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APPROXIMATED TRAVELING WAVE SOLUTIONS TO GENERALIZED HOPF EQUATION

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Abstract. We give a direct method of characterizing traveling wave solutions to a Riemann problem within the Colombeau space $\mathcal{G}$. Solutions are represented by nets of approximated solutions. The propagation of singularities is also described.

Although we assume that f in (1) is smooth, while the usual assumption is $f \in C^2(\mathbf{R})$ (cf. [28]), our approach is simple, flexible, includes generalized function solutions and give possibilities for a unified treatment of various different approaches.

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1. Introduction

Our aim is to analyze in the frame of the Colombeau space $\mathcal{G}$ the solutions to

(1) $$u_t + (F(u))_x = u_t + f(u)u_x \approx 0, \quad t > t_0, x \in \mathbf{R}, t_0 \leq 0$$

(2) $$u(x,0) = u_0(x) + \sum_{i=1}^{s}(u_i(x) - u_{i-1}(x))H(x - x_i), \quad x \in \mathbf{R}$$

which are given in the form of traveling wave solutions, where f is a smooth function, H is the Heaviside function, $u_i \in C^\infty(\mathbf{R})$, $i = 0, 1, \ldots, s$,

(3) $$x_1 \leq x_2 \leq \ldots \leq x_s.$$

The term "approximated solution" used in the title means in fact a solution in the sense of a weak limit i.e. "associated solution" in Colombeau sense; an associated solution to an equation with "=" is a solution to an equation with "$\approx$".

Our aim in this paper is to extend the approach of using Colombeau's generalized functions to some parts of the very rich and vast classical theory of

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conservation laws. This approach is close to Maslov's method of finding solutions via asymptotic expansion ([10], [11], [20], [21]). It is made within the space $\mathcal{G}$ of the Colombeau generalized functions, which enables us to perform operations of composition $f(u)$ as well as of product $f(u)u_x$, where f is a smooth function. Note the Colombeau theory is used in the analysis of nonlinear problems as well as for linear problems with singular coefficients. We refer to [3], [4], [8], [9], [13], [17], [19], [23], [24], [27] for Colombeau theory and some of applications of this theory. Inviscid Burgers equation (here called Hopf's equation), Korteweg de Vries equation and (1) within Colombeau generalized functions are considered in [1], [3], [4], [5], [6], [14], [17], [22], [24], [25].

Equation (1), (2) in $\mathcal{G}$ is considered in [7] and [25]. In [25], viscosity type solutions to the corresponding parabolic equation are analyzed, while in [7] this problem is considered with one discontinuity line.

We extend the analysis started in [7] by considering the traveling wave solutions with various types of discontinuities. We characterize in Propositions 2, 3 and 4 the solutions to (1), (2). Solutions are the Colombeau generalized functions, which are associated to shock wave solutions, if we add the entropy condition. They are represented by the corresponding nets, elements of $\mathcal{E}_M(\mathbf{R}^2)$ (defined in Section 2), which appear in the proofs of the propositions and outline the essential connection with the classically formulated problems ([15], [18], [28]). Nets of approximate smooth solutions are suitable for explaining the phenomena known in the theory. The different feature of the solutions to (1) and solutions to $u_t + uu_x \approx 0$ with the same initial data is given in Proposition 1.

Note that in [12] we studied the Lie symmetry groups of (1), (2) in an associated sense. We refer to [2], [16], [26] and to [17] for this theory within the space $\mathcal{G}$.

2. Notions

We recall the basic definitions of the Colombeau theory (cf. [3], [8], [9], [13], [24] and [27]).

As usual, denote by $C^\infty(\Omega)$, Ω being open in $\mathbf{R}^n$, the space of smooth complex valued functions φ on Ω such that

$$\mu_k(\varphi) = \sup\{|\varphi^{(i)}(x)| \mid x \in \Omega_k, |i| \leq k\} < \infty, \quad k \in \mathbf{N},$$

where $\Omega_k \subset\subset \Omega_{k+1}$ for $k \in \mathbf{N}$ and $\bigcup_{k=1}^{\infty} \Omega_k = \Omega$. We define $\mathcal{E}_M$ as the set of families $(R_\varepsilon)_{\varepsilon\in(0,1)} \in C^\infty(\Omega)^{(0,1)}$, such that for every $k \in \mathbf{N}$ there exists $a \in \mathbf{R}$ such that $\mu_k(R_\varepsilon) = \mathcal{O}(\varepsilon^a)$, where $\mathcal{O}(\varepsilon^a)$ means that the left-hand side is smaller than $C\varepsilon^a$ for some $C > 0$ and every $\varepsilon \in (0,1)$. By $\mathcal{N}$ is denoted the space of all elements (R_ε) in $\mathcal{E}_M$ such that $\mu_k(R_\varepsilon) = \mathcal{O}(\varepsilon^a)$ for every $k \in \mathbf{N}$ and $a \in \mathbf{R}$. The quotient space $\mathcal{G} = \mathcal{E}_M/\mathcal{N}$ is Colombeau's space of generalized functions.

