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# ONE VERSION OF MIRON'S GEOMETRY IN $Osc^3M$

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**Abstract.** R. Miron and Gh. Atanasiu in [15], [16], [17] studied the geometry of $Osc^kM$. Among many various problems they solved the authors introduced the adapted basis and $d$-connection and gave its curvature theory. Different structures as almost product structure and metric structure were determined.

Here, the attention is restricted onto the variational problem and integrability conditions on $E = Osc^3M$, and the transformation group is slightly different from that used in [15]. This resulted in a different theory.

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## 1. Adapted basis in $T(Osc^3M)$ and $T^*(Osc^3M)$

Let $E = Osc^3M$ be a $4n$ dimensional $C^\infty$ manifold. In a local chart $(U, \varphi)$ a point $u \in E$ has the coordinates

$$(x^a, y^{1a}, y^{2a}, y^{3a}) = (y^{0a}, y^{1a}, y^{2a}, y^{3a}) = (y^{\alpha a}),$$

where $x^a = y^{0a}$ and

$$a, b, c, d, e, \ldots = 1, 2, \ldots, n, \quad \alpha, \beta, \gamma, \delta, \kappa, \ldots = 0, 1, 2, 3.$$

If in some other chart $(U', \varphi')$ the point $u \in E$ has the coordinates $(x^{a'}, y^{1a'}, y^{2a'}, y^{3a'})$, then in $U \cap U'$ the allowable coordinate transformations are given by:

$$\text{(1.1)} \quad \begin{aligned} &\text{(a)} \quad x^{a'} = x^{a'}(x^1, x^2, \ldots, x^n) \\ &\text{(b)} \quad y^{1a'} = \frac{\partial x^{a'}}{\partial x^a} y^{1a} = \frac{\partial y^{0a'}}{\partial y^{0a}} y^{1a} \\ &\text{(c)} \quad y^{2a'} = \frac{\partial y^{1a'}}{\partial y^{0a}} y^{1a} + \frac{\partial y^{1a'}}{\partial y^{1a}} y^{2a} \\ &\text{(d)} \quad y^{3a'} = \frac{\partial y^{2a'}}{\partial y^{0a}} y^{1a} + \frac{\partial y^{2a'}}{\partial y^{1a}} y^{2a} + \frac{\partial y^{2a'}}{\partial y^{2a}} y^{3a}. \end{aligned}$$

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A nice example of the space $E$ can be obtained if the points $(x^a) \in M$ $(dim M = n)$ are considered as the points of the curve $x^a = x^a(t)$ and $y^{\alpha a}$, $\alpha = 1, 2, 3$, are defined by

$$y^{1a} = \frac{dx^a}{dt},\ y^{2a} = \frac{d^2x^a}{dt^2} = \frac{dy^{1a}}{dt},\ y^{3a} = \frac{d^3x^a}{dt^3} = \frac{dy^{2a}}{dt}.$$

$M$ is the base manifold and $(x^a) \in M$ is the projection of $(x^a, y^{1a}, y^{2a}, y^{3a}) \in E$ on $M$. In [15], [16] $y^{\alpha a} = \frac{1}{\alpha!}\frac{d^\alpha x^a}{dt^\alpha}$, $\alpha = 1, \dots, k$ and the transformations (1.1) have different form. If in $U \cap U'$ the equation

$$x^{a'} = x^{a'}(x^1(t), x^2(t), \dots, (x^n(t))$$

is valid, then it is easy to see that

$$\begin{aligned} y^{1a'} &= \frac{dx^{a'}}{dt} = y^{1a'}(x^a, y^{1a}), \\ y^{2a'} &= \frac{dy^{1a'}}{dt} = y^{2a'}(x^a, y^{1a}, y^{2a}), \\ y^{3a'} &= \frac{dy^{2a'}}{dt} = y^{3a'}(x^a, y^{1a}, y^{2a}, y^{3a}), \end{aligned} \tag{1.2}$$

satisfy (1.1b), (1.1c) and (1.1d) respectively and the explicite form of (1.1) is the following:

$$\begin{aligned} x^{a'} &= x^{a'}(x^1, x^2, \dots, x^n) \\ y^{1a'} &= \frac{\partial x^{a'}}{\partial x^a} y^{1a}, \\ y^{2a'} &= \frac{\partial^2 x^{a'}}{\partial x^a \partial x^b} y^{1a} y^{1b} + \frac{\partial x^{a'}}{\partial x^a} y^{2a}, \\ y^{3a'} &= \frac{\partial^3 x^{a'}}{\partial x^a \partial x^b \partial x^c} y^{1a} y^{1b} y^{1c} + 3 \frac{\partial^2 x^{a'}}{\partial x^a \partial x^b} y^{1a} y^{2b} + \frac{\partial x^{a'}}{\partial x^a} y^{3a}. \end{aligned} \tag{1.3}$$

**Theorem 1.1.** *The transformations determined by (1.1) form a group.*

By determining of the group of allowable coordinate transformations the first step in constructing a geometry is made. The second important step is the construction of the adapted basis in $T(E)$, which depends on the choice of the coefficients of the nonlinear connections, here denoted by $N$ and $M$.

