $\mathbf{d}$ in the following way:

$$c_i \in \varrho_3 \Rightarrow d_i = c_i$$
$$c_i \notin \varrho_3 \overset{(1)}{\Rightarrow} (\exists d_i \in E_k) d_i \in \varrho_3 \ \wedge \ (c_i, d_i) \in \varrho_1$$

for all $i \in N_n$. However, $f \in M_1$ and $(c_i, d_i) \in \varrho_1$ for all $i \in N_n$ implies $(f(\mathbf{c}), f(\mathbf{d})) \in \varrho_1$.

From $(a_i, d_i) \in \varrho_2$ for all $i \in N_n$ and $f \in M_2$ follows $(f(\mathbf{a}), f(\mathbf{d})) \in \varrho_2$, i.e. by (2) we have $(f(\mathbf{a}), f(\mathbf{d})) \in \varrho_1$.

$(f(\mathbf{c}), f(\mathbf{d})) \in \varrho_1$ and $(f(\mathbf{a}), f(\mathbf{d})) \in \varrho_1$ implies $(f(\mathbf{c}), f(\mathbf{a})) \in \varrho_1$ for each $\mathbf{c} \in \mathrm{E}_{\mathrm{k}}^{\mathrm{n}}$. It means that for all vectors $\mathbf{x}, \mathbf{y} \in E_k^n$ $(f(\mathbf{x}), f(\mathbf{y})) \in \varrho_1$. From $\varrho_1 \subset \varrho_4$ follows $f \in M_4$. □

Theorem 2.13. *Let $\varrho_2 \in R_5^{(1)}$, $\varrho_1 \in R_4$, $\varrho_3, \varrho_4 \in R_5^{(2)}$ and holds:*

$$\begin{aligned}
&(1) \quad (a,b) \notin \varrho_3 \Rightarrow (\forall c \notin \varrho_2)(a,c) \notin \varrho_4 \ \vee \ (b,c) \notin \varrho_4 \\
&(2) \quad C \text{ is equivalent class of } \varrho_1 \Rightarrow C \cap \varrho_2 \neq \emptyset \\
&(3) \quad (a,b) \notin \varrho_4 \ \wedge \ (b,c) \in \varrho_1 \Rightarrow (a,c) \notin \varrho_4 \\
&(4) \quad a \in \varrho_2 \ \wedge \ b \in \varrho_2 \Rightarrow (a,b) \in \varrho_4
\end{aligned}$$

$$\text{then} \qquad M_1 \overline{M}_2 \overline{M}_3 \subset \overline{M}_4.$$

Proof. Suppose $f \in M_1 \overline{M}_2 \overline{M}_3$.

From $f \in \overline{M}_3$ follows that there exist $\mathbf{a}, \mathbf{b} \in \mathrm{E}_{\mathrm{k}}^{\mathrm{n}}$ such that $(a_i, b_i) \in \varrho_3$ for all $i \in N_n$ and $(f(\mathbf{a}), f(\mathbf{b})) \notin \varrho_3$.

From $f \in \overline{M}_2$ follows that there exists $\mathbf{c} \in \mathrm{E}_{\mathrm{k}}^{\mathrm{n}}$ such that $c_i \in \varrho_2$ and $f(\mathbf{c}) \notin \varrho_2$. The condition (1) implies $(f(\mathbf{a}), f(\mathbf{c})) \notin \varrho_4$ or $(f(\mathbf{b}), f(\mathbf{c})) \notin \varrho_4$.

Let $(f(\mathbf{b}), f(\mathbf{c})) \notin \varrho_4$. (Analogously for the case $(f(\mathbf{a}), f(\mathbf{c})) \notin \varrho_4$).

From the condition (2) follows that there exists $d_i \in \varrho_2$ for all $i \in N_n$ such that $(b_i, d_i) \in \varrho_1$ and $f \in M_1$ implies $(f(\mathbf{b}), f(\mathbf{d})) \in \varrho_1$. Now we have

$$(f(\mathbf{c}), f(\mathbf{b})) \notin \varrho_4 \ \wedge \ (f(\mathbf{b}), f(\mathbf{d})) \in \varrho_1 \overset{(3)}{\Rightarrow} (f(\mathbf{c}), f(\mathbf{d})) \notin \varrho_4 \quad \square$$
$$c_i \in \varrho_2 \ \wedge \ d_i \in \varrho_2 \overset{(4)}{\Rightarrow} (c_i, d_i) \in \varrho_4 \Rightarrow f \notin M_4.$$

Theorem 2.14. *If $\varrho_3, \varrho_4 \in R_5^{(1)}$, $\varrho_1, \varrho_2 \in R_4$, and*

$$\begin{aligned}
&(1) \quad (\forall c \in \varrho_4)(\forall d \in \varrho_3)(\exists a \in E_k)(a,c) \in \varrho_1 \ \wedge \ (a,d) \in \varrho_2 \\
&(2) \quad a \notin \varrho_3 \ \wedge \ b \notin \varrho_4 \Rightarrow (\forall c \in E_k)(a,c) \notin \varrho_2 \ \vee \ (b,c) \notin \varrho_1
\end{aligned}$$

$$\text{then} \qquad M_1 M_2 \overline{M}_3 \subset M_4.$$

Proof. Suppose that $f \in M_1M_2\overline{M}_3\overline{M}_4$. From $f \notin M_4$ follows that there exists $\mathbf{c} \in \mathrm{E}_\mathrm{k}^\mathrm{n}$ such that for all $i \in N_n$ $c_i \in \varrho_4$ and $f(\mathbf{c}) \notin \varrho_4$.

From $f \in \overline{M}_3$ follows that there exists $\mathbf{d} \in \mathrm{E}_\mathrm{k}^\mathrm{n}$ such that $d_i \in \varrho_3$ for all $i \in N_n$ and $f(\mathbf{d}) \notin \varrho_3$.

The condition (1) implies that there exists $a_i \in E_k$ for all $i \in N_n$ such that $(a_i, c_i) \in \varrho_1$ and $(a_i, d_i) \in \varrho_2$. However, $f \in M_1M_2$ implies $(f(\mathbf{a}), f(\mathbf{c})) \in \varrho_1$ and $(f(\mathbf{a}), f(\mathbf{d})) \in \varrho_2$, which is a contradiction with the condition (2). □

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