NOVI SAD J. MATH.
VOL. 29, NO. 1, 1999, 55-59

A NOTE ON n-GROUPS FOR $n \geq 3$

Janez Ušan[1]

Abstract. A part of Theorem 1.4 in [2] is the following Proposition: Let $n \geq 3$ and let (Q, A) be an n-semigroup [:1.2]. Then (Q, A) is an n-group [:1.2] iff for arbitrary $i \in \{2, \ldots, n-1\}$ for every $a_1^n \in Q$ [:1.1] there is exactly one $x \in Q$ such that the following equality holds $A(a_1^{i-1}, x, a_i^{n-1}) = a_n$ [:1.1]. In the present paper the following proposition is proved: Let $n \geq 3$ and let (Q, A) be an n-groupoid. Then (Q, A) is an n-group iff for an arbitrary $i \in \{2, \ldots, n-1\}$ the following condition hold: (a) the $\langle i-1, i\rangle$-associative law holds in (Q, A); (b) the $\langle i, i+1\rangle$-associative law holds in (Q, A); and (c) for every $a_1^n \in Q$ there is exactly one $x \in Q$ such that the following equality holds $A(a_1^{i-1}, x, a_i^{n-1}) = a_n$. In addition, for $n = 3$ [$i = 2$] the conditions (a) and (b) are equivalent to the condition that (Q, A) is a 3-semigroup [:1.2].

AMS Mathematics Subject Classification(1991): 20N15

Key words and phrases: n-groupoids, n-semigroups, n-quasigroups, n-groups.

1. Preliminaries

1.1 About the expression a_p^q

Let $p \in \mathbf{N}$, $q \in \mathbf{N} \cup \{0\}$ and let a be a mapping of the set $\{i |\ i \in \mathbf{N} \wedge i \geq p \wedge i \leq q\}$ into the set S; $\emptyset \notin S$. Then:

$$a_p^q \text{ stands for } \begin{cases} a_p, \ldots, a_q; & p < q \\ a_p; & p = q \\ \text{empty sequence } (= \emptyset); & p > q. \end{cases}$$

Besides, in some situations *instead of* a_p^q *we write* $(a_i)_{i=p}^q$ [briefly: $(a_i)_p^q$]. For example:

$$(\forall x_i \in Q)_1^q$$

for $q > 1$ stands for

$$\forall x_1 \in Q \ldots \forall x_q \in Q$$

[usually, we write: $(\forall x_1 \in Q) \ldots (\forall x_2 \in Q)$], for $q = 1$ it stands for

$$\forall x_1 \in Q$$

[1]Institute of Mathematics, University of Novi Sad Trg D. Obradovića 4, 21000 Novi Sad, Yugoslavia

[usually, we write: $(\forall x_1 \in Q)$], and for $q = 0$ it stands for an empty sequence $(= \emptyset)$.

In *some cases*, instead of a_p^q only, we write: sequence a_p^q (sequence a_p^q over a set S). For example: ... for every sequence a_p^q over a set S ... . And if $p \leq q$, we usually write: $a_p^q \in S$.

1.2 On n-group

Let $n \geq 2$ and let (Q, A) be an n-groupoid. Then: (a) we say that (Q, A) is an n-**semigroup** iff for every $i, j \in \{1, \dots, n\}$, $i < j$, the following law holds

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-1})$$

[: $\langle i, j\rangle$-**associative law**]; (b) we say that (Q, A) is an n-**quasigroup** iff for every $i \in \{1, \dots, n\}$ and for every $a_1^n \in Q$ there is exactly one $x_i \in Q$ such that the following equality holds

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n; \text{ and}$$

(c) we say that (Q, A) is a Dörnte n-**group** [briefly: n-group] iff (Q, A) is an n-semigroup and an n-quasigroup as well.

A notion of an n-group was introduced by W. Dörnte in [1] as a generalization of the notion of a group.