The following abbreviations

$$\partial_{\alpha a} = \frac{\partial}{\partial y^{\alpha a}},\ \alpha = 1, 2, 3, \text{ and } \partial_a = \partial_{0a} = \frac{\partial}{\partial x^a} = \frac{\partial}{\partial y^{0a}}$$

will be used. From (1.3) it follows

$$(1.4)\quad \partial_{0a}y^{0a'} = \partial_{1a}y^{1a'} = \partial_{2a}y^{2a'} = \partial_{3a}y^{3a'} = \frac{\partial x^{a'}}{\partial x^a} = A_a^{a'},$$

$$\frac{dA_a^{a'}}{dt} = \partial_{0a}y^{1a'} = \frac{1}{2}\partial_{1a}y^{2a'} = \frac{1}{2}\frac{2}{3}\partial_{2a}y^{3a'} = \frac{\partial^2 x^{a'}}{\partial x^a \partial x^b}y^{1b} = B_a^{a'},$$

$$\frac{dB_a^{a'}}{dt} = \partial_{0a}y^{2a'} = \frac{1}{3}\partial_{1a}y^{3a'} = \frac{\partial^3 x^{a'}}{\partial x^a \partial x^b \partial x^c}y^{1b}y^{1c} + \frac{\partial^2 x^{a'}}{\partial x^a \partial x^b}y^{2b} = C_a^{a'},$$

$$\frac{dC_a^{a'}}{dt} = \partial_{0a}y^{3a'} = D_a^{a'}.$$

The natural basis $\bar{B}$ of $T(E)$ is

$$(1.5)\quad \bar{B} = \{\partial_{0a}, \partial_{1a}, \partial_{2a}, \partial_{3a}\} = \{\partial_{\alpha a}\}$$

The elements of $\bar{B}$ with respect to (1.1) are not transformed as $d$-tensors. They satisfy the following relations:

$$\begin{aligned}
\partial_{0a} &= (\partial_{0a}y^{0a'})\partial_{0a'} + (\partial_{0a}y^{1a'})\partial_{1a'} + (\partial_{0a}y^{2a'})\partial_{2a'} + (\partial_{0a}y^{3a'})\partial_{3a'}\\
\partial_{1a} &= (\partial_{1a}y^{1a'})\partial_{1a'} + (\partial_{1a}y^{2a'})\partial_{2a'} + (\partial_{1a}y^{3a'})\partial_{3a'}\\
\partial_{2a} &= (\partial_{2a}y^{2a'})\partial_{2a'} + (\partial_{2a}y^{3a'})\partial_{3a'}\\
\partial_{3a} &= (\partial_{3a}y^{3a'})\partial_{3a'}.
\end{aligned}$$

(1.6)

The natural basis $\bar{B}^*$ of $T^*(E)$ is

$$(1.7)\quad \bar{B}^* = \{dx^a, dy^{1a}, dy^{2a}, dy^{3a}\} = \{dy^{\alpha a}\}.$$

The elements of $\bar{B}^*$ with respect to (1.1) are transformed in the following way (see (1.2)):

$$(1.8)\quad dx^{a'} = \frac{\partial x^{a'}}{\partial x^a}dx^a \Leftrightarrow dy^{0a'} = (\partial_{0a}y^{0a'})dy^{0a}$$

$$dy^{1a'} = (\partial_{0a}y^{1a'})dy^{0a} + (\partial_{1a}y^{1a'})dy^{1a}$$

$$dy^{2a'} = (\partial_{0a}y^{2a'})dy^{0a} + (\partial_{1a}y^{2a'})dy^{1a} + (\partial_{2a}y^{2a'})dy^{2a}$$

$$dy^{3a'} = (\partial_{0a}y^{3a'})dy^{0a} + (\partial_{1a}y^{3a'})dy^{1a} + (\partial_{2a}y^{3a'})dy^{2a} + (\partial_{3a}y^{3a'})dy^{3a}.$$

The adapted basis $B^*$ of $T^*(E)$ is given by:

$$(1.9)\quad B^* = \{\delta y^{0a}, \delta y^{1a}, \delta y^{2a}, \delta y^{3a}\},$$

where

$$(1.10)\quad \delta y^{0a} = dx^a = dy^{0a}$$

$$\delta y^{1a} = dy^{1a} + M^{1a}_{0b} dy^{0b}$$
$$\delta y^{2a} = dy^{2a} + M^{2a}_{1b} dy^{1b} + M^{2a}_{0b} dy^{0b}$$
$$\delta y^{3a} = dy^{3a} + M^{3a}_{2b} dy^{2b} + M^{3a}_{1b} dy^{1b} + M^{3a}_{0b} dy^{0b}.$$

**Theorem 1.2.** *The necessary and sufficient conditions that $\delta y^{\alpha a}$ are transformed as d-tensor field, i.e.*

$$\delta y^{\alpha a'} = \frac{\partial x^{a'}}{\partial x^a} \delta y^{\alpha a}, \ \alpha = 0, 1, 2, 3,$$

*are the following equations:*

$$\begin{aligned}
&(1.11)(a) \quad M^{1a}_{0b} \partial_{1a} y^{1a'} = M^{1a'}_{0b'} \partial_{0b} y^{0b'} + \partial_{0b} y^{1a'} \\
&(b) \quad M^{2a}_{1b} \partial_{2a} y^{2a'} = M^{2a'}_{1c'} \partial_{1b} y^{1c'} + \partial_{1b} y^{2a'} \\
&(c) \quad M^{2a}_{0b} \partial_{2a} y^{2a'} = M^{2a'}_{0c'} \partial_{0b} y^{0c'} + M^{2a'}_{1c'} \partial_{0b} y^{1c'} + \partial_{0b} y^{2a'} \\
&(d) \quad M^{3a}_{2b} \partial_{3a} y^{3a'} = M^{3a'}_{2c'} \partial_{2b} y^{2c'} + \partial_{2b} y^{3a'} \\
&(e) \quad M^{3a}_{1b} \partial_{3a} y^{3a'} = M^{3a'}_{1c'} \partial_{1b} y^{1c'} + M^{3a'}_{2c'} \partial_{1b} y^{2c'} + \partial_{1b} y^{3a'} \\
&(f) \quad M^{3a}_{0b} \partial_{3a} y^{3a'} = M^{3a'}_{0c'} \partial_{0b} y^{0c'} + M^{3a'}_{1c'} \partial_{0b} y^{1c'} + M^{3a'}_{2c'} \partial_{0b} y^{2c'} + \partial_{0b} y^{3a'}.
\end{aligned}$$