Its elements are denoted by $\mathrm{cl}\,[R_\varepsilon]$. The Colombeau space of complex numbers $\bar{\mathbf{C}}$ is defined by $\bar{\mathbf{C}} = \mathcal{E}_0/\mathcal{N}_0$, where $\mathcal{E}_0$ (resp. $\mathcal{N}_0$) is the set of functions $\varepsilon \mapsto R_\varepsilon$, $(0,1) \to \mathbf{C}$ such that $|R_\varepsilon| = \mathcal{O}(\varepsilon^a)$ for some $a \in \mathbf{R}$ (resp. for every $a \in \mathbf{R}$). Its subspace $\bar{\mathbf{R}}$ is defined in a similar way. Note, $\mathcal{G}(\Omega)$ is a differential algebra and $\bar{\mathbf{C}}$ can be considered as a subalgebra of $\mathcal{G}(\Omega)$.

Let $\phi \in \mathcal{S}(\mathbf{R}^n)$ be such that ϕ is even, $\mathcal{F}(\phi) = \hat{\phi} \in \mathcal{D}(\mathbf{R}^n)$ and $\hat{\phi} \equiv 1$ on a neighborhood of zero, where $\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{D}(\mathbf{R}^n)$ are Schwartz's test function spaces. Put $\phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi(\frac{x}{\varepsilon})$, $x \in \mathbf{R}^n$, $\varepsilon \in (0,1)$. Let $\psi \in \mathcal{D}(\mathbf{R}^n)$. Then $N_\varepsilon(x) = (\psi * \phi_\varepsilon(x) - \psi(x)) \in \mathcal{N}(\mathbf{R})$. Put $I_\phi(\psi) = \mathrm{cl}\,[\psi * \phi_\varepsilon]$. It can be easily verified that if $\psi_1, \psi_2 \in \mathcal{D}(\mathbf{R}^n)$, then

$$I_\phi(\psi_1 \cdot \psi_2) = I_\phi(\psi_1) \cdot I_\phi(\psi_2).$$

If $T \in \mathcal{E}'(\Omega) \subset \mathcal{E}'(\mathbf{R}^n)$ (the space of compactly supported distributions), we put $I_\phi(T) = \mathrm{cl}\,[T_\varepsilon]$, where T_ε is defined by $T_\varepsilon(x) = T*\phi_\varepsilon(x)$ for $x \in \Omega$, $\varepsilon \in (0,1)$. After proving that the presheaf $\Omega \mapsto \mathcal{G}(\Omega)$ (Ω is open in $\mathbf{R}^n$) is a sheaf, one can prove that the above embeddings can be extended to the embeddings of $C^\infty(\Omega)$ and Schwartz's space of distributions $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$. The support of a generalized function H, $\mathrm{supp}_g\, H$, is defined as the complement of the largest open subset Ω' such that $H_{|\Omega'} = 0$. This notion is coherent with the embedding I_ϕ because if $T \in \mathcal{D}'(\Omega)$, then $\mathrm{supp}\, T = \mathrm{supp}_g(I_\phi T)$.

If G is a generalized function with a compact support $K \subset\subset \Omega$ ($G \in \mathcal{G}_c(\Omega)$) and $G_\varepsilon(x)$ is a representative of G, then its integral is defined by

$$\int G\, dx = \mathrm{cl}\left[\int \psi(x) G_\varepsilon(x)\, dx\right],$$

where $\psi \in \mathcal{D}(\mathbf{R}^n)$, $\psi = 1$ on K. This definition does not depend on ψ.

Let $F, G \in \mathcal{G}(\Omega)$. Then:

(i) They are equal in the distribution sense, $G \overset{\mathcal{D}'}{=} F$ if

$$\int (G - F)\psi dx = 0 \in \overline{\mathbf{C}}, \quad \text{for any} \quad \psi \in \mathcal{D}(\Omega).$$

(ii) They are associated $G \approx F$ if there exist representatives G_ε and F_ε of G and F, respectively, such that

$$\lim_{\varepsilon \to 0} \int (G_\varepsilon(x) - F_\varepsilon(x))\psi(x) dx = 0, \text{ for any } \psi \in \mathcal{D}(\Omega).$$

Clearly, the definition (ii) does not depend on the choice of representatives.

