NOVI SAD J. MATH.
VOL. 29, NO. 1, 1999, 61-67

ADMISSIBLE SOLUTIONS OF A SYSTEM OF COMPLEX HIGHER-ORDER DIFFERENTIAL EQUATIONS [1]

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Abstract. Using Nevanlinna theory of the value distribution of meromorphic functions, we investigate the problem of the existence of admissible meromorphic solutions of a type of a system of algebraic differential equations,which has not been discussed previously.

AMS Mathematics Subject Classification (1991):30D05,30D35,34A34

Key words and phrases: admissible solutions,systems of differential equations,finite accumulations.

1. Introduction and the main result

This paper needs some familiarity with the Nevanlinna theory,see,e.g.[1] for notations and basic results.

Recently, several authors have investigated the problem of the existence of admissible solutions or m components-admissible solutions of a system of algebraic differential equations and have obtained some results (see [3-8]). However, they have not considered the system of algebraic differential equations of the form

$$(1)\qquad \begin{cases} \sum_{(i)\in I} a_{(i)}(z) \prod_{k=1}^{2} w_k^{i_{k0}}(w_k')^{i_{k1}} \dots (w_k^{(n)})^{i_{kn}} = H_1(z,w_1), \\ \sum_{(j)\in J} b_{(j)}(z) \prod_{k=1}^{2} w_k^{j_{k0}}(w_k')^{j_{k1}} \dots (w_k^{(n)})^{j_{kn}} = H_2(z,w_2), \end{cases}$$

where $\{a_{(i)}(z)\}$ and $\{b_{(j)}(z)\}$ are meromorphic functions,I, J are two finite sets of multi-indices $I = (i_{10}, i_{20}, \dots, i_{1n}, i_{2n})$ for $a_{(i)} \neq 0$ and $J = (j_{10}, j_{20}, \dots, j_{1n}, j_{2n})$ for $b_{(j)} \neq 0$ respectively, $H_1(z, w_1)$ is the quotient of entire function in variables z and w_1, $H_2(z, w_2)$ is the quotient of entire function in variables z and w_2.

N. Steinmetz([2]) had considered first the differential equation:

$$(2)\qquad \sum_{(i)} a_{(i)}(z) w^{i_0}(w')^{i_1} \dots (w^{(n)})^{i_n} = H(z,w),$$

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where the letf-side of (2) is a differential polynomial with meromorphic coefficients, $H(z,w)$ is a quotient of entire function in variables z and w.

In this paper we consider the system of algebraic differential equations (1) using Steinmetz's idea. Obviously, there is an essential generalzation of the equation (2).

For differential polynomial $\sum\limits_{(i)\in I} a_{(i)}(z)\prod\limits_{k=1}^{2} w_k^{i_{k0}}(w_k')^{i_{k1}}\dots(w_k^{(n)})^{i_{kn}}$,we adopt the notation:

$$\lambda_k=\max\{i_{k0}+i_{k1}+\dots+i_{kn}\}, u_k=\max\{i_{k1}+2i_{k2}+\dots+ni_{kn}\},$$

$$\Delta_k=\max\{i_{k0}+2i_{k1}+\dots+(n+1)i_{kn}\}.(k=1,2)$$

Similarly, for $\sum\limits_{(j)\in J} b_{(j)}(z)\prod\limits_{k=1}^{2} w_k^{j_{k0}}(w_k')^{j_{k1}}\dots(w_k^{(n)})^{j_{kn}}$,we can note $\overline{\lambda}_k,\overline{u}_k.\overline{\Delta}_k.(k=1,2)$

Definition. *Let*

$$S_3(r)=\sum_{(i)}T(r,a_{(i)})+\sum_{(j)}T(r,b_{(j)}),$$

$$S_1(r)=T(r,H_1(z,c_{1i})),S_2(r)=T(r,H_2(z,c_{2i})),c_{li}\in\mathbf{C}.$$

$E_1=\{c_{1i}\},E_2=\{c_{2i}\}(\in\mathbf{C},E_1\cap E_2=\emptyset)$ be two finite accumulation sets,(w_1,w_2) be a meromorphic solution of (1). For every such $c_{li}\in E_l,(l=1,2)$, if the following conditions are satisfied:

$$S_3(r)+S_1(r)=o(T(r,w_i)),S_3(r)+S_2(r)=o(T(r,w_i)),$$

possibly outside a set of r of finite linear measure, we say that w_i is an admissible component of solution of (1).