From (1.11) and (1.4) it follows that (1.11) is a system in which equations of second, third and fourth order appeared, so there are infinity functions

$$\begin{aligned}
(1.12) \qquad & M^{1a}_{0b} = M^{1a}_{0b}(x, y^1), \ M^{2a}_{1b} = M^{2a}_{1b}(x, y^1), \ M^{3a}_{2b} = M^{3a}_{2b}(x, y^1), \\
& M^{2a}_{0b} = M^{2a}_{0b}(x, y^1, y^2), \ M^{3a}_{1b} = M^{3a}_{1b}(x, y^1, y^2), \\
& M^{3a}_{0b} = M^{3a}_{0b}(x, y^1, y^2, y^3),
\end{aligned}$$

which are the solutions of (1.11). From the choice of $M$ depends the adapted basis $B^*$ ((1.9)).

Let us denote the adapted basis of $T(E)$ by $B$, where

$$(1.13) \qquad B = \{\delta_{0a}, \delta_{1a}, \delta_{2a}, \delta_{3a}\} = \{\delta_{\alpha a}\},$$

and

$$(1.14) \qquad \begin{aligned}
\delta_{0a} &= \partial_{0a} - N^{1b}_{0a} \partial_{1b} - N^{2b}_{0a} \partial_{2b} - N^{3b}_{0a} \partial_{3b}, \\
\delta_{1a} &= \partial_{1a} - N^{2b}_{1a} \partial_{2b} - N^{3b}_{1a} \partial_{3b} \\
\delta_{2a} &= \partial_{2a} - N^{3b}_{2a} \partial_{3b} \\
\delta_{3a} &= \partial_{3a}.
\end{aligned}$$

**Theorem 1.3.** *The necessary and sufficient conditions that $B$ ((1.13)) be dual to $B^*$ ((1.9)), (when $\bar{B}$ ((1.5)) is dual to $\bar{B}^*$ ((1.7)) i.e.*

$$< \delta_{\alpha a} \delta y^{\beta b} >= \delta^\beta_\alpha \delta^b_a$$

*are the following relations:*

(1.15)
$$\begin{aligned} N^{1b}_{0a} &= M^{1b}_{0a} \\ N^{2b}_{0a} &= M^{2b}_{0a} - M^{2b}_{1c}N^{1c}_{0a} \\ N^{3b}_{0a} &= M^{3b}_{0a} - M^{3b}_{1c}N^{1c}_{0a} - M^{3b}_{2c}N^{2c}_{0a} \\ N^{2b}_{1a} &= M^{2b}_{1a} \\ N^{3b}_{1a} &= M^{3b}_{1a} - M^{3b}_{2c}N^{2c}_{1a} \\ N^{3b}_{2a} &= M^{3b}_{2a}, \end{aligned}$$

*or equivalently*

(1.16)
$$\begin{aligned} M^{1b}_{0a} &= N^{1b}_{0a} \\ M^{2b}_{0a} &= N^{2b}_{0a} + N^{2b}_{1c}N^{1c}_{0a} \\ M^{3b}_{0a} &= N^{3b}_{0a} + N^{3b}_{1c}N^{1c}_{0a} + N^{3b}_{2c}N^{2c}_{0a} + N^{3b}_{2d}N^{2d}_{1c}N^{1c}_{0a} \\ M^{2b}_{1a} &= N^{2b}_{1a} \\ M^{3b}_{1a} &= N^{3b}_{1a} + N^{3b}_{2c}N^{2c}_{1a} \\ M^{3b}_{2a} &= N^{3b}_{2a}. \end{aligned}$$

From (1.15) and (1.14) it follows

**Theorem 1.4.** *The necessary and sufficient conditions that $\delta_{\alpha a}$ with respect to (1.1) are transformed as d-tensors, i.e.*

(1.17)
$$\delta_{\alpha a'} = \frac{\partial x^a}{\partial x^{a'}}\delta_{\alpha a}, \ \alpha = 0, 1, 2, 3,$$

*are the following formulae:*

(1.18)
$$\begin{aligned} N^{1b'}_{0a'}\partial_{0a}y^{0a'} &= N^{1c}_{0a}\partial_{1c}y^{1b'} - \partial_{0a}y^{1b'} \\ N^{2b'}_{0a'}\partial_{0a}y^{0a'} &= N^{2c}_{0a}\partial_{2c}y^{2b'} + N^{1c}_{0a}\partial_{1c}y^{2b'} - \partial_{0a}y^{2b'} \\ N^{3b'}_{0a'}\partial_{0a}y^{0a'} &= N^{3c}_{0a}\partial_{3c}y^{3b'} + N^{2c}_{0a}\partial_{2c}y^{3b'} + N^{1c}_{0a}\partial_{1c}y^{3b'} - \partial_{0a}y^{3b'} \\ N^{2b'}_{1a'}\partial_{1a}y^{1a'} &= N^{2c}_{1a}\partial_{2c}y^{2b'} - \partial_{1a}y^{2b'} \\ N^{3b'}_{1a'}\partial_{1a}y^{1a'} &= N^{3c}_{1a}\partial_{3c}y^{3b'} + N^{2c}_{1a}\partial_{2c}y^{3b'} - \partial_{1a}y^{3b'} \\ N^{3b'}_{2a'}\partial_{2a}y^{2a'} &= N^{3b}_{2a}\partial_{3b}y^{3b'} - \partial_{2a}y^{3b'}. \end{aligned}$$

