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CUBIC SPLINE DIFFERENCE SCHEME ON A MESH OF BAKHVALOV TYPE

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Abstract

The cubic spline difference scheme for solving singularly perturbed boundary value problem is considered. The non-uniform mesh of Bakhvalov type is used in order to avoid the problem of stability. The second order of the uniform convergence in respect to perturbation parameter is obtained. The result is better than the one obtained on Shishkin's mesh.

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1. Introduction

We consider a numerical solution of the singularly perturbed boundary value problem:

$$-\varepsilon^2 y'' + c(x,y) = 0, \quad x \in [0,1], \quad \varepsilon \in (0,\varepsilon_0), \quad \varepsilon_0 \ll 1, \tag{1}$$

$$y(0) = y(1) = 0$$

where ε is a small perturbation parameter.
This problem has been treated numerically in many papers.
We consider the problem (1) under the following conditions:

$$c \in \mathcal{C}^2(I \times \mathcal{R}), \quad I = [0,1], \tag{2}$$

(3) $$0 < \gamma^2 \leq c_y(x,y), \quad (x,y) \in I \times \mathcal{R},$$

For small ε, the solution of (1) has in general two boundary layers with $\mathcal{O}(\varepsilon)$ near to $x = 0$ and $x = 1$. That can be seen from the properties of the exact solution of the problem (1) described in Lemma, which will be given in another section.

It is known that the standard cubic spline difference scheme on the regular mesh gives unsatisfactory results for our problem.

In [3] this difference scheme is used on Shihkin's mesh and the obtained estimate is $\|y - u\|_\infty \leq Mn^{-2}\ln^2 n$.

In this paper we shall use the mesh of Bakhvalov type given in [2] and [8], and prove that the cubic spline difference scheme applied on such mesh is the second order accurate in the discrete maximum norm, uniformly in the perturbation parameter ε, i. e. the obtained estimate is $\|y - u\|_\infty \leq Mn^{-2}$.

2. Discretization mesh and difference scheme

In this section we consider the cubic spline difference scheme defined in [3] and applied on discretization mesh of Bakhvalov type given in [2].

A special non-equidistant mesh is used in order to obtain more mesh points in the region of boundary layers.

This fact and the estimates of derivatives of the solution determine the mesh generating function $\lambda(t)$ in [2], which is given by:

(4) $$\lambda(t) = \begin{cases} \mu(t) := a\varepsilon t/(q-t), & \text{if} \quad t \in [0,\tau] \\ \mu(\tau) + \mu'(\tau)(t-\tau), & \text{if} \quad t \in [\tau, 0.5] \\ 1 - \lambda(1-t), & \text{if} \quad t \in [0.5, 1] \end{cases}$$

where a and q are the constants independent of ε, such as

(5) $$q \in (0, 0.5), \quad a\varepsilon_0 \leq q.$$

τ is a unique point from $(0,q)$, where

(6) $$\tau = \frac{q - (a\varepsilon q(1 - 2q + 2a\varepsilon))^{1/2}}{1 + 2a\varepsilon}.$$

A class of mesh generating function suitable for discretization of our type is given by:

$$\mu(t) = a\varepsilon t/(q-t).$$

Using this function we form the mesh

$$I_h = \{x_i = \lambda(t_i), t_i = ih : i = 0, ..., n\}, \quad n \in \mathcal{N}, \quad h = 1/n, \tag{7}$$

and we also suppose $n = 2m$, $m \in \mathcal{N}$
We form the descrete analogue of the problem (1) using a cubic spline difference scheme.

Let $h_i = x_i - x_{i-1}$, $i = 1, ..., n$, $\bar{h}_i = (h_i + h_{i+1})/2$, $i = 1, ..., n-1$. The difference scheme is defined by:

$$\begin{aligned} 0 = \quad & -\varepsilon^2(r_i^- y_{i-1} + r_i y_i + r_i^+ y_{i+1}) + \\ & b_i^- c(x_{i-1}, y_{i-1}) + b_i c(x_i, y_i) + \\ & b_i^+ c(x_{i+1}, y_{i+1}), \quad i = 1, ..., n-1, \\ & y_0 = y_n = 0, \end{aligned} \tag{8}$$

where $y \in \mathcal{R}^{n+1}$,

$$r_i^- = \frac{3}{h_i \bar{h}_i}, \quad r_i^+ = \frac{3}{h_{i+1}\bar{h}_i}, \quad r_i = \frac{-6}{h_i h_{i+1}},$$

$$b_i^- = \frac{h_i}{2\bar{h}_i}, \quad b_i^+ = \frac{h_{i+1}}{2\bar{h}_i}, \quad b_i = 2.$$