2. Result

Theorem: *Let $n \geq 3$ and let (Q, A) be an n-groupoid. Then the following statements are equivalent:*

(i) (Q, A) is an n-group [: 1.2]; *and*

(ii) For arbitrary $i \in \{2, \dots, n-1\}$ the following conditions hold: (a) the $\langle i-1, i\rangle$-associative law holds in (Q, A); (b) the $\langle i, i+1\rangle$-associative law holds in (Q, A); and (c) for every $a_1^n \in Q$ [: 1.1] *there is exactly one $x \in Q$ such that the following equality holds $A(a_1^{i-1}, x, a_i^{n-1}) = a_n$.*[2]

Proof. 1) $\Rightarrow$:

Let (i) holds. Then the implication (i) $\Rightarrow$ (ii) holds tautologically.

2) $\Leftarrow$:

Let (ii) holds. We prove respectively that the following propositions hold:

1° (Q, A) is an n-semigroup;

2° For every $a_1^n \in Q$ there is exactly one $x \in Q$ such that the following equality holds $A(x, a_1^{n-1}) = a_n$;

3° For every $a_1^n \in Q$ there is exactly one $y \in Q$ such that the following equality holds $A(a_1^{n-1}, y) = a_n$; and

[2] For $n = 3$ [$i = 2$] the conditions (a) and (b) are equivalent to the condition that (Q, A) is a 3-semigroup.

4° For every $a, b, c \in Q$, for every sequence a_1^{n-3} over Q and for every $j \in \{1, \ldots, n-2\}$ there is exactly one $z \in Q$ such that the following equality holds $A(a, a_1^{j-1}, z, a_j^{n-3}, b) = c$ [3].

Proof of 1°:

a) Let i be from (ii) $[i \in \{2, \ldots, n-1\}]$ and let $k \in \mathbf{N}$ satisfying

$$i \leq k < n - 1. \tag{1}$$

In addition, suppose that the $\langle k, k+1 \rangle$-associative law holds in (Q, A) [for $k = i$ it holds: (b)]. Then, by (b) and (c), we conclude that for every $a_1^{2n-1}, b_1^{n-1} \in Q$ the following sequence of implications holds:

$$\begin{aligned}
&A(a_1^{k-1}, A(a_k^{k+n-1}), a_{k+n}^{2n-1}) = A(a_1^k, A(a_{k+1}^{k+n}), a_{k+n+1}^{2n-1}) \Rightarrow \\
&A(b_1^i, A(a_1^{k-1}, A(a_k^{k+n-1}), a_{k+n}^{2n-1}), b_{i+1}^{n-1}) = \\
&\qquad A(b_1^i, A(a_1^k, A(a_{k+1}^{k+n}), a_{k+n+1}^{2n-1}), b_{i+1}^{n-1}) \Rightarrow \\
&A(b_1^{i-1}, A(b_i, a_1^{k-1}, A(a_k^{k+n-1}), a_{k+n}^{2n-2}), a_{2n-1}, b_{i+1}^{n-1}) = \\
&\qquad A(b_1^{i-1}, A(b_i, a_1^k, A(a_{k+1}^{k+n}), a_{k+n+1}^{2n-2}), a_{2n-1}, b_{i+1}^{n-1}) \Rightarrow \\
&A(b_i, a_1^{k-1}, A(a_k^{k+n-1}), a_{k+n}^{2n-2}) = A(b_i, a_1^k, A(a_{k+1}^{k+n}), a_{k+n+1}^{2n-2}),
\end{aligned}$$

whence we conclude that: if the $\langle k, k+1 \rangle$-associative law holds in (Q, A) and $k \in \mathbf{N}$ satisfies (1), then also the $\langle k+1, k+2 \rangle$-associative law holds in (Q, A).

b) Let i be from (ii) $[i \in \{2, \ldots, n-1\}]$ and let $l \in \mathbf{N}$ satisfies the following condition

$$2 < l \leq i. \tag{2}$$

In addition, suppose that the $\langle l-1, l \rangle$-associative law holds in (Q, A) [for $l = i$ it holds: (a)]. Then, by (a) and (c), we conclude that for every $a_1^{2n-1}, b_1^{n-1} \in Q$ the following sequence of implications holds:

$$\begin{aligned}
&A(a_1^{l-2}, A(a_{l-1}^{l+n-2}), a_{l+n-1}^{2n-1}) = A(a_1^{l-1}, A(a_l^{l+n-1}), a_{l+n}^{2n-1}) \Rightarrow \\
&A(b_1^{i-2}, A(a_1^{l-2}, A(a_{l-1}^{l+n-2}), a_{l+n-1}^{2n-1}), b_{i-1}^{n-1}) = \\
&\qquad A(b_1^{i-2}, A(a_1^{l-1}, A(a_l^{l+n-1}), a_{l+n}^{2n-1}), b_{i-1}^{n-1}) \Rightarrow \\
&A(b_1^{i-2}, a_1, A(a_2^{l-2}, A(a_{l-1}^{l+n-2}), a_{l+n-1}^{2n-1}, b_{i-1}), b_i^{n-1}) = \\
&\qquad A(b_1^{i-2}, a_1, A(a_2^{l-1}, A(a_l^{l+n-1}), a_{l+n}^{2n-1}, b_{i-1}), b_i^{n-1}) \Rightarrow \\
&A(a_2^{l-2}, A(a_{l-1}^{l+n-2}), a_{l+n-1}^{2n-1}, b_{i-1}) = A(a_2^{l-1}, A(a_l^{l+n-1}), a_{l+n}^{2n-1}, b_{i-1}),
\end{aligned}$$

whence we conclude that: if the $\langle l-1, l \rangle$-associative law holds in (Q, A) and $l \in \mathbf{N}$ satisfies (2), then also the $\langle l-2, l-1 \rangle$-associative law holds in (Q, A).

[3]By (c) from (ii), this holds starting with $n \geq 4$.

Proof of 2°:

$\bar{a}$) By the fact that the sequences c_1^{i-1} and c_i^{n-1} over Q are not empty [$:i \in \{2,\ldots,n-1\}$; 1.1] and also by 1° and (c) [from (ii)], we conclude that for all $x, y, a_1^{n-1}, c_1^{n-1} \in Q$ the following sequence of implications holds:

$$\begin{aligned}
&A(x, a_i^{n-2}, a_{n-1}, a_1^{i-1}) = A(y, a_i^{n-2}, a_{n-1}, a_1^{i-1}) \Rightarrow \\
&A(c_1^{i-1}, A(x, a_i^{n-2}, a_{n-1}, a_1^{i-1}), c_i^{n-1}) = \\
&\qquad A(c_1^{i-1}, A(y, a_i^{n-2}, a_{n-1}, a_1^{i-1}), c_i^{n-1}) \Rightarrow \\
&A(c_1^{i-1}, x, a_i^{n-2}, A(a_{n-1}, a_1^{i-1}, c_i^{n-1})) = \\
&\qquad A(c_1^{i-1}, y, a_i^{n-2}, A(a_{n-1}, a_1^{i-1}, c_i^{n-1})) \Rightarrow \\
&x = y,
\end{aligned}$$

whence we conclude that the following statement holds

(3) $\quad (\forall a_i \in Q)_1^{n-1}(\forall x \in Q)(\forall y \in Q)(A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Rightarrow x = y),$

i.e., taking into account the monotonicity, that also the following statement holds:

(4) $\quad (\forall a_i \in Q)_1^{n-1}(\forall x \in Q)(\forall y \in Q)(A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Leftrightarrow x = y).$

Further on, by (4) we conclude that for all $x, a_1^n, c_1^{n-1} \in Q$ the following equivalence holds

$$\begin{aligned}
&A(x, a_i^{n-1}, a_1^{i-1}) = a_n \Leftrightarrow \\
&A(c_1^{i-1}, x, a_i^{n-2}, A(a_{n-1}, a_1^{i-1}, c_i^{n-1})) = A(c_1^{i-1}, a_n, c_i^{n-1}),
\end{aligned}$$

whence by (c) [from (ii)] we conclude that for every $a_1^n \in Q$ there is at **least one** $x \in Q$ such that the following equality holds

(5) $$A(x, a_1^{n-1}) = a_n$$

[c_1^{n-1} over Q is arbitrary]. In addition, by (3), the equation (5) over the unknown x for every $a_1^n \in Q$ has **at most one** solution. Hence: the statement 2° holds.

Similarly, it is possible to prove the statement 3°.