Heaviside's function H and δ–distribution have to be embedded into $\mathcal{G}(\mathbf{R})$ as $\mathrm{cl}\,[H * \phi_\varepsilon]$ and $\mathrm{cl}\,[\delta * \phi_\varepsilon]$. Since we will deal with approximated shock wave solutions we introduce Heaviside's generalized function and δ–generalized function as follows. Heaviside's generalized function is

$$H(y) = \mathrm{cl}\,[H_\varepsilon(y)], \quad \text{where} \quad H_\varepsilon(y) = \int_{-\infty}^{y/\varepsilon} \theta(p)\,dp, \quad y \in \mathbf{R}$$

and

$$\delta(y) = \mathrm{cl}\,[\delta_\varepsilon(y)], \quad \text{where} \quad \delta_\varepsilon(y) = \frac{1}{\varepsilon}\theta\left(\frac{y}{\varepsilon}\right), \quad y \in \mathbf{R},\ \theta \in \mathcal{D},\ \int \theta = 1.$$

3. Some equalities in associated sense

The next two lemmas are of their own interests.

Lemma 1. *Let $c_1, c_2 \in \mathbf{R}$, $n \in \mathbf{N}$. Then, for $(x,t) \in \mathbf{R}^2$,*

(a) $\delta(x - c_1t)H^n(x - c_2t) \approx H(t)\delta(x - c_1t)$ *if* $c_2 < c_1$.

(b) $\delta(x - c_1t)H^n(x - c_2t) \approx H(-t)\delta(x - c_1t)$ *if* $c_2 > c_1$.

(c) $H(t)\delta(x - c_1t) + H(-t)\delta(x - c_1t) \approx \delta(x - c_1t)$.

(d) *With* $c_1 > c_2$, $x_1 < x_2$,

$$\begin{aligned} \delta(x - c_1t - x_1)H(x - c_2t - x_2) &\approx H\left(t - \frac{x_2 - x_1}{c_1 - c_2}\right)\delta(x - c_1t - x_1) \\ \delta(x - c_2t - x_2)H(x - c_1t - x_1) &\approx H\left(-t + \frac{x_2 - x_1}{c_1 - c_2}\right)\delta(x - c_2t - x_2). \end{aligned}$$

(e)

$$\begin{aligned} \delta(x - ct - x_0)H(x - ct) &\approx \delta(x - ct - x_0), \\ \delta(x - ct)H(x - ct - x_0) &\approx 0, \quad x_0 > 0. \end{aligned}$$

(f)

$$\delta(x - c_1t)H(x - c_1t)H(x - c_2t) \approx \begin{cases} H(t)\delta(x - c_1t)/2, & c_1 > c_2, \\ (1 - H(t))\delta(x - c_1t)/2, & c_1 < c_2. \end{cases}$$

Proof. We will prove only (a) and (f).

(a) Let $\psi(x,t) \in \mathcal{D}(\mathbf{R}^2)$. Then, in the form of representatives, we have to prove

$$J_\varepsilon = \iint \frac{1}{\varepsilon}\theta\left(\frac{x-c_1t}{\varepsilon}\right)\left(\left(\int_{-\infty}^{\frac{x-c_2t}{\varepsilon}} \theta(p)\,dp\right)^n - \int_{-\infty}^{\frac{t}{\varepsilon}} \theta(p)\,dp\right) \times$$
$$\psi(x,t)\,dx\,dt \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

By putting $\frac{x-c_1t}{\varepsilon} = \xi$, $t = \tau$, we have

$$J_\varepsilon = \iint \theta(\xi)\left(\left(\int_{-\infty}^{\xi+\frac{(c_1-c_2)\tau}{\varepsilon}} \theta(p)\,dp\right)^n - \int_{-\infty}^{\frac{\tau}{\varepsilon}} \theta(p)\,dp\right) \times$$
$$\psi(\varepsilon\xi + c_1\tau, \tau)\,d\xi\,d\tau.$$

The integrand is bounded on a compact set $\operatorname{supp}\theta(\xi) \cap \operatorname{supp}\psi(\varepsilon\xi + c_1\tau, \tau)$ and for $\tau \neq 0$ it converges to 0 as $\varepsilon \to 0$. Lebesgue's theorem implies $J_\varepsilon \to 0$ as $\varepsilon \to 0$.