Let $z := [z_0, z_1, ..., z_n]^\top \in \mathcal{R}^{n+1}$ Let A be a tridiagonal matrix, $A \in \mathcal{R}^{n+1,n+1}$, defined by

$$A = \begin{bmatrix} -\varepsilon^{-2} & 0 & 0 & & & \\ r_1^- & r_1 & r_1^+ & & & \\ & \ddots & \ddots & \ddots & & \\ & & & r_{n-1}^- & r_{n-1} & r_{n-1}^+ \\ & & & 0 & 0 & -\varepsilon^{-2} \end{bmatrix},$$

B is a mapping $B : \mathcal{R}^{n+1} \longrightarrow \mathcal{R}^{n+1}$ given by

$$(Bz)_i = \begin{cases} 0, & i = 0 \\ b_i^- c(x_{i-1}, z_{i-1}) + b_i c(x_i, z_i) + b_i^+ c(x_{i+1}, z_{i+1}), & i = 1, ..., n-1 \\ 0, & i = n \end{cases}$$

then our scheme is $Fu = 0$, where

$$F = -\varepsilon^2 A + B. \tag{9}$$

The solution $u = [u_0, u_1, ..., u_n]^\top$ of (8), is the approximation to the exact solution y of (1).

3. Truncation error estimate

In this section we shall demonstrate that the cubic spline difference scheme applied on described mesh is the second order accurate in the discrete maximum norm, uniformly in the perturbation parameter ε. In order to do that we use the following Lemma.

Lemma 1. *There exists a unique solution $y \in C^4(I \times \mathcal{R})$ of the problem (1). This solution satisfies:*

$$|y^{(j)}(x)| \leq M(1 + \varepsilon^{-j}(e^{-\gamma x/\varepsilon} + e^{-\gamma(1-x)/\varepsilon}))$$

for $x \in I$ and $j = 0, 1, 2, 3, 4$.

It is easy to see that F, as defined in (9), is continiously differentiable on $\mathcal{R}^{n+1}$. The Frechet derivative $F'(z)$ of F at any $z = [z_0, z_1, ..., z_n]^\top \in \mathcal{R}^{n+1}$ is the tridiagonal matrix

$$F'(z) = \begin{bmatrix} 1 & 0 & 0 & & & \\ f_1^- & f_1 & f_1^+ & & & \\ & \ddots & \ddots & \ddots & & \\ & & f_{n-1}^- & f_{n-1} & f_{n-1}^+ \\ & & 0 & 0 & 1 \end{bmatrix},$$

where for $i = 1, 2, ..., n-1$,

$$f_i^- = -\varepsilon^2 r_i^- + b_i^- c_y(x_{i-1}, z_{i-1}),$$

$$f_i = -\varepsilon^2 r_i + b_i c_y(x_i, z_i),$$

$$f_i^+ = -\varepsilon^2 r_i^+ + b_i^+ c_y(x_{i+1}, z_{i+1}).$$

Set

$$\mu = \min_{1 \leq i \leq n-1} \{|f_i| - |f_i^-| - |f_i^+|\}.$$

Then there exists a constant $\mu^* > 0$ which is independent of n and ε, such that $\mu \geq \mu^* > 0$ for all $z \in S$, where S is an open ball in $\mathcal{R}^{n+1}$. According to the theorem given in [5], the previous inequality implies that $F'(z)^{-1}$ exists and

$$\|F'(z)^{-1}\|_\infty \leq \frac{1}{\min\{1, \mu^*\}}, \quad \text{for} \quad z \in S.$$

For a given $z^0 \in \mathcal{R}^{n+1}$ and $r > 0$, the open ball $\{z \in \mathcal{R}^{n+1} : \|z - z^0\|_\infty < r\}$ in $\mathcal{R}^{n+1}$ is denoted by $S(z^0, r)$.

Theorem 1. *Assume that (1) and (3) hold. Then there exists a constant $M_0 > 0$ which is independent of ε and n, and a constant n_0 which depends on M_0 but is independent of ε, such that the scheme (8) has a solution $u \in \mathcal{R}^{n+1}$ satisfying $\|y-u\|_\infty \le M_0 n^{-2}$ for $n \ge n_0$. Moreover, this solution u is the only solution of (8) that lies in $S(y, M_0 n^{-2})$.*

Proof. The truncation error Fy for the function exact solution obtained in [3] has the form:

$$(Fy)_i = 6\varepsilon^2 \Phi_i/(h_i + h_{i+1})$$

where

$$\Phi_i = -\frac{1}{6}h_{i+1}R_{2,i+1} - \frac{1}{3}h_i R_{2,i} - R_{1,i} + \frac{R_{0,i+1}}{h_{i+1}}.$$

$$R_{k,i} = \frac{1}{(4-k)!}h_i^{4-k}y^{iv}(\eta_{k,i}), \qquad x_{i-1} \le \eta_{k,i} \le x_i$$