Our main result is:

Theorem 1. *Let (w_1,w_2) be a meromorphic solutions of (1).If the following condition is satisfied:*

$$\left\{\begin{array}{l}\deg_{w_1}H_1(z,w_1)\ge\Delta_1,\\ \deg_{w_2}H_2(z,w_2)\ge\overline{\Delta}_2,\end{array}\right.\tag{3}$$

then both w_1 and w_2 are either admissible, or inadmissible.

Remark. If (w_1,w_2) is an entire solution of (1), then it ought to replace the $\deg_{w_1}H_1(z,w_1)\ge\lambda_1,\deg_{w_2}H_2(z,w_2)\ge\overline{\lambda}_2$ by the condition (3) of Theorem 1.

2. Proof of Theorem 1

Let (w_1, w_2) be a meromorphic solution of (1) and let

$$\sum_{(i)\in I} a_{(i)}(z) \prod_{k=1}^{2} w_k^{i_{k0}} (w_k')^{i_{k1}} \dots (w_k^{(n)})^{i_{kn}} =$$

$$= \Omega_1, \sum_{(j)\in J} b_{(j)}(z) \prod_{k=1}^{2} w_k^{j_{k0}} (w_k')^{j_{k1}} \dots (w_k^{(n)})^{j_{kn}} = \Omega_2.$$

For $c_{11} \in E_1$,set

$$(4) \qquad (w_2 - c_{11})\varphi_1(z; c_{11}) = \frac{\Omega_1 - H_1(z, c_{11})}{w_1 - c_{11}} = \frac{\Omega_1}{w_1 - c_{11}} - \frac{H_1(z, c_{11})}{w_1 - c_{11}}.$$

Because (w_1, w_2) is a meromorphic solution of (1),by (4),we know the zeroes of $w_1 - c_{11}$ with the multiplicity τ_1 are the poles of $(w_2 - c_{11})\varphi_1(z; c_{11})$ with multiplicity at most $\tau_1 - 1$.Now we take $c_{11}, c_{12} \in E_1, c_{11} \neq c_{12}$ and set

$$\begin{aligned} \varphi_2(z; c_{11}, c_{12}) = & \tfrac{1}{c_{11}-c_{12}}\{(w_2 - c_{11})\varphi_1(z; c_{11}) - (w_2 - c_{12})\varphi_1(z; c_{12})\} \\ = & \tfrac{\Omega_1}{(w_1-c_{11})(w_1-c_{12})} - \tfrac{1}{c_{11}-c_{12}} \tfrac{H_1(z,c_{11})}{w_1-c_{11}} + \tfrac{1}{c_{11}-c_{12}} \tfrac{H_1(z,c_{12})}{w_1-c_{12}} \\ = & \tfrac{\Omega_1 - Q_2(z,w_1)}{(w_1-c_{11})(w_1-c_{12})}. \end{aligned}$$

It is evident that they are poles of $\varphi_2(z; c_{11}, c_{12})$ with multiplicity at most $\tau_j - 1$ when the zeroes of $w_1 - c_{1i}$ with multiplicity τ_j are not poles of $a_{(i)}$ and $H_j(z)(j = 1, 2)$.

In general,we take distinct $c_{11}, c_{12}, \dots, c_{1k} \in E_1$ and set

$$(5) \qquad \begin{aligned} \varphi_k(z; c_{11}, \dots, c_{1k}) = & \tfrac{1}{c_{1,k-1}-c_{1k}}\{\varphi_{k-1}(z; c_{11}, \dots, c_{1,k-1}) \\ & -\varphi_{k-1}(z; c_{11}, \dots, c_{1,k-2}, c_{1k})\} \\ = & \frac{\Omega_1}{\prod_{j=1}^{k}(w_1 - c_{1j})} + \sum_{j=1}^{k} \overline{c}_{1j} \frac{H_j(z)}{w_1 - c_{1j}} \\ = & \frac{\Omega_1 - Q_k(z, w_1)}{\prod_{j=1}^{k}(w_1 - c_{1j})}, \end{aligned}$$

where $Q_k(z, w_1)$ is a polynomial of degree $k - 1$ in w_1,its coefficients are linear combination with $H_j(z)(j = 1, 2, \dots, k)$,$\overline{c}_{1j}$ is a constant which depends on c_{1j}.By induction, it is evident from (5) that they are poles of $\varphi_k(z; c_{11}, \dots, c_{1k})$ with multiplicity at most $\tau_j - 1$ when zeroes of $w_1 - c_{1i}$ with multiplicity τ_j are not poles of $a_{(i)}$ and $H_j(z)$.