(f) We will prove (f) in case $c_1 > c_2$. In order to prove it, we start with

$$\frac{\partial}{\partial x}\left(\left(H_\varepsilon^{n+1}(x-c_1t) - H_\varepsilon(x-c_1t)\right)H_\varepsilon(x-c_2t)\right)$$
$$= (n+1)H_\varepsilon^n(x-c_1t)\delta_\varepsilon(x-c_1t)H_\varepsilon(x-c_2t) - \delta_\varepsilon(x-c_1t)H_\varepsilon(x-c_2t) + \left(H_\varepsilon^{n+1}(x-c_1t) - H_\varepsilon(x-c_1t)\right)\delta_\varepsilon(x-c_2t).$$

So if we prove that for every $\psi \in \mathcal{D}(\mathbf{R}^2)$

$$(4)\quad I_{1\varepsilon} = \iint \left(H_\varepsilon^{n+1}(x-c_1t) - H_\varepsilon(x-c_1t)\right)\delta_\varepsilon(x-c_2t)\psi(x,t)\,dx\,dt \to 0,$$
$$I_{2\varepsilon} = \iint \frac{\partial}{\partial x}\left(\left(H_\varepsilon^{n+1}(x-c_1t) - H_\varepsilon(x-c_1t)\right)H_\varepsilon(x-c_2t)\right)\psi(x,t)\,dx\,dt$$
$$= -\iint \left(H_\varepsilon^{n+1}(x-c_1t) - H_\varepsilon(x-c_1t)\right) \times$$
$$(5)\quad H_\varepsilon(x-c_2t)\frac{\partial}{\partial x}\psi(x,t)\,dx\,dt \to 0,$$

as $\varepsilon \to 0$, it will follow

$$J_\varepsilon = I_{2\varepsilon} - I_{1\varepsilon} = \iint \Big((n+1)H_\varepsilon^n(x-c_1t)\delta_\varepsilon(x-c_1t)H_\varepsilon(x-c_2t) -$$
$$(6)\quad \delta_\varepsilon(x-c_1t)H_\varepsilon(x-c_2t)\Big)\psi(x,t)\,dx\,dt \to 0, \quad \text{as } \varepsilon \to 0,\ \psi \in \mathcal{D}(\mathbf{R}^2).$$

Now (6) implies the assertion since we have

$$\iint \Big((n+1)H_\varepsilon^n(x-c_1t)\delta_\varepsilon(x-c_1t)H_\varepsilon(x-c_2t) -$$

$$\delta_\varepsilon(x-c_1t)H_\varepsilon(t)\Big)\psi(x,t)\,dx\,dt = J_\varepsilon + \iint\Big(\delta_\varepsilon(x-c_1t)H_\varepsilon(t) -$$

$$\delta_\varepsilon(x-c_1t)H_\varepsilon(x-c_2t)\Big)\psi(x,t)\,dx\,dt = J_\varepsilon + R_\varepsilon$$

and we know by (a) that $R_\varepsilon \to 0$ as $\varepsilon \to 0$.

So, let us prove (4). But we have that (4) follows from the assertions (a) and (b). By the similar arguments, we prove (5). □

Lemma 2. *Let F be a smooth function on $\mathbf{R}$. Then*

$$F(u_0+\sum_{i=1}^{s}(u_i-u_{i-1})H(x-c_it)) \approx H(t)(F(u_0)+\sum_{i=1}^{s}(F(u_i)-F(u_{i-1}))H(x-$$

$$c_it)) + H(-t)(F(\tilde{u}_0)+\sum_{i=1}^{s}(F(\tilde{u}_i)-F(\tilde{u}_{i-1}))H(x-c_it)).$$

$$(x,t)\in\mathbf{R}\times(t_0,\infty),\quad t_0<0,$$

where $u_i, \tilde{u}_i \in \mathbf{R}$, $\tilde{u}_i = u_0+u_s-u_i$, $i=0,1,\ldots,s$, $c_i\in\mathbf{R}$, $i=1,\ldots,s$.

Proof. Under the assumption $c_1\le c_2\le\ldots\le c_s$ we have

$$\text{meas}\left(\mathbf{R}^2\setminus\left(\bigcup_{i=0}^{s}(D_{i+1}\setminus\bar{D}_i)\cup(D_i\setminus\bar{D}_{i+1})\right)\right)=0,$$

where $D_i=\{x<c_it\}$, $i=1,\ldots,s$, $D_0=\{t<0\}$, $D_{s+1}=\{t>0\}$ and $\bar{D}$ denotes the closure of a set $D\subset\mathbf{R}^2$. The claim follows from

$$\lim_{\varepsilon\to0}F(u_0+\sum_{i=1}^{s}(u_i-u_{i-1})H_\varepsilon(x-c_it))=\begin{cases}F(u_i) & \text{on } D_{i+1}\setminus\bar{D}_i\\ F(\tilde{u}_i) & \text{on } D_i\setminus\bar{D}_{i+1}\end{cases},\quad i=0,1,\ldots,s,$$

$$\lim_{\varepsilon\to0}H_\varepsilon(t)(F(u_0)+\sum_{i=1}^{s}(F(u_i)-F(u_{i-1}))H_\varepsilon(x-c_it))=$$
$$\begin{cases}F(u_i) & \text{on } D_{i+1}\setminus\bar{D}_i\\ 0 & \text{on } D_i\setminus\bar{D}_{i+1}\end{cases},\quad i=0,1,\ldots,s,$$

$$\lim_{\varepsilon\to0}H_\varepsilon(-t)(F(\tilde{u}_0)+\sum_{i=1}^{s}(F(\tilde{u}_i)-F(\tilde{u}_{i-1}))H_\varepsilon(x-c_it))=$$
$$\begin{cases}0 & \text{on } D_{i+1}\setminus\bar{D}_i\\ F(\tilde{u}_i) & \text{on } D_i\setminus\bar{D}_{i+1}\end{cases},\quad i=0,1,\ldots,s.$$

□

For the composition we need the following simple lemma.