Using Lemma and analyzing the truncation error and exact solution we have

$$(Fy)_i \le M\varepsilon^2 \max\{h_i^2, h_{i+1}^2\}(1 + \varepsilon^{-4}e^{-\frac{\gamma x_{i-1}}{\varepsilon}})$$

where M is a positive constant independent of ε and n.
In [8] it is obtained that

$$\|Fy\|_\infty \le M_1 n^{-2},$$

where M_1 is a positive constant independent of ε and n.
Set $M_0 = \frac{2M_1}{\min\{1,\gamma^2/2\}}$.
We now prove that there exists a positive integer n_0, depending on M_0 but being independent of ε, such that for $n \ge n_0$

$$\|F'(z)^{-1}\|_\infty \le \frac{1}{\min\{1, \gamma^2/2\}},$$

for all $z \in S(y, M_0 n^{-2})$.
For $i = 1, 2, ..., n-1$ we have

$$\begin{aligned}
\Delta_i &= |f_i| - |f_i^+| - |f_i^-| \\
&\ge 2c_y(x_i, z_i) - \frac{2}{\bar{h}_i}(h_{i+1}c_y(x_{i+1}, z_{i+1}) + h_i c_y(x_{i-1}, z_{i-1})) \\
&= c_y(x_i, z_i) - \frac{2h_{i+1}}{\bar{h}_i}[h_{i+1}c_{xy}(\bar{x}_i, \bar{z}_i) + (z_{i+1} - z_i)c_{yy}(\bar{x}_i, \bar{z}_i)] \\
&- \frac{2h_i}{\bar{h}_i}[-h_i c_{xy}(\tilde{x}_i, \tilde{z}_i) + (z_i - z_{i-1})c_{yy}(\tilde{x}_i, \tilde{z}_i)]
\end{aligned} \tag{10}$$

where $(\tilde{x}_i, \tilde{z}_i)$ is between (x_{i-1}, z_{i-1}) and (x_i, z_i), and $(\bar{x}_i, \bar{z}_i)$ is between (x_i, z_i) and (x_{i+1}, z_{i+1}).
By Lemma, $|y(x)| \leq M_2$, for some positive constant M_2. Now

(11) $|c_{xy}(x,z)| + |c_{yy}(x,z)| \leq M, \quad \text{for} \quad (x,z) \in I \times [-M_2 - 1, M_2 + 1].$

Choose n_1 such that $M_0 n^{-2} \leq 1$ for $n \geq n_1$. Let $z \in S(y, M_0 n^{-2})$. Then $|z_i| \leq M_2 + 1$, for $i = 0, ..., n$. Consequently, for $n \geq n_1$

$$\max\{|\bar{z}_i|, |\tilde{z}_i|\} \leq M_2 + 1, \quad \text{for} \quad i = 1, ..., n-1. \tag{12}$$

Therefore, we estimate $|z_i - z_{i-1}|$ for $n \geq n_1$ and $i = 1, 2, ..., n$ with

$$|z_i - z_{i-1}| \leq |z_i - y_i| + |y_i - y_{i-1}| + |y_{i-1} - z_{i-1}| \leq 2M_0 n^{-2} + |y_i - y_{i-1}|.$$

Since,

for $t_i \leq \tau$

$$|y_i - y_{i-1}| = |\int_{x_{i-1}}^{x_i} y'(x)dx| = |\int_{\lambda(t_{i-1})}^{\lambda(t_i)} y'(x)dx| \leq \frac{M_3 h \varepsilon}{(q - t_{i-1})^2}(1 + \frac{1}{\varepsilon} e^{-\frac{a\gamma t_{i-1}}{q - t_{i-1}}}) \leq Mn^{-1}$$

for $t_{i-1} \geq \tau$

$$|y_i - y_{i-1}| \leq Mh_i + \frac{M}{\varepsilon}\int_{\lambda(\tau)}^{1/2} e^{-\frac{\gamma x}{\varepsilon}} dx \leq Mh + Me^{-\frac{\gamma}{\sqrt{\varepsilon}}} \leq Mn^{-1},$$

for $t_{i-1} < \tau < t_i$

$$|y_i - y_{i-1}| \leq |\int_{x_{i-1}}^{\lambda(\tau)} y'(x)dx| + |\int_{\lambda(\tau)}^{x_i} y'(x)dx| \leq Mn^{-1}.$$