By the condition $\deg_{w_1} H_1(z, w_1) = k \geq \Delta_1$ and the first fundamental Theorem of Nevanlinna,it follows that

$$(6) \qquad \begin{aligned} T(r, w_1) = & T(r, w_1 - c_{1,k+1}) + O(1) \\ \leq & T(r, (w_1 - c_{1,k+1})\varphi_{k+1}) + T(r, \varphi_{k+1}) + O(1). \end{aligned}$$

Now we estimate $T(r,(w_1-c_{1,k+1})\varphi_{k+1})$ and $T(r,\varphi_{k+1})$.

$$m(r,\varphi_k)\leq m(r,\frac{\Omega_1}{\prod\limits_{j=1}^{k}(w_1-c_{1j})})+m(r,\frac{Q_k(z,w_1)}{\prod\limits_{j=1}^{k}(w_1-c_{1j})})+O(1).$$

Note that

$$\begin{aligned}|\tfrac{w_1}{w_1-c_{1j}}|\leq\ & 1+\tfrac{|c_{1j}|}{|w_1-c_{1j}|}\leq(1+|c_{1j}|)(\tfrac{1}{|w_1-c_{1j}|})^+\\ \leq\ & c(\tfrac{1}{|w_1-c_{1j}|})^+,\end{aligned}\tag{7}$$

where $|a|^+=\max\{1,|a|\}$,$c=\max\{1+|c_{1j}|\}$.

$$(w_2)^{i_{20}}(w_2')^{i_{21}}\ldots(w_2^{(n)})^{i_{2n}}=(w_2)^{i_{20}+\ldots+i_{2n}}(\frac{w_2'}{w_2})^{i_{21}}\ldots(\frac{w_2^{(n)}}{w_2})^{i_{2n}}.$$

Thus

$$\begin{aligned}|\frac{\Omega_1}{\prod\limits_{j=1}^{k}(w_1-c_{1j})}|\leq\ & c^k\sum|a_{(i)}(z)|(\prod_j|\tfrac{w_1'}{w_1-c_{1j}}|)\ldots(\prod_j|\tfrac{w_1^{(n)}}{w_1-c_{1j}}|)\\ & (\prod_j|\tfrac{1}{w_1-c_{1j}}|)^+|w_2|^{i_{20}+\ldots+i_{2n}}|\tfrac{w_2'}{w_2}|^{i_{21}}\ldots|\tfrac{w_2^{(n)}}{w_2}|^{i_{2n}},\end{aligned}$$

where $\prod\limits_j|\frac{w_1^{(\alpha)}}{w_1-c_{1j}}|$ is product of $i_{1\alpha}$ factors,$\prod\limits_j(|\frac{1}{w_1-c_{1j}}|)^+$ is product of $k-\lambda_i-i_{10}$ factors.

So

$$\begin{aligned}m(r,\frac{\Omega_1}{\prod\limits_{j=1}^{k}(w_1-c_{1j})})\leq\ & \sum_{j=1}^{k}m(r,\tfrac{1}{w_1-c_{1j}})+\lambda_2m(r,w_2)+\sum_{(i)}m(r,a_{(i)})\\ +\ & O\{\sum\sum m(r,\tfrac{w_1^{(\alpha)}}{w_1-c_{1j}})\}+O\{\sum m(r,\tfrac{w_2^{(\alpha)}}{w_2})\}\end{aligned}\tag{8}$$

$$m(r,\frac{Q_k(z,w_1)}{\prod\limits_{j=1}^{k}(w_1-c_{1j})})\leq\sum_{j=1}^{k}m(r,\frac{1}{w_1-c_{1j}})+\sum_{j=1}^{k}m(r,H_j)+O(1).\tag{9}$$