From (1.13) and (1.14) it follows

(1.19)
$$\begin{aligned} \partial_{3a} &= \delta_{3a} \\ \partial_{2a} &= \delta_{2a} + M^{3b}_{2a}\delta_{3b} \\ \partial_{1a} &= \delta_{1a} + M^{2b}_{1a}\delta_{2b} + M^{3b}_{1a}\delta_{3b} \\ \partial_{0a} &= \delta_{0a} + M^{1b}_{0a}\delta_{1b} + M^{2b}_{0a}\delta_{2b} + M^{3b}_{0a}\delta_{3b}. \end{aligned}$$

From (1.12) and (1.15) it follows that

$$
\begin{aligned}
&N^{1b}_{0a} = N^{1b}_{0a}(x, y^1),\ N^{2b}_{1a} = N^{2b}_{1a}(x, y^1),\ N^{3b}_{2a} = N^{3b}_{2a}(x, y^1)\\
&N^{2b}_{0a} = N^{2b}_{0a}(x, y^1, y^2),\ N^{3b}_{1a} = N^{3b}_{1a}(x, y^1, y^2)\\
&N^{3b}_{0a} = N^{3b}_{0a}(x, y^1, y^2, y^3).
\end{aligned}
\tag{1.20}
$$

## 2. Decomposition of $T(E)$. Integrability conditions

Let us denote by $T_H$, $T_{V_1}$, $T_{V_2}$, $T_{V_3}$ the subspaces of $T(E)$ spanned by

$$\{\delta_{0a}\},\ \{\delta_{1a}\},\ \{\delta_{2a}\},\ \{\delta_{3a}\}$$

respectively. Then we have

$$T(E) = T_H \oplus T_{V_1} \oplus T_{V_2} \oplus T_{V_3}.$$

**Proposition 2.1.** *The horizontal distribution $T_H$ is integrable if all $\bar{K}_{0a}{}^{\alpha d}{}_{0b}$, $\alpha = 1, 2, 3$ determined by (2.2) are equal to zero.*

*Proof.* By direct calculation taking into account (1.20) one obtains

$$[\delta_{0a}, \delta_{0b}] = \bar{K}_{0a}{}^{1d}{}_{0b}\partial_{1d} + \bar{K}_{0a}{}^{2d}{}_{0b}\partial_{2d} + \bar{K}_{0a}{}^{3d}{}_{0b}\partial_{3d}, \tag{2.1}$$

where

$$
\begin{aligned}
\bar{K}_{0a}{}^{1d}{}_{0b} &= [(\partial_{0b} - N^{1c}_{0b}\partial_{1c})N^{1d}_{0a}] - [a/b]\\
\bar{K}_{0a}{}^{2d}{}_{0b} &= [(\partial_{0b} - N^{1c}_{0b}\partial_{1c} - N^{2c}_{0b}\partial_{2c})N^{2d}_{0a}] - [a/b]\\
\bar{K}_{0a}{}^{3d}{}_{0b} &= [(\partial_{0b} - N^{1c}_{0b}\partial_{1c} - N^{2c}_{0b}\partial_{2c} - N^{3c}_{0b}\partial_{3c})N^{3d}_{0b}] - [a/b]
\end{aligned}
\tag{2.2}
$$

In (2.1) $[\delta_{0a}, \delta_{0b}]$ is expressed in $\bar{B}$. Its components in $B$ have the form:

$$[\delta_{0a}, \delta_{0b}] = K_{0a}{}^{1d}{}_{0b}\delta_{1d} + K_{0a}{}^{2d}{}_{0b}\delta_{2d} + K_{0a}{}^{3d}{}_{0b}\delta_{3d}, \tag{2.3}$$

where

$$
\begin{aligned}
K_{0a}{}^{1d}{}_{0b} &= \bar{K}_{0a}{}^{1d}{}_{0b}\\
K_{0a}{}^{2d}{}_{0b} &= \bar{K}_{0a}{}^{2d}{}_{0b} + \bar{K}_{0a}{}^{1c}{}_{0b}M^{2d}_{1c}\\
K_{0a}{}^{3d}{}_{0b} &= \bar{K}_{0a}{}^{3d}{}_{0b} + \bar{K}_{0a}{}^{2c}{}_{0b}M^{3d}_{2c} + \bar{K}_{0a}{}^{1c}{}_{0b}M^{3d}_{1c}.
\end{aligned}
\tag{2.4}
$$

(2.4) is obtained from (2.1) using (1.19). □

**Proposition 2.2.** *$T_{V_1}$ is integrable distribution if $\bar{K}_{1a}{}^{\alpha d}{}_{1b}$ $\alpha = 2, 3$ determined by (2.6) are equal to zero.*

*Proof.* By direct calculation, taking into account (1.20) one obtains

(2.5) $$[\delta_{1a}, \delta_{1b}] = \bar{K}_{1a}{}^{2d}{}_{1b}\partial_{2d} + \bar{K}_{1a}{}^{3d}{}_{1b}\partial_{3d},$$

where

(2.6) $$\bar{K}_{1a}{}^{2d}{}_{1b} = \partial_{1b}N^{2d}_{1a} - \partial_{1a}N^{2d}_{1b}$$
$$\bar{K}_{1a}{}^{3d}{}_{1b} = [(\partial_{1b} - N^{2c}_{1b}\partial_{2c})N^{3d}_{1a}] - [a/b].$$

$[\delta_{1a}, \delta_{1b}]$ expressed in the basis $B$ has the form:

(2.7) $$[\delta_{1a}, \delta_{1b}] = K_{1a}{}^{2d}{}_{1b}\delta_{2d} + K_{1a}{}^{3d}{}_{1b}\delta_{3d},$$

where

(2.8) $$K_{1a}{}^{2d}{}_{1b} = \bar{K}_{1a}{}^{2d}{}_{1b}$$
$$K_{1a}{}^{3d}{}_{1b} = \bar{K}_{1a}{}^{3d}{}_{1b} + \bar{K}_{1a}{}^{2c}{}_{1b}M^{3d}_{2c}.$$ □