From (10)-(12) and the estimates of $|z_i - z_{i-1}|$ and $|y_i - y_{i-1}|$ we obtain for $n \geq n_1$ that

$$\Delta_i \geq \gamma^2 - M_3 n^{-1}, \quad i = 1, 2, ...n - 1,$$

where M_3 is a positive constant which depends on M_0 but is independent of ε and n. Choose n_2 such that $M_3 n^{-1} \leq \gamma^2/2$ for $n \geq n_2$. Then for $n \geq n_0 = \max\{n_1, n_2\}$ we obtain $\Delta_i \geq \gamma^2/2$, for $i = 1, 2, ..., n-1$.
Thus

$$\Delta_i \geq \min\{1, \gamma^2/2\}, \quad i = 0, 1, ..., n.$$

This yields

$$\|F'(z)^{-1}\|_\infty \le \frac{1}{\min\{1, \gamma^2/2\}}, \qquad z \in S(y, M_0 n^{-2})$$

by the theorem given in [5].
This implies that the system $Fu = 0$ can have at most one solution u in $S(y, M_0 n^{-2})$.

4. Numerical results

In this section we shall present some numerical results obtained using cubic spline difference scheme on the non-equidistant mesh described in the previous section.
We shall consider the nonlinear test problem

$$-\varepsilon^2 y'' + (y^2 + y - 0.75)(y^2 + y - 3.75) = 0, \quad y(0) = y(1) = 0$$

for which the exact solution y_ε is unknown. The mesh I_h with $a = 1$, $q = 0.48$ and $\gamma = 1$ is used.

The numerical solving of the nonlinear singularly perturbed boundary value problem becomes the solving of a system of linear equations, in every iteration step, using Newton's method given in [2], which guarantees the local convergence.

Let

$$E_n = \|y_\varepsilon - u\|_\infty,$$

where y_ε is the exact solution and u is the solution of the discrete analogue. Also, we define in the usual way the order of convergence Ord for two successive values of n with respective errors E_n and E_{2n}:

$$Ord = \frac{\ln E_n - \ln E_{2n}}{\ln 2}.$$

As the exact solution of our problem is not known, the approximate solution with $n = 1024$ points is used instead of it.

The following tables illustrate that the cubic spline difference scheme used on the decsribed non-uniform mesh of Bakhvalov type is the second order uniformly convergent.

$n\backslash\varepsilon$	2^{-3}	2^{-4}	2^{-5}	2^{-6}	
8	0.0354202 —	0.0505455 —	0.0523618 —	0.0530485 —	E_n Ord
16	0.0117025 1.59775	0.0130909 1.94901	0.0131023 1.99869	0.013104 2.01731	E_n Ord
32	0.00301112 1.95844	0.00323265 2.01778	0.00323298 2.01888	0.00323298 2.01907	E_n Ord
64	7.50417(−4) 2.00454	8.11532(−4) 1.994	8.11602(−4) 1.99402	8.11602(−4) 1.99402	E_n Ord
128	1.85434(−4) 2.01679	2.00457(−4) 2.01736	2.0047(−4) 2.01739	2.0047(−4) 2.01739	E_n Ord
256	4.4221(−5) 2.0681	4.7772(−5) 2.06905	4.77753(−5) 2.06905	4.77753(−5) 2.06905	E_n Ord
512	8.84452(−6) 2.32188	9.55383(−6) 2.32201	9.55449(−6) 2.32202	9.55449(−6) 2.32202	E_n Ord

Table 1.

$n\backslash\varepsilon$	2^{-7}	2^{-8}	2^{-9}	2^{-10}	
8	0.0533898 —	0.0535595 —	0.0536441 —	0.0536862 —	E_n Ord
16	0.0131042 2.02654	0.0131043 2.0311	0.0131043 2.03337	0.0131044 2.03451	E_n Ord
32	0.00323298 2.01909	0.00323298 2.0191	0.00323298 2.01911	0.00323298 2.01911	E_n Ord
64	8.11602(−4) 1.99402	8.11602(−4) 1.99402	8.11602(−4) 1.99402	8.11602(−4) 1.99402	E_n Ord
128	2.0047(−4) 2.01739	2.0047(−4) 2.01739	2.0047(−4) 2.01739	2.0047(−4) 2.01739	E_n Ord
256	4.77753(−5) 2.06905	4.77753(−5) 2.06905	4.77753(−5) 2.06905	4.77753(−5) 2.06905	E_n Ord
512	9.55449(−6) 2.32202	9.55449(−6) 2.32202	9.55449(−6) 2.32202	9.55449(−6) 2.32202	E_n Ord

Table 2.

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