By (7),(8),(9) and logarithmic derivative lemma,we have

$$\begin{aligned}m(r,\varphi_k)\leq\ & 2\sum_{j=1}^{k}m(r,\tfrac{1}{w_1-c_{1j}})+\lambda_2m(r,w_2)+\sum_{(i)}m(r,a_{(i)})\\ & +\sum m(r,H_j)+S(r,w_1)+S(r,w_2),\end{aligned}\tag{10}$$

where $S(r,w_i)=O\{\log(rT(r,w_i))\}$. Moreover,

$$\varphi_{k+1}(w_1-c_{1,k+1})=\frac{\Omega_1}{\prod_{j=1}^{k}(w_1-c_{1j})}+\sum_{j=1}^{k+1}\bar{c}_{1j}\frac{(w_1-c_{1,k+1})H_j(z)}{w_1-c_{1j}}.$$

Note that

$$\begin{aligned}|\tfrac{w_1(z)-c_{1,k+1}}{w_1(z)-c_{1j}}|\leq & \ 1+\tfrac{|c_{1j}|}{|w_1-c_{1j}|}\leq(1+|c_{1,k+1}-c_{1j}|)(\tfrac{1}{|w_1-c_{1j}|})^+\\ \leq & \ c(\tfrac{1}{|w_1-c_{1j}|})^+,\end{aligned}$$

and

$$\begin{aligned}m(r,\varphi_{k+1}(w_1-c_{1,k+1}))\leq & \ m(r,\frac{\Omega_1}{\prod_{j=1}^{k}(w_1-c_{1j})})\\ & +m(r,\sum_{j=1}^{k+1}\bar{c}_{1j}\frac{(w_1-c_{1,k+1})H_j(z)}{w_1-c_{1j}})+O(1).\end{aligned}$$

Similarly,
(11)

$$\begin{aligned}m(r,\varphi_{k+1}(w_1-c_{1,k+1}))\leq & \ 2\sum_{j=1}^{k}m(r,\tfrac{1}{w_1-c_{1j}})+\lambda_2 m(r,w_2)+\sum_{(i)}m(r,a_{(i)})\\ & +\sum_{j=1}^{k+1}m(r,H_j)+S(r,w_1)+S(r,w_2).\end{aligned}$$

Now we estimate $N(r,\varphi_{k+1})$ and $N(r,(w_1-c_{1,k+1})\varphi_{k+1})$.

At first,the poles of φ_{k+1} may arise from the following cases:

(i): The poles of $\{a_{(i)}(z)\}$,whose contribution to $N(r,\varphi_{k+1})$ is $\sum N(r,a_{(i)})$.

(ii): The poles of $\{H_j(z)\}$,whose contribution to $N(r,\varphi_{k+1})$ is $\sum N(r,H_j)$.

(iii): The zeroes of w_1-c_{1j} but not the cases (i) and (ii). According to the above discussion,each zero with multiplicity τ_j are the poles of φ_{k+1} with multiplicity at most τ_j-1,thus, its contribution is at most $\sum_{j=1}^{k+1}N_1(r,\frac{1}{w_1-c_{1j}})$,where $N_1(r,\frac{1}{w_1-c_{1j}})$ is the count function of zeros of w_1-c_{1j} and the zeroes with multiplicity τ_j count only τ_j-1 times.

(iv): The poles of w_1 but not any pole of w_2.In this case,if z_0 is a pole of w_1 with multiplicity τ,then it is the poles of the denominator of $\varphi_{\Delta_1+1}(z)$ with multiplicity $(\Delta_1+1)\tau$,but z_0 is at most the poles of Ω_1 and $Q_{\Delta_1+1}(z,w_1)$ of the numerator of $\varphi_{\Delta_1+1}(z)$, hence,z_0 is a zero of $\varphi_{\Delta_1+1}(z)$,it follows that the poles of $w_1(z)$ do not arise from the poles of $\varphi_{\Delta_1+1}(z)$.

(v): The poles of w_2 but not any pole of w_1.In this case, its contribution to $N(r,\varphi_{k+1})$ is $\lambda_2 N(r,w_2)+u_2\overline{N}(r,w_2)$.