Lemma 3. *Let f be a smooth function. Assume that $u = \mathrm{cl}\,[u_\varepsilon(x)] \in \mathcal{G}(\mathbf{R}^n)$ such that for every $K \subset\subset \mathbf{R}^n$ $\sup\{|u_\varepsilon(y)| \mid y \in K\} = \mathcal{O}(1)$. Then*

(i) For every representative $(u_\varepsilon)_\varepsilon$, $f(u_\varepsilon) \in \mathcal{E}_M(\mathbf{R}^n)$.

(ii) If $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ are representatives of u, then

$$f(u_\varepsilon) - f(\tilde{u}_\varepsilon) \in \mathcal{N}(\mathbf{R}^n).$$

Lemma 3 justifies the following definition.

Definition 1. *Let f be a smooth function. Then the composition*

$$f\left(u_0(x,t) + \sum_{i=1}^{s}(u_i(x,t) - u_{i-1}(x,t))H(x - c_i(t) - x_i)\right)$$

is defined by

$$\mathrm{cl}\left[f\left(u_0(x,t) + \sum_{i=1}^{s}(u_i(x,t) - u_{i-1}(x,t))H_\varepsilon(x - c_i(t) - x_i)\right)\right],$$

where x_i are constants, $u_i(x,t)$ and $c_i(t)$, $i = 1, \ldots, s$, are smooth functions.

Remark 1. *In the sequel, we will use the notation in accordance to this definition without referring that the above composition is an element of $\mathcal{G}(\mathbf{R}^2)$.*

4. Relations with Hopf's equation

In order to relate solutions of

$$u_t + f(u)u_x \approx 0, \quad t > 0, x \in \mathbf{R} \quad \text{and} \tag{7}$$

$$u_t + uu_x \approx 0, \quad t > 0, x \in \mathbf{R} \tag{8}$$

with the same initial condition

$$u(x,0) = u_l + (u_r - u_l)H(x) \quad x \in \mathbf{R},\ (u_l, u_r \in \mathbf{R}) \tag{9}$$

we have to use Lemma 2.

Proposition 1. *If u is of the form*

$$u(x,t) = u_l + (u_r - u_l)H(x - ct),$$

$$c = \frac{1}{u_r - u_l} \int_{u_l}^{u_r} f(u)\,du, \quad u_l, u_r \in \mathbf{R}, \quad (x,t) \in \mathbf{R} \times (0,\infty)$$

and solves (7), (9), then $\tilde{u} = f(u)$ *solves*

$$\tilde{u}_t + \tilde{u}\tilde{u}_x \approx 0, \quad \tilde{u}(x,0) \approx f(u_l) + (f(u_r) - f(u_l))H(x),$$

$(x,t) \in \mathbf{R} \times (0,\infty)$, *but this* $\tilde{u}$ *is not of the form*

$$f(u_l) + (f(u_r) - f(u_l))H(x - \tilde{c}t), \quad (x,t) \in \mathbf{R} \times (0,\infty)$$

where $\tilde{c} = \frac{f(u_r)+f(u_l)}{2}$. *Conversely, let*

$$\tilde{u}(x,t) = f(u_l) + (f(u_r) - f(u_l))H(x - \tilde{c}t), \quad (x,t) \in \mathbf{R} \times (0,\infty)$$

($\tilde{c} = (f(u_r) + f(u_l))/2$) *be a solution to (8), (9) and let* f^{-1} *exist and be smooth in a neighborhood of*

$$\{\tilde{u}_l + (\tilde{u}_r - \tilde{u}_l)H_\varepsilon(x - ct) \mid (x,t) \in \mathbf{R} \times (0,\infty),\ \varepsilon \in (0,1)\}.$$

Then $u = f^{-1}(\tilde{u}_l + (\tilde{u}_r - \tilde{u}_l)H(x - \tilde{c}t))$ *is a solution to*

$$u_t + f(u)u_x \approx 0, \quad u(x,0) \approx f^{-1}(u_l) + (f^{-1}(u_r) - f^{-1}(u_l))H(x),$$

$(x,t) \in \mathbf{R} \times (0,\infty)$, *but this* u *is not of the form*

$$f^{-1}(u_l) + (f^{-1}(u_r) - f^{-1}(u_l))H(x - ct), \quad (x,t) \in \mathbf{R} \times (0,\infty),$$

where $c = \frac{1}{f^{-1}(u_r) - f^{-1}(u_l)} \int_{f^{-1}(u_l)}^{f^{-1}(u_r)} f(u)\,du.$

The above proposition is analogous to Example 9 in [28], Ch. 5.