**Proposition 2.3.** *$T_{V_2}$ is integrable distribution.*

*Proof.* We have

(2.9) $$[\delta_{2a}, \delta_{2b}] = [(\partial_{2b} - N^{3c}_{2b}\partial_{3c})N^{3d}_{2a}] - [a/b],$$

but using (1.20) the above equation reduces to the form

(2.10) $$[\delta_{2a}, \delta_{2b}] = 0.$$ □

**Proposition 2.4.** *$T_{V_3}$ is integrable distribution*

(2.11) $$[\delta_{3a}, \delta_{3b}] = 0.$$

**Proposition 2.5.** *For $[\delta_{0a}, \delta_{1b}]$ we have:*

(2.12) $$[\delta_{0a}, \delta_{1b}] = \bar{K}_{0a}{}^{1c}{}_{1b}\partial_{1c} + \bar{K}_{0a}{}^{2c}{}_{1b}\partial_{2c} + K_{0a}{}^{3c}{}_{1b}\partial_{3c},$$

*where*

(2.13) $$\bar{K}_{0a}{}^{1c}{}_{1b} = \partial_{1b}N^{1c}_{0a}$$
$$\bar{K}_{0a}{}^{2c}{}_{1b} = (\partial_{1b} - N^{2d}_{1b}\partial_{2d})N^{2c}_{0a} - (\partial_{0a} - N^{1d}_{0a}\partial_{1d})N^{2c}_{1b}$$
$$\bar{K}_{0a}{}^{3c}{}_{1b} = (\partial_{1b} - N^{2d}_{1b}\partial_{2d} - N^{3d}_{1b}\partial_{3d})N^{3c}_{0a} - (\partial_{0a} - N^{1d}_{0a}\partial_{1d} - N^{2d}_{0a}\partial_{2d})N^{3c}_{1b}.$$

*$[\delta_{0a}, \delta_{1b}]$ in the basis $B$ has the form*

(2.14) $$[\delta_{0a}, \delta_{1b}] = K_{0a}{}^{1c}{}_{1b}\delta_{1c} + K_{0a}{}^{2c}{}_{1b}\delta_{2c} + K_{0a}{}^{3c}{}_{1b}\delta_{3c},$$

*where*

(2.15) $$K_{0a}{}^{1c}{}_{1b} = \bar{K}_{0a}{}^{1c}{}_{1b}$$
$$K_{0a}{}^{2c}{}_{1b} = \bar{K}_{0a}{}^{2c}{}_{1b} + \bar{K}_{0a}{}^{1d}{}_{1b}M^{2c}_{1d}$$
$$K_{0a}{}^{3c}{}_{1b} = \bar{K}_{0a}{}^{3c}{}_{1b} + \bar{K}_{0a}{}^{2d}{}_{1b}M^{3c}_{2d} + \bar{K}_{0a}{}^{1d}{}_{1b}M^{3c}_{1d}.$$

**Proposition 2.6.** *For* $[\delta_{0a}, \delta_{2b}]$ *we have*

$$(2.16)\qquad [\delta_{0a}, \delta_{2b}] = \bar{K}_{0a}{}^{2c}{}_{2b}\partial_{2c} + K_{0a}{}^{3c}{}_{2b}\partial_{3c},$$

*where*

$$(2.17)\qquad \begin{aligned} \bar{K}_{0a}{}^{2c}{}_{2b} &= \partial_{2b}N^{2c}_{0a} \\ \bar{K}_{0a}{}^{3c}{}_{2b} &= (\partial_{2b} - N^{3d}_{2b}\partial_{3d})N^{3c}_{0a} - (\partial_{0a} - N^{1d}_{0a}\partial_{1d})N^{3c}_{2b}. \end{aligned}$$

*In the basis B (2.16) has the form*

$$(2.18)\qquad [\delta_{0a}, \delta_{2b}] = K_{0a}{}^{2c}{}_{2b}\delta_{2c} + K_{0a}{}^{3c}{}_{2b}\delta_{3c},$$

*where*

$$(2.19)\qquad \begin{aligned} K_{0a}{}^{2c}{}_{2b} &= \bar{K}_{0a}{}^{2c}{}_{2b} \\ K_{0a}{}^{3c}{}_{2b} &= \bar{K}_{0a}{}^{3c}{}_{2b} + \bar{K}_{0a}{}^{2d}{}_{2b}M^{3c}_{2d}. \end{aligned}$$

**Proposition 2.7.** *For* $[\delta_{1a}, \delta_{2b}]$ *we have*

$$(2.20)\qquad [\delta_{1a}, \delta_{2b}] = \bar{K}_{1a}{}^{3c}{}_{2b}\partial_{3c} = K_{1a}{}^{3c}{}_{2b}\delta_{3c},$$

*where*

$$(2.21)\qquad K_{1a}{}^{3c}{}_{2b} = \bar{K}_{1a}{}^{3c}{}_{2b} = \partial_{2b}N^{3c}_{1a} - \partial_{1a}N^{3c}_{2b}.$$

**Proposition 2.8.** *We have*

$$(2.22)\qquad [\delta_{1a}, \delta_{3b}] = 0.$$

The proof is obtained by direct calculation using (1.20).