Form the cases (i)-(v), we have

$$
\begin{aligned}
N(r,\varphi_{k+1}) \le\ & \sum_{j=1}^{k+1} N_1(r,\tfrac{1}{w_1-c_{1j}}) + \lambda_2 N(r,w_2) + u_2\overline{N}(r,w_2) \\
& + \sum_{j=1}^{k+1} N_1(r,H_j) + \sum_{(i)} N(r,a_{(i)}).
\end{aligned} \tag{12}
$$

Similarly,
(13)

$$
\begin{aligned}
N(r,(w_1-c_{1,k+1})\varphi_{k+1}) \le\ & \sum_{j=1}^{k} N_1(r,\tfrac{1}{w_1-c_{1j}}) + \lambda_2 N(r,w_2) + u_2\overline{N}(r,w_2) \\
& + \sum_{j=1}^{k+1} N_1(r,H_j) + \sum_{(i)} N(r,a_{(i)}).
\end{aligned}
$$

Combining (10),(11),(12) and (13),we obtain
(14)

$$
\begin{aligned}
T(r,w_1) \le\ & 4\sum_{j=1}^{k+1} m(r,\tfrac{1}{w_1-c_{1j}}) + 2\sum_{j=1}^{k+1} N_1(r,\tfrac{1}{w_1-c_{1j}}) + 2\lambda_2 N(r,w_2) \\
& + 2u_2\overline{N}(r,w_2) + \sum_{j=1}^{k+1} N_1(r,H_j) + \sum_{(i)} T(r,a_{(i)}) + S(r,w_1) + S(r,w_2).
\end{aligned}
$$

We choose 9 systems which differ from each other $\{c_{1j}\}(j=1,2,\ldots,9(k+1))$,apply the inequality (14) to every system, combining the above 13 inequlities,we deduce

$$
\begin{aligned}
9T(r,w_1) \le\ & 4\sum_{j=1}^{9(k+1)} m(r,\tfrac{1}{w_1-c_{1j}}) + 2\sum_{j=1}^{9(k+1)} N_1(r,\tfrac{1}{w_1-c_{1j}}) + 18\lambda_2 T(r,w_2) \\
& + 18u_2\overline{N}(r,w_2) + 2\sum_{j=1}^{9(k+1)} N_1(r,H_j) + 18\sum_{(i)} T(r,a_{(i)}) \\
& + S(r,w_1) + S(r,w_2).
\end{aligned}
$$

By the second fundamental theorem of Nevanlinna,we have

$$
\begin{aligned}
9T(r,w_1) \le\ & 8T(r,w_1) + 18\lambda_2 T(r,w_2) + 18u_2\overline{N}(r,w_2) + 2\sum_{j=1}^{9(k+1)} N_1(r,H_j) \\
& + 18\sum_{(i)} T(r,a_{(i)}) + S(r,w_1) + S(r,w_2),
\end{aligned}
$$

i.e.

$$
\begin{aligned}
T(r,w_1) \le\ & 18\lambda_2 T(r,w_2) + 18u_2\overline{N}(r,w_2) + 2\sum_{j=1}^{9(k+1)} N_1(r,H_j) \\
& + 18\sum_{(i)} T(r,a_{(i)}) + S(r,w_1) + S(r,w_2), (k \ge \Delta_1).
\end{aligned} \tag{15}
$$

In a similar fashion,we have for the second equation of (1),

$$(16)\qquad \begin{aligned} T(r,w_2) \leq \; & 18\overline{\lambda}_1 T(r,w_1) + 18\overline{u}_1 \overline{N}(r,w_1) + 2\sum_{j=1}^{9(l+1)} N_1(r,H_j) \\ & +18\sum_{(j)} T(r,b_{(j)}) + S(r,w_1) + S(r,w_2), (l \geq \overline{\Delta}_2). \end{aligned}$$

If w_1 is admissible,w_2 is non-admissible,by the inequality (15) and $\limsup\limits_{r\to\infty} \frac{T(r,w_2)}{T(r,w_1)} = 0$ we have

$$1 \leq 0.$$

This is a contradiction.

In a similar way, we can prove the case that w_1 is admissible,w_2 is non-admissible.

This completes the proof of Theorem 1.

Acknowledgements I should like to express my greatitude to Prof. Xiao Xiuzhi for several comments and discussions.

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Received by the editors November 2, 1997.