Proof of 4°:

Let $j \in \{2,\ldots,n-1\}$. Then by 1°-3°, we conclude that for every $x, y, a_1^{n-1}, c_1^{n-1} \in Q$ the following sequence of implications holds:

$$\begin{aligned}
&A(a_1^{j-1}, x, a_j^{n-1}) = A(a_1^{j-1}, y, a_j^{n-1}) \Rightarrow \\
&A(c_j^{n-1}, A(a_1^{j-1}, x, a_j^{n-1}), c_1^{j-1}) = A(c_j^{n-1}, A(a_1^{j-1}, y, a_j^{n-1}), c_1^{j-1}) \Rightarrow \\
&A(A(c_j^{n-1}, a_1^{j-1}, x), a_j^{n-1}, c_1^{j-1}) = A(A(c_j^{n-1}, a_1^{j-1}, y), a_j^{n-1}, c_1^{j-1}) \Rightarrow \\
&A(c_j^{n-1}, a_1^{j-1}, x) = A(c_j^{n-1}, a_1^{j-1}, y) \Rightarrow \\
&x = y,
\end{aligned}$$

whence we conclude that for an arbitrary $j \in \{2, \ldots, n-1\}$ the following statement holds

$$(6) \qquad (\forall a_i \in Q)_1^{n-1}(\forall x \in Q)(\forall y \in Q)(A(a_1^{j-1}, x, a_j^{n-1}) = A(a_1^{j-1}, y, a_j^{n-1}) \Rightarrow x = y).$$

whence, taking into account the monotonicity, we conclude that for every $t \in \{2, \ldots, n-1\}$ the following formula holds:

$$(7) \qquad (\forall a_i \in Q)_1^{n-1}(\forall x \in Q)(\forall y \in Q)(A(a_1^{t-1}, x, a_t^{n-1}) = A(a_1^{t-1}, y, a_t^{n-1}) \Leftrightarrow x = y).^4$$

Further on, by (7) and by 1°, we conclude that for an arbitrary $j \in \{2, \ldots, n-1\}$, for every $x, a_1^n, c_1^{n-1} \in Q$ the following sequence of equivalences holds:

$$A(a_1^{j-1}, x, a_j^{n-1}) = a_n \Leftrightarrow$$
$$A(c_j^{n-1}, A(a_1^{j-1}, x, a_j^{n-1}), c_1^{j-1}) = A(c_j^{n-1}, a_n, c_1^{j-1}) \Leftrightarrow$$
$$A(A(c_j^{n-1}, a_1^{j-1}, x), a_j^{n-1}, c_1^{j-1}) = A(c_j^{n-1}, a_n, c_1^{j-1}),$$

i.e., the following equivalences

$$A(a_1^{j-1}, x, a_j^{n-1}) = a_n \Leftrightarrow$$
$$A(A(c_j^{n-1}, a_1^{j-1}, x), a_j^{n-1}, c_1^{j-1}) = A(c_j^{n-1}, a_n, c_1^{j-1}),$$

whence, by 2° and 3°, we conclude that for every $a_1^n \in Q$ there is at least one $x \in Q$ such that the following equality holds

$$(8) \qquad A(a_1^{j-1}, x, a_j^{n-1}) = a_n.$$

In addition, since (6) holds for any $j \in \{2, \ldots, n-1\}$, the equation (8) over the unknown x, for every $a_1^n \in Q$ has at **most one** solution. Hence: for every $j \in \{2, \ldots, n-1\}$, for every $a_1^n \in Q$ there is **exactly one** $x \in Q$ such that the following equality holds $A(a_1^{j-1}, x, a_j^{n-1}) = a_n$.

References

[1] Dörnte, W., Untersuchengen über einen verallgemeinerten Gruppenbegriff, Math. Z., 29 (1928), 1-19.

[2] Monk, J.D., Sioson, F.M., On the general theory of m-groups, Fund. Math. 72 (1971), 233-244.

Received by the editors October 23, 1997.

[4] Since $t \in \{2, \ldots, n-1\}$ and $n \geq 3$, the sequences a_1^{t-1} and a_t^{n-1} over Q are nonvoid.