5. Associated solutions

We will assume in the sequel that F and f are smooth functions on $\mathbf{R}$ and that $F' = f$.

First we consider (1) with the initial condition

$$u(x,0) = u_0 + \sum_{i=1}^{s} (u_i - u_{i-1})H(x - x_i), \quad x \in \mathbf{R} \tag{10}$$

where $u_i \in \mathbf{R}$, $i = 0, 1, \ldots, s$, $x_1 \leq x_2 \leq \ldots \leq x_s$. We are going to analyze a solution to (1), (10) in $\mathcal{G}$ of the form

$$u(x,t) = u_0 + \sum_{i=1}^{s} (u_i - u_{i-1})H(x - c_i(t) - x_i),\ (x,t) \in \mathbf{R} \times (t_0,\infty), \tag{11}$$

where $t_0 \leq 0$ and $c_i(t)$, $i = 1, \ldots, s$, are smooth functions which can mutually intersect at a discrete set of points in $\mathbf{R} \times (t_0,\infty)$.

Proposition 2. *Assume $F'' > 0$. Then*

$$u(x,t) = u_0 + \sum_{i=1}^{s}(u_i - u_{i-1})H(x - c_i t),$$

$$(x,t) \in \mathbf{R} \times (t_0, \infty),\ t_0 < 0,\ c_i \in \mathbf{R},\ i = 1, \dots, s,$$

is a solution to (1) and

$$u(x,0) \quad = \quad u_0 + (u_s - u_0)H(x), \quad x \in \mathbf{R}$$

if and only if $c_0 = \ldots = c_s = c$ and c satisfies the Rankine–Hugoniot condition

$$c = \frac{F(u_s) - F(u_0)}{u_s - u_0}, \quad i = 1, \dots, s.$$

Proof. By employing the Rankine–Hugoniot condition on the line $x = c_i t$

$$\begin{array}{rcl} c_i(u_i - u_{i-1}) & = & F(u_i) - F(u_{i-1}) \\ c_i(\tilde{u}_i - \tilde{u}_{i-1}) & = & F(\tilde{u}_i) - F(\tilde{u}_{i-1}) \end{array}, \quad i = 1, \dots, s$$

with $\tilde{u}_i$ as in Lemma 2, we have

$$F(u_i) - F(u_{i-1}) = F(\tilde{u}_{i-1}) - F(\tilde{u}_i), \quad i = 1, \dots, s.$$

By the convexity of F it follows $u_i = \tilde{u}_{i-1}$ for $i = 1, \dots, s$. Thus, $u_i = u_0 + u_s - u_{i-1}$ for $i = 1, \dots, s$. So $u_i + u_{i-1}$ for $i = 1, \dots, s$ does not depend on i and hence $u_{i+1} = u_{i-1}$ for $i = 1, \dots, s-1$. It follows

$$c_{i+1} = \frac{F(u_{i-1}) - F(u_i)}{u_{i-1} - u_i} = c_i, \quad i = 1, \dots, s-1,$$

$$u(x,t) = u_0 + (u_s - u_0)H(x - ct)$$

and

$$c = \frac{F(u_s) - F(u_0)}{u_s - u_0}.$$

□

Proposition 3. *Assume $F'' > 0$.*

(i) If $s = 1$ in (10), then (1), (10) has a solution of the form (11) if and only if

$$c(t) = ct, \quad t > t_0, \quad \text{and} \quad c = \frac{F(u_1) - F(u_0)}{u_1 - u_0}. \tag{12}$$

(ii) If $s > 1$ in (10), then (1), (10) has a solution of the form (11) if and only if the constants u_i, and functions $c_i(t)$ satisfy:

(a) $c_i(t) = c_i t$, $t > t_0$,

$$c_i = \frac{F(u_i) - F(u_{i-1})}{u_i - u_{i-1}}, \quad i = 1, \dots, s, \tag{13}$$

for those i for which $c_i(t) + x_i$ has no intersection with any $c_j(t) + x_j$, $j \neq i$;

(b) if $c_k(t) + x_k$ intersects $c_r(t) + x_r$ for some $t > t_0$, then

$$c_k(t) = c_r(t) = ct, \quad t > t_0,$$

where

$$c = \frac{F(u_k) - F(u_{k-1})}{u_k - u_{k-1}} = \frac{F(u_r) - F(u_{r-1})}{u_r - u_{r-1}}.$$

Remark 2. *Concerning (ii) we know ([28]) that $f' > 0$ and $u_0 > u_1 > \ldots > u_s$ and the Rankine–Hugoniot conditions imply the unicity of the weak solution.*

Proof.