**Proposition 2.9.** *We have*

$$(2.23)\qquad [\delta_{2a}, \delta_{3b}] = 0.$$

## 3. Variational problem of the Lagrangian of order three

**Definition 3.1.** *A differentiable Lagrangian of order three on a* $C^\infty$ *manifold* $E$ *is a function* $L : E \to R$ *differentiable on* $\tilde{E}(rank[y^{1a}] = 1)$ *and continuous at the points of* $E$*, where* $y^{1a}$ *are equal to zero.*

From this definition it follows that

$$(3.1)\qquad g_{ab}(x, y^1, \dots, y^3) = \frac{1}{2}\partial_{3a}\partial_{3b}L^2$$

is a symmetric $d$-tensor field of type $(0, 2)$ on $\tilde{E}$. We say that the Lagrangian $L$ is regular if $rank[g_{ab}] = n$ on $\tilde{E}$.

**Definition 3.2.** *We call a Lagrange space of order three a pair* $L^{(3)n} = (E, L)$, *where* $L$ *is a regular* $C^\infty$ *Lagrangian of order* 3 *and the d-tensor field* $g_{ab}$ *from (3.1) has a constant signature on* $\tilde{E}$.

If the metric tensor $G$ on $T(E)$ is defined by:

$$G = g_{ab}\delta y^{0a} \otimes \delta y^{0b} + g_{ab}\delta y^{1a} \otimes \delta y^{1b} + g_{ab}\delta y^{2a} \otimes \delta y^{2b} + g_{ab}\delta y^{3a} \otimes \delta y^{3b},$$

then $T_H, T_{V_1}, T_{V_2}, T_{V_3}$ with respect to $G$ are mutually orthogonal to each other.

Let $L : E \to R$ be a differentiable Lagrangian of order three and $c : t \in [0,1] \to (x^a(t))\partial_a \in M$ a smooth parametrized curve, such that $Im c \subset U$. $U$ being the domain of a local chart of the differentiable manifold $M$.

The extension $c^*$ (of $c$) to $\tilde{E}$ is given by

$$c^* : t \in [0,1] \to x^a(t)\partial_a + d_t^1 x^a(t)\partial_{1a} + d_t^2 x^a(t)\partial_{2a} + d_t^3 x^a(t)\partial_{3a}$$

where the notations:

$$d_t^\alpha = \frac{d^\alpha}{dt^\alpha},\ y^{\alpha a} = d_t^\alpha x^a, \quad \alpha = 1,2,3$$

are used.

The integral of the action of the Lagrangian $L$ along the curve $c^*$ is given by

$$I_{(c^*)} = \int_0^1 L(x, d_t^1 x, d_t^2 x, d_t^3 x)dt = \int_0^1 L(x, y^1, y^2, y^3)dt. \tag{3.2}$$

We consider the curves $c_\varepsilon^*$ on $\tilde{E}$:

$$\begin{aligned} c_\varepsilon^* : t \in [0,1] \to & \\ & (x^a(t) + \varepsilon v^a(t))\partial_{0a} + (y^{1a}(t) + \varepsilon v^{1a}(t))\partial_{1a} + (y^{2a}(t) + \varepsilon v^{2a}(t))\partial_{2a} + \\ & (y^{3a}(t) + \varepsilon v^{3a}(t))\partial_{3a}, \end{aligned}$$

where

$$\begin{aligned} & v^a(t) = v^a(x^1(t), \ldots, x^n(t)), \\ & y^{\alpha a} = d_t^\alpha x^a,\ v^{\alpha a} = d_t^\alpha v^a,\ \alpha = 1,2,3, \end{aligned}$$

$v^a(t)$ are $C^\infty$ functions along $c_\epsilon^*$ and $\varepsilon$ is a real number sufficiently small in absolute value, such that

$$x^a + \varepsilon v^a \in U \subset M.$$

We assume that

$$v^a(0) = v^a(1) = 0,\ d_t^\alpha v^a(0) = d_t^\alpha v^a(1) = 0,\ \alpha = 1,2. \tag{3.3}$$

The integral of action of the Lagrangian $L$ along $c^*_\varepsilon$ is

$$I_{(c^*_\varepsilon)} = \int_0^1 L(x+\varepsilon v, d^1_t(x+\varepsilon v), d^2_t(x+\varepsilon v), d^3_t(x+\varepsilon v))dt. \tag{3.4}$$

A necessary condition that $I_{(c^*)}$ be an extremal value for $I_{(c^*_\varepsilon)}$ is

$$\left.\frac{dI_{(c^*_\varepsilon)}}{d\varepsilon}\right|_{\varepsilon=0} = 0. \tag{3.5}$$

Using the regularity, the operators $\frac{d}{d\varepsilon}$ and $\int$ can be permuted, i.e. we get

$$\frac{dI_{(c^*_\varepsilon)}}{d\varepsilon} = \int_0^1 \frac{d}{d\varepsilon} L(x+\varepsilon v, d^1_t(x+\varepsilon v), d^2_t(x+\varepsilon v), d^3_t(x+\varepsilon v))dt = \tag{3.6}$$
$$\int_0^1 [(\partial_{0a}L)v^a + (\partial_{1a}L)d^1_t v^a + (\partial_{2a}L)d^2_t v^a + (\partial_{3a}L)d^3_t v^a]dt.$$

As

$$\begin{aligned}
(\partial_{1a}L)d^1_t v^a &= d^1_t((\partial_{1a}L)v^a) - (d^1_t\partial_{1a}L)v^a, \\
(\partial_{2a}L)d^2_t v^a &= d^1_t((\partial_{2a}L)d^1_t v^a) - d^1_t((d^1_t\partial_{2a}L)v^a) + (d^2_t\partial_{2a}L)v^a \\
(\partial_{3a}L)d^3_t v^a &= d^1_t((\partial_{3a}L)d^2_t v^a) - d^1_t((d^1_t\partial_{3a}L)d^1_t v^a) + \\
&\quad d^1_t((d^2_t\partial_{3a}L)v^a) - (d^3_t\partial_{3a}L)v^a
\end{aligned}$$

the substitution of the above equations into (3.6) results in

$$\frac{dI_{(c^*_\varepsilon)}}{d\varepsilon} = \int_0^1 \{(\partial_{0a}L - d^1_t\partial_{1a}L + d^2_t\partial_{2a}L - d^3_t\partial_{3a}L)v^a + \tag{3.7}$$
$$d^1_t[(\partial_{1a}L - d^1_t\partial_{2a}L + d^2_t\partial_{3a}L)v^a + (\partial_{2a}L - d^1_t\partial_{3a}L)d^1_t v^a + \partial_{3a}Ld^2_t v^a]\}\, dt.$$