(i) As in Lemma 2 one can prove

$$F(u_0 + (u_1 - u_0)H(x - c(t) - x_1)) \approx F(u_0) + (F(u_1) - F(u_0))H(x - c(t) - x_1).$$

Thus

$$(F(u_0) + (F(u_1) - F(u_0))H(x - c(t) - x_1))_x = (F(u_1) - F(u_0))\delta(x - c(t) - x_1)$$

and we have

$$-c'(t)\delta(x - c(t) - x_1)(u_1 - u_0) + (F(u_1) - F(u_0))\delta(x - c(t) - x_1) \approx 0.$$

This implies

$$c'(t) = \frac{F(u_1) - F(u_0)}{u_1 - u_0} = c \quad \text{and} \quad c(t) = ct.$$

(ii) First we show that the proof follows from the proof of (i) and Proposition 2.

Denote $\Gamma_i = \{(x,t) \in \mathbf{R} \times (t_0, \infty) \mid x = c_i(t) + x_i\}$, $i = 1, \dots, s$. For every $i \in \{1, \dots, s\}$ and every point $(x,t) \in \Gamma_i$ which does not belong to any other $\Gamma_j, j \neq i$, there exists an open set $U_i \subset \mathbf{R} \times (t_0, \infty)$ such that

$$(x,t) \in U_i \quad \text{and} \quad U_i \cap \Gamma_j = \emptyset, \quad j = 1, \dots, i-1, i+1, \dots, s.$$

By choosing test functions $\phi \in C_0^\infty(U_i)$ we can simply prove that the assertion (i) implies (13).

Note, a function $c_i(t) + x_i$ is smooth and it is a straight line on any part where it does not intersect with any other function $c_k(t) + x_k$, $k \neq i$. Thus it is a straight line for $t > t_0$.

Now assume $(\xi, \tau) = P = \Gamma_i \cap \Gamma_j \cap \mathbf{R} \times (t_0, \infty)$, $i < j$, for some i and j and that there exists an open set $V_{ij} \subset \mathbf{R} \times (t_0, \infty)$ such that

$$P \in V_{ij} \quad \text{and} \quad V_{ij} \cap \Gamma_k = \emptyset, \quad k \neq i,\ k = j.$$

So in V_{ij} we have $\tilde{x} = c_i t + x_i = c_j t + x_j$ for $t = \tau$ and

$$u(x,t) = u_0 + (u_i - u_{i-1})H(x - \tilde{x} - c_i(t-\tau)) + u_0 + (u_j - u_{j-1})H(x - \tilde{x} - c_j(t-\tau))$$

satisfies

$$u_t - F(u)_x \approx 0,$$

$$u(x,\tau) = u_0 + ((u_i - u_{i-1}) + (u_j - u_{j-1}))\, H(x - \tilde{x}), \quad (x,\tau) \in V_{ij}.$$

By the same arguments as in the proof of Proposition 2 we have that

$$c_i = c_j.$$

The same reasoning implies that the assumption $\bigcap_{k=1}^{r} \Gamma_{i_k} = \{P\}$, $\Gamma_i \not\ni P$ for $i \neq i_k$, $k = 1, \ldots, r$, implies $c_{i_1} = \ldots = c_{i_r}$.

This implies the proof of (ii). □

Example 1. *We will show that when $f(u) = u^2$ then different velocities c_i can appear. More precisely, we prove that the Cauchy problem*

$$u_t + u^2 u_x \approx 0, \quad u(x,0) = u_0 + (u_2 - u_0)H(x), \quad (x,t) \in \mathbf{R}^2$$

has a solution of the form

$$u(x,t) = u_0 + (u_1 - u_0)H(x - c_1 t) + (u_2 - u_1)H(x - c_2 t),$$

$(x,t) \in \mathbf{R} \times (0,\infty)$, *iff $u_0 + u_2 = 0$ holds and in that case we have*

$$c_1 = \frac{u_1^2 + u_0^2 + u_1 u_0}{3}, \quad c_2 = \frac{u_1^2 + u_0^2 - u_1 u_0}{3},$$

u_1 *being an arbitrary real number subject to condition $u_0 u_1 < 0$.*

Note that c_1 and c_2 need not to be equal, subject to the choice of u_1.