Using the notations:

$$\begin{aligned}
&\text{(a)} \quad E^0_a = \partial_a - d^1_t\partial_{1a} + d^2_t\partial_{2a} - d^3_t\partial_{3a} \\
&\text{(b)} \quad E^1_a = \partial_{1a} - d^1_t\partial_{2a} + d^2_t\partial_{3a} \\
&\text{(c)} \quad E^2_a = \partial_{2a} - d^1_t\partial_{3a} \\
&\text{(d)} \quad E^3_a = \partial_{3a}
\end{aligned} \tag{3.8}$$

(3.7) can be written in the form:

$$\frac{DI_{(c^*_\varepsilon)}}{d\varepsilon} = \int_0^1 [E^0_a(L)v^a + d^1_t[E^1_a(L)v^a + E^2_a(L)d^1_t v^a + E^3_a(L)d^2_t v^a)]dt = \tag{3.9}$$
$$\int_0^1 E^0_a(L)v^a dt + [(E^1_a(L)v^a + E^2_a(L)d^1_t v^a + E^3_a(L)d^2_t v^a]\Big|_{t=0}^{t=1}.$$

The comparison of (3.6) and (3.9) gives the following important formula:

$$(\partial_{0a}L)v^a + (\partial_{1a}L)d_t^1 v^a + (\partial_{2a}L)d_t^2 v^a + (\partial_{3a}L)d_t^3 v^a = E_a^0(L)v^a + d_t^1[E_a^1(L)v^a + E_a^2(L)d_t^1 v^a + E_a^3(L)d_t^2 v^a]. \tag{3.10}$$

According to (3.3) the last part of (3.9) vanishes and we obtain

$$\frac{dI_{(c_\varepsilon^*)}}{d\varepsilon} = \int_0^1 E_a^0(L)v^a dt = 0.$$

As $v^a(t)$ are arbitrary functions we get

**Theorem 3.1.** *In order the integral of action $I(c^*)$ be an extremal value for the functionals $I(c_\varepsilon^*)$, it is necessary that the following Euler-Lagrange equations hold:*

$$E_a^0(L) = \partial_a L - d_t^1 \partial_{1a} L + d_t^2 \partial_{2a} L - d_t^3 \partial_{3a} L = 0, \tag{3.11}$$

$$y^{1a} = \frac{dx^a}{dt},\ y^{2a} = \frac{d^2x^a}{dt^2},\ y^{3a} = \frac{d^3x^a}{dt^3}. \tag{3.12}$$

From (1.6) and (1.4) it follows that $E_a^3$ is a covariant $d$-field, i.e.

$$E_a^3 = (\partial_{3a}y^{3a'})E_{a'}^3 = (\partial_a x^{a'})E_{a'}^3. \tag{3.13}$$

Furhter we have:

$$E_a^2 = \partial_{2a} - d_t^1\partial_{3a} = (\partial_{2a}y^{2a'})\partial_{2a'} + (\partial_{2a}y^{3a'})\partial_{3a'} - d_t^1(\partial_{3a}y^{3a'})\partial_{3a'} - (\partial_{3a}y^{3a'})d_t^1\partial_{3a'}. \tag{3.14}$$

From (1.4) and (3.14) it follows

$$E_a^2 = (\partial_a x^{a'})E_{2a'} + 2(\partial_{0a}y^{1a'})E_{a'}^3. \tag{3.15}$$

Using (1.4) and (3.8b) we get

$$E_a^1 = (\partial_{0a}y^{0a'})E_{a'}^1 + (\partial_{0a}y^{1a'})E_{a'}^2 + (\partial_{0a}y^{2a'})E_{a'}^3. \tag{3.16}$$

from which follows that $E_a^1$ is not a $d$-tensor.

In a similar way and using the relations (1.4) and (3.8a) we get

$$E_a^0 = (\partial_a x^{a'})E_{a'}^0. \tag{3.17}$$

From (3.13), (3.15), (3.16) and (3.18) it follows that $E_a^3$ and $E_a^0$ are $d$-tensors, but $E_a^1$ and $E_a^2$ are not $d$-tensor fields.

**Theorem 3.2.** *The equation (3.10) is invariant with respect to the change of coordinates of type (1.3) if and only if the functions $v^a(x)$ are transformed as $d$-tensors, if i.e. $v^{a'} = (\partial_a x^{a'})v^a$.*

*Proof.* Let us suppose that the condition holds. Then, using the notations from (1.4) we have

$$(3.18)\qquad \begin{aligned} v^{a'} &= A^{a'}_a v^a,\ d^1_t v^{a'} = B^{a'}_a v^a + A^{a'}_a d^1_t v^a \\ d^2_t v^{a'} &= C^{a'}_a v^a + 2B^{a'}_a d^1_t v^a + A^{a'}_a d^2_t v^a, \\ d^3_t v^{a'} &= D^{a'}_a v^a + 3C^{a'}_a d^1_t v^a + 3B^{a'}_a d^2_t v^a + A^{a'}_a d^3_t v^a. \end{aligned}$$

The substitution of (3.18) into

$$\begin{aligned} &v^{a'}\partial_{0a'} + (d^1_t v^{a'})\partial_{1a'} + (d^2_t v^{a'})\partial_{2a'} + (d^3_t v^{a'})\partial_{3a'} = \\ &v^a E^0_a + d^1_t[v^a E^1_a + (d^1_t v^a)E^2_a + (d^2_t v^a)E^3_a] \end{aligned}$$

results in the following equations

$$(3.19)\qquad \begin{aligned} E^0_a + d^1_t E^1_a &= A^{a'}_a \partial_{a'} + B^{a'}_a \partial_{1a'} + C^{a'}_a \partial_{2a'} + D^{a'}_a \partial_{3a'} \\ E^1_a + d^1_t E^2_a &= A^{a'}_a \partial_{1a'} + 2B^{a'}_a \partial_{2a'} + 3C^{a'}_a \partial_{3a'} \\ E^2_a + d^1_t E^3_a &= A^{a'}_a \partial_{2a'} + 3B^{a'}_a \partial_{3a'} \\ E^3_a &= A^{a'}_a \partial_{3a'}. \end{aligned}$$

From (3.8) it follows

$$(3.20)\qquad E^0_a + d^1_t E^1_a = \partial_a,\ E^1_a + d^1_t E^2_a = \partial_{1a},\ E^2_a + d^1_t E^3_a = \partial_{3a}.$$

If we substitute (3.20) and (1.4) into (3.19) we obtain (1.6). The proof in the opposite direction shows that (3.18) is a necessary condition. □

**Theorem 3.3.** *If the expression*

$$(a)\qquad v^a E^0_a + d^1_t[v^a E^1_a + (d^1_t v^a)E^2_a + (d^2_t v^a)E^3_a]$$

*is a scalar field with respect to the transformation group determined by (1.3), then* $E^3_a$, $E^2_a$, $E^1_a$ *and* $E^0_a$ *transform as is prescribed by (3.13), (3.15), (3.16) and (3.17) respectively.*

*Proof.* In Theorem 3.2 it was proved that if (a) is a scalar field, then (3.18) is satisfied, and we have

$$(3.21)\qquad \begin{aligned} &v^a E^0_a + d^1_t[v^a E^1_a + (d^1_t v^a)E^2_a + (d^2_t v^a)E^3_a] = \\ &A^{a'}_a E^0_{a'} + d^1_t[A^{a'}_a v^a E^1_{a'} + (B^{a'}_a v^a + A^{a'}_a d^1_t v^a)E^2_{a'} + \\ &(C^{a'}_a v^a + 2B^{a'}_a d^1_t v^a + A^{a'}_a d^2_t v^a)E^3_{a'}]. \end{aligned}$$

One solution of the above equation can be obtained if we put $v^a E^0_a = A^{a'}_a v^a E^0_{a'}$, then we have:

$$(3.22)\qquad \begin{aligned} E^0_a &= A^{a'}_a E^0_a \\ E^1_a &= A^{a'}_a E^1_{a'} + B^{a'}_a E^2_a + C^{a'}_a E^3_{a'} \\ E^2_a &= A^{a'}_a E^2_{a'} + 2B^{a'}_a E^3_{a'} \\ E^3_a &= A^{a'}_a E^3_{a'}. \end{aligned}$$

Using (1.4) it is easy to see that (3.22) is equivalent to (3.17), (3.16), (3.15) and (3.13), but these equations were obtained using only the definition of $E^{\alpha}_{a}$ $\alpha = 0, 1, 2, 3$ and (1.3), so (3.22) is the unique solution of (3.21). □

**Remark.** $E^{\alpha}_{a}$ $\alpha = 0, 1, 2, 3$ defined by (3.8) corresponds to the Craig-Synge covectors from [15], but in this geometry they have different form.

**Theorem 3.4.** *With respect to the coordinate transformation (1.3) the Liouville vector fields have the form*

$$\begin{aligned} \Gamma_{(1)} &= y^{1a}\partial_{3a}, \quad \Gamma_{(2)} = y^{1a}\partial_{2a} + 3y^{2a}\partial_{3a}, \\ \Gamma_{(3)} &= y^{1a}\partial_{1a} + 2y^{2a}\partial_{2a} + 3y^{3a}\partial_{3a}. \end{aligned} \tag{3.23}$$

In the geometry where Miron's transformation group is used ([15], [16], [17]) $\Gamma_{(1)}$ and $\Gamma_{(3)}$ are the same as here, but $\Gamma_{(2)} = y^{1a}\partial_{2a} + 2y^{2a}\partial_{3a}$.

The vector fields $\Gamma_{(\alpha)}$, $\alpha = 1, 2, 3$ given by (3.23) in the basis $B$ have the form

$$\begin{aligned} \Gamma_{(1)} &= z_1^{3a}\delta_{3a}, \quad \Gamma_{(2)} = z_2^{2a}\delta_{2a} + z_2^{3a}\delta_{3a}, \\ \Gamma_{(3)} &= z_3^{1a}\delta_{1a} + z_3^{2a}\delta_{2a} + z_3^{3a}\delta_{3a}. \end{aligned} \tag{3.24}$$

The relation between the components is given by:

$$\begin{aligned} z_1^{3a} &= y^{1a}, \quad z_2^{2a} = y^{1a}, \quad z_2^{3a} = 3y^{2a} + y^{1b}M^{3a}_{2b} \\ z_3^{1a} &= y^{1a}, \quad z_3^{2a} = 2y^{2a} + y^{1b}M^{2a}_{1b} \\ z_3^{3a} &= 3y^{3a} + 2y^{2b}M^{3a}_{2b} + y^{1b}M^{3a}_{1b}. \end{aligned} \tag{3.25}$$

The proof is obtained by (1.19). All $z$ from (3.25) with respect to (1.3) are transformed as tensors of the type (1,0).

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