Proof. In order to shorten formulas, we will use the notion

$$\tilde{u}_1 = u_1 - u_0, \quad \tilde{u}_2 = u_2 - u_1.$$

By using (f) of Lemma 1 we have

$$\begin{aligned} u_t + u^2 u_x &= -c_1\tilde{u}_1\delta(x-c_1t) - c_2\tilde{u}_2\delta(x-c_2t) + \\ &\quad (u_0 + \tilde{u}_1 H(x-c_1t) + \tilde{u}_2 H(x-c_2t))^2 (\tilde{u}_1\delta(x-c_1t) + \tilde{u}_2\delta(x-c_2t)) \\ &= -c_1\tilde{u}_1\delta(x-c_1t) - c_2\tilde{u}_2\delta(x-c_2t) + \Big(u_0^2 + \tilde{u}_1^2 H^2(x-c_1t) + \\ &\quad \tilde{u}_2^2 H^2(x-c_2t) + 2u_0\tilde{u}_1 H(x-c_1t) + 2u_0\tilde{u}_2 H(x-c_2t) + \\ &\quad 2\tilde{u}_1\tilde{u}_2 H(x-c_1t)H(x-c_2t)\Big)(\tilde{u}_1\delta(x-c_1t) + \tilde{u}_2\delta(x-c_2t)) \\ &\approx \Big(-c_1\tilde{u}_1 + u_0^2\tilde{u}_1 + \frac{\tilde{u}_1^3}{3} + \tilde{u}_1\tilde{u}_2^2(1-H(t)) + u_0\tilde{u}_1^2 + \\ &\quad 2u_0\tilde{u}_1\tilde{u}_2(1-H(t)) + \tilde{u}_1^2\tilde{u}_2(1-H(t))\Big)\delta(x-c_1t) + \\ &\quad \Big(-c_2\tilde{u}_2 + u_0^2\tilde{u}_2 + \frac{\tilde{u}_2^3}{3} + \tilde{u}_2\tilde{u}_1^2 H(t) + \\ &\quad u_0\tilde{u}_2^2 + 2u_0\tilde{u}_1\tilde{u}_2 H(t) + \tilde{u}_1\tilde{u}_2^2 H(t)\Big)\delta(x-c_2t), \quad (x,t)\in\mathbf{R}\times(0,\infty). \end{aligned}$$

By equating all the coefficients with zero, we obtain the system

$$\begin{aligned} \tilde{u}_1\tilde{u}_2^2 + 2u_0\tilde{u}_1\tilde{u}_2 + \tilde{u}_1^2\tilde{u}_2 &= 0, \\ c_1 &= u_0^2 + \frac{\tilde{u}_1^2}{3} + \tilde{u}_2^2 + u_0\tilde{u}_1 + 2u_0\tilde{u}_2 + \tilde{u}_1\tilde{u}_2, \\ c_2 &= u_0^2 + \frac{\tilde{u}_2^2}{3} + u_0\tilde{u}_2, \end{aligned}$$

which simplifies to

$$u_0 + \frac{\tilde{u}_1 + \tilde{u}_2}{2} = 0, \quad c_1 = \frac{\tilde{u}_1^2}{4} + \frac{\tilde{u}_2^2}{12}, \quad c_2 = \frac{\tilde{u}_1^2}{12} + \frac{\tilde{u}_2^2}{4}$$

and gives us the necessary conclusions. □

Proposition 4. *Assume that* $u_0(x,t),\dots,u_s(x,t) \in C^\infty(\mathbf{R}\times[t_0,\infty))$, $t_0 < 0$, *such that* $u_i(x,0) = u_i(x)$ *and that (3) holds. Let*

$$u(x,t) = u_0(x,t) + \sum_{i=1}^{s}(u_i(x,t) - u_{i-1}(x,t))H(x - c_it - x_i)$$

be a solution to (1), (2) and $c_i \in \mathbf{R}$.

If $c_it + x_i$ *intersects some* $c_jt + x_j$, *these lines must coincide for* $t > t_0$.

Assuming that the lines of discontinuities do not have intersections, we have $u_i(x,t)$, $i = 1,\dots,s-1$ *are smooth solutions to (1) on*

$$\{(x,t) \mid t > t_0,\ c_{i-1}t + x_{i-1} < x < c_it + x_i\},$$

and u_0 and u_s are smooth solutions on the domains left of $c_1 t + x_1$ and right of $c_s t + x_s$, respectively ($t > t_0$).

Moreover,

$$c_i = \frac{F(u_i(c_i t + x_i, t)) - F(u_{i-1}(c_i t + x_i, t))}{u_i(c_i t + x_i, t) - u_{i-1}(c_i t + x_i, t)}, \quad t > t_0,\ i = 1, \ldots, s. \tag{14}$$

The proof of the theorem is a straightforward application of the arguments given in Propositions 2 and 3, and therefore is omitted.

Remark 3. *Proposition 4 with $c_i(t)$ instead of $c_i t$ is a difficult task and it will be part of our further investigations